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BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS IN A LIE GROUP

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Abstract We study backward stochastic differential equations where the solution process lives in a finite dimensional Lie group. The group structure makes this problem easier to deal with than in a general manifold, but the geometry still imposes interesting conditions. The main tools are the stochastic exponential and logarithm of Lie groups, used to change group-valued martingales into $\mathbb{R}^d$-valued martingales. We are first interested in getting a group-valued martingale with prescribed terminal value: existence and uniqueness are proved for nilpotent Lie groups by a constructive method; also a recursive construction of the solution is given and uniqueness is obtained for groups where a convex barycenter can be defined. We then study more general backward stochastic equations with a drift term.

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Introduction

Our aim is to solve backward stochastic differential equations for processes living in a finite dimensional Lie group G. More precisely, to obtain an adapted continuous G-valued process with prescribed terminal value, prescribed drift term and prescribed Brownian perturbation.

Those backward equations are of great interest in many applied problems (financial markets, controlled systems, ...) and also raise many interesting theoretical problems (representation of martingales, anticipative calculus, ...). Therefore a lot of papers are devoted to these equations; most of them deal with vector-valued processes and few of them with manifold-valued processes. As far as we know no one deals with Lie group-valued processes. Of course, Lie groups are specific manifolds, but with a more elaborate structure, more elaborate results are to be obtained.

In the domain of backward stochastic differential equations, the Pardoux and Peng paper [9], where processes are $\mathbb{R}^d$-valued, is to be considered a pioneer work. In a Lie group the non-linear structure raises new difficulties, but the main three steps of the method in [9] can be kept: they consist in solving the backward equation with no drift term first, then with a drift depending only on time, and last with a general drift term.

The first step is equivalent to the problem of finding a martingale with prescribed terminal value. This is straightforward when the martingale lives in a Euclidean structure by using conditional expectation. In a manifold, things are much more difficult. Different authors worked on it but positive results were obtained only under restricted conditions for the manifold. Some authors as W.S. Kendall [8] gave an answer for bounded manifolds such as small balls, or J. Picard [10] for manifolds with curvature bounded from below, and in [11] -as in [3]- for compact manifolds with convex geometry. In [4], Darling extends the previous answer to manifolds obtained as increasing limit of compact submanifolds with convex geometry and stated a conjecture concerning manifolds with compactly supported connection. Arnaudon [2] also brought a solution by constraining the martingale to stay in a compact convex subset of the manifold. Existence and uniqueness of a martingale with prescribed terminal value is solved here for non compact Lie groups of two types: the (Γ)-groups, which are more or less flat in a specific coordinate system, and the simply connected nilpotent Lie groups.

The literature does not deal with general backward stochastic equations in a general manifold, although Darling [3] solves the prescribed terminal value martingale problem by using backward equation. In a Lie group, the second step of the procedure in [9] (drift depending only on time) is easy to deal with using the stochastic exponential of Lie groups: a Girsanov type formula reduces the problem to a martingale problem. The third and last step is solved for (Γ)-groups only where Stratonovich equations become Itô equations; so, the usual method with Picard iteration is available.

The paper is organized as follows. First section introduces the main tools. As usual in a manifold, a notion of connection is needed to describe the manifold valued-martingales. Here specific connections in subsection 1.1 and associated specific martingales in subsection 1.2 are introduced to be adapted to the left-invariant structure of Lie groups. Subsection 1.3 deals with the stochastic exponential and logarithm of Lie groups. First defined by Hakim-Dowek and Lépingle [6], they allow exchanges between Lie group-valued semimartingales and $\mathbb{R}^d$-valued semimartingales.

In the second section, we solve the prescribed terminal value martingale problem: in section 2.1 for the 3-dimensional Heisenberg group; in section 2.2 for (Γ)-groups, and in section 2.3 for nilpotent Lie groups.

The backward stochastic differential equations in Lie groups are studied in the last section: subsection 3.1 is devoted to backward equations where the drift depends only on time, and subsection 3.2 to general drifts.

Notations

When not specified a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq 1}, \mathbb{P})$ is given. All processes are continuous and, if M is a semimartingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq 1}, \mathbb{P})$ taking its values in $\mathbb{R}^d$ and H is an adapted process taking its values in $\mathbb{R}^{k\times d}$, the usual Itô stochastic integral of H along M—when defined—is denoted by $\int H\, dM$, whereas the Stratonovich stochastic integral is denoted by $\int H \circ dM$. The space of all local martingales taking values in $\mathbb{R}^d$ is denoted by $\mathcal{M}(\mathbb{R}^d)$ ($\mathcal{M}$ when $d = 1$).

The Einstein summation convention is used throughout.

1 Geometry of G and G-martingales

1.1 Choice of a connection

Let G be a d-dimensional Lie group with identity e. For all g in G, we denote by L_g the left translation:

$$L_g \ : \ x \in G \ \mapsto \ L_g(x) = gx.$$

The tangent vector space T_eG with the bracket rule $[A, B] = AB - BA$ is an algebra, **the Lie algebra of** G, denoted by $\mathcal{G}$. We identify $\mathcal{G}$ with the algebra of the left invariant vector fields on G through: $A \in \mathcal{G} \ \mapsto \ \tilde{A}$ where $\tilde{A}$ is the vector field on G defined by

$$\forall g \in G \ ; \ \tilde{A}(g) = (dL_g)_e A$$

and denoted below as $\tilde{A}(g) = g.A$.

We first introduce a connection on G such that

$$\forall (A, B) \in \mathcal{G} \times \mathcal{G} \ , \ \nabla_{\tilde{A}} \tilde{B} = \tilde{\alpha}(A, B) \tag{1}$$

where α is a bilinear alternate mapping on $\mathcal{G}\times\mathcal{G}$ with values in $\mathcal{G}$. By [7] (prop.II.1.4), equation (1) is equivalent to the determination of an affine left invariant connection on G (i.e. for all g in G and for all vector fields X and Y on G, $\nabla_{dL_g(Y)} dL_g(Y) = dL_g(\nabla_Y X)$) such that the geodesics starting from e are exactly the maps $t \ \mapsto \ \exp tA$ where A ranges over $\mathcal{G}$.

The choice of a left invariant connection whose geodesics are the exponential curves seems to be natural. In the following we will deal with two "natural" connections corresponding to two different mappings α. Arguments in favour of one or the other will be developped when necessary.

Definition 1.1 *We call* **(–)-connection** *the connection on G defined by (1) with*

$$\alpha \equiv 0$$

and **(0)-connection** *the connection on G defined by (1) with*

$$\forall (A,B) \in \mathcal{G} \times \mathcal{G}\ ,\ \alpha(A,B) = \frac{1}{2}[A,B].$$

Remark: The name (–) and (0)-connection can be found in [7] p.104, taken from an earlier work of E. Cartan. It is related with the torsion: the torsion of the (–)-connection is equal to $-\widetilde{[A,B]}$ when computed at $(\tilde{A},\tilde{B})$ whereas the (0)-connection is torsion free. Actually both connections have the same symmetric part.

Recall that if $(x^i)_{1\le i\le d}$ is a local coordinate system on a domain U of the manifold G and if ∇ is an affine connection on G, then **the Christoffel symbols** $(\Gamma_{ij}^k)_{1\le i,j,k\le d}$ of the connection are functions on U defined by

$$\nabla_{D_i}(D_j) = \Gamma_{ij}^k\ D_k$$

where D_i denotes the derivation along the i-th coordinate: $D_i = \frac{\partial}{\partial x^i}$. The Christoffel symbols of the (–)-connection are described in the next lemma.

Lemma 1.2 *If $(x^i)_{1\le i\le d}$ is a local coordinate system on G, then the Christoffel symbols of the (–)-connection are given by the following: for all $x \in G$ and for $i = 1,\ldots,d$*

$$\Gamma_{i.}^{.}(x) = -\ \mathcal{H}(x)^{-1} \times D_i(\mathcal{H})(x)$$

where $\Gamma_{i.}^{.}(x)$ and $\mathcal{H}(x)$ are the $d\times d$-matrices

$$\Gamma_{i.}^{.}(x) = (\Gamma_{ij}^k(x))_{1\le j,k\le d} \text{ and } \mathcal{H}(x) = (\tilde{H}_\alpha x^i)_{1\le\alpha,i\le d}$$

for any basis $(H_\alpha)_{1\le\alpha\le d}$ of $\mathcal{G}$.

Proof: Let us split $\tilde{H}_\alpha(x) = (\tilde{H}_\alpha x^i)\ D_i(x)$ and use the calculus axioms of a connection to write that the right-hand-side of relation (1) is zero with $A = H_\alpha$ and $B = H_\beta$ for any $\alpha,\beta \in \{1,\ldots,d\}$ and $x \in G$,

$$\begin{aligned} 0 &= \nabla_{(\tilde{H}_\alpha x^i)\ D_i}\left((\tilde{H}_\beta x^j)\ D_j\right)(x) \\ &= (\tilde{H}_\alpha x^i)(\tilde{H}_\beta x^j)\nabla_{D_i}(D_j)(x) + (\tilde{H}_\alpha x^i)D_i(\tilde{H}_\beta x^j)\ D_j(x) \\ &= \left((\tilde{H}_\alpha x^i)(\tilde{H}_\beta x^j)\Gamma_{ij}^k(x) + (\tilde{H}_\alpha x^i)D_i(\tilde{H}_\beta x^k)\right)\ D_k(x) \end{aligned}$$

and then, for all $k \in \{1, \ldots, d\}$

$$(\tilde{H}_\alpha x^i)\left((\tilde{H}_\beta x^j)\Gamma^k_{ij}(x) + D_i(\tilde{H}_\beta x^k)\right) = 0.$$

Hence for all $\beta, i, k \in \{1, \ldots, d\}$

$$(\tilde{H}_\beta x^j)\Gamma^k_{ij}(x) = -D_i(\tilde{H}_\beta x^k)$$

and introducing the matrices $\Gamma^{\cdot}_{i\cdot}(x)$ and $\mathcal{H}(x)$ gives the result.

□

1.2 G-valued martingales

Any affine connection ∇ on the manifold G being fixed, a notion of G-valued martingales is available through the definition of a Hessian (see for example [5] Chapter IV).

Definition 1.3 *A G-valued semimartingale is a G-***martingale** *if for all f in $C^\infty(G)$,*

$$f(X) - f(X_0) - \frac{1}{2}\int_0^{\cdot} Hess\ f(dX, dX) \tag{2}$$

is a local martingale, where $Hess\ f$ is the bilinear form given by

$$Hess\ f(A, B) = AB\ f\ -\ \nabla_A B\ f$$

for all vector fields A and B on G.

Note that the process (2) only depends of the symmetric part of the Hessian, this leads to the usefull next lemma. More general considerations about the links between connection and martingales on Lie groups are developped in [1].

Lemma 1.4 *The (-)-connection and the (0)-connection both induce the same G-martingales.*

In the sequel "G-martingale" will undifferently refer to the (0)- or (−)-connection on G. The notion of a G-martingale is clearly a local notion and can be written in local coordinates $(x^i)_{1\leq i\leq d}$. With the above notations, computing for $i, j, k = 1, \ldots, d$,

$$Hess\ x^k(D_i, D_j) = D_i D_j(x^k) - \nabla_{D_i} D_j(x^k) = -\Gamma^k_{ij} \tag{3}$$

gives the following lemma.

Lemma 1.5 *Let $(x^i)_{1\leq i\leq d}$ be a local coordinate system on G and $(\Gamma^k_{ij})_{1\leq i,j,k\leq d}$ be the associated Christoffel symbols of the connection. A semimartingale X in G with coordinates $(X^1,\dots,X^d)$ is a G-martingale if and only if, for all $k=1,\dots,d$,*

$$X^k - X_0^k + \frac{1}{2}\int_0^{\cdot} \Gamma^k_{ij}(X_s)d\langle X^i, X^j\rangle_s \in \mathcal{M}. \tag{4}$$

We will sometimes assume that the following hypothesis is realized.

Definition 1.6 *Let G be a Lie group equipped with a connection. We say that* **hypothesis** (Γ) *is realized, or that G is a* (Γ)**-group**, *if there exists a system of global coordinates Φ on G such that $Hess\ \Phi$ vanishes.*

Looking at the process (2) it is clear that if hypothesis (Γ) holds then the G-martingales are exactly the G-valued semimartingales X for which the coordinate process $\Phi(X)$ is a local martingale.

Let us make some comments about hypothesis (Γ) :

1. Using (3), hypothesis (Γ) is equivalent to: *the Christoffel symbols of the connection expressed in coordinates Φ are identically zero.* By the way, in a (Γ)-group the connection is always torsion free. Therefore a (Γ)-group will always be a Lie group with (0)-connection.

2. In particular, hypothesis (Γ) implies that the curvature tensor field vanishes (see [7] p.45); G is then a locally flat manifold.

3. The 3-dimensional Heisenberg group with the (0)-connection satisfies hypothesis (Γ); this example will be developped in next section.

1.3 The stochastic exponential and logarithm

We recall here part of the results obtained by M. Hakim and D. Lépingle in [6]. The stochastic exponential of Lie groups and its converse the stochastic logarithm are the main tools of this paper to establish a one-to-one correspondence between G-valued semimartingales and $\mathcal{G}$-valued semimartingales. These tranformations are close to the development and the lift of a manifold semimartingale described in chapter VII of [5], but are specific to Lie groups.

Let $(H_i)_{1\leq i\leq d}$ be a basis of $\mathcal{G}$.

Proposition 1.7 *([6] th.1) Given a $\mathcal{G}$-valued semimartingale $M = M^iH_i$ and an $\mathcal{F}_0$-mesurable random variable X_0 in G, the Stratonovich differential equation:*

$$\forall f \in \mathcal{C}^\infty(G)\ ,\ f(X) = f(X_0) + \int_0^{\cdot} \tilde{H}_i f(X_s) \circ dM_s^i \tag{5}$$

has a unique solution $X=(X_t)_{t\geq 0}$, G-valued semimartingale.

Equation (5) is clearly independent of the basis $(H_i)_{1\leq i\leq d}$ of $\mathcal{G}$ and will sometimes be written

$$dX = X \circ dM$$

The solution starting from $X_0 = e$ is denoted $\mathcal{E}(M)$ and named the **stochastic exponential** of M.

Lemma 1.8 *When G satisfies hypothesis (Γ) with the (0)-connection and coordinate system Φ, then (5) is equivalent to*

$$\Phi(X) = \Phi(X_0) + \int_0^{\cdot} \tilde{H}_i\Phi(X_s)\, dM_s^i$$

where it is important to note that the Stratonovich integral has been replaced by an Itô integral.

Proof: Firstly, suppose G satisfies hypothesis (Γ). Then for all semimartingale M taking its values in $\mathcal{G}$, the process $X = X_0\mathcal{E}(M)$ is solution of the Itô stochastic differential equation

$$\Phi(X_t) = \Phi(X_0) + \int_0^t \tilde{H}_i\Phi(X_s)dM_s^i$$

since $Hess\ \Phi = 0$ cancels the Stratonovich second order term in (5) with $f = \Phi$.

Conversely, for any regular function f, let $\hat{f} = f \circ \Phi^{-1}$ and Itô formula get:

$$f(X_t) = \hat{f}(\Phi(X_t)) = f(X_0) + \int_0^t \partial\hat{f}(\Phi(X_s)) \circ d\Phi(X_s).$$

But, $\tilde{H}f(x)$ is nothing but $\partial\hat{f}(\Phi(x))(\tilde{H}\Phi)(x)$ and $d\Phi(X_s) = \tilde{H}_i\Phi(X_s)dM_s^i$, yielding (5). □

Conversely to proposition 1.7, a unique $\mathcal{G}$-semimartingale $M = M^iH_i$ starting from 0 is associated with each G-semimartingale X such that $X = X_0\mathcal{E}(M)$ ([6] th.4). It is denoted by $M = \mathcal{L}(X)$ and named **the stochastic logarithm** of X.

When the connection on G is chosen to be the (−)- or the (0)-connection (definition 1.1), the correspondence between G-semimartingales and $\mathcal{G}$-semimartingales is respectfull towards our notion of martingale as claimed in the following proposition (see [6] section 4 and also [1] prop.3).

Proposition 1.9 *With the (−)- or the (0)-connection on G, the G-martingales are exactly the processes $X_0\ \mathcal{E}(M)$, where M is a local martingale in $\mathcal{G}$ and X_0 a G-valued $\mathcal{F}_0$-mesurable random variable.*

One more result concerning the stochastic exponential and logarithm of Lie groups will be usefull in the following. It gives a computation rule and can be interpreted as a stochastic version of the Campbell-Hausdorff formula.

Let us first recall the adjoint representation of G. For all g in G, the automorphism on $\mathcal{G}$ $Ad(g)$ is defined by $Ad(g) = (dI_g)_e$ where I_g denotes the inner automorphism of G : $x \mapsto I_g(x) = gxg^{-1}$.

Proposition 1.10 *([6] prop.5) If M and N are $\mathcal{G}$-valued semimartingales then*

$$\mathcal{E}(M+N) = \mathcal{E}\left(\int_0^\cdot Ad(\mathcal{E}(N)_s) \circ dM_s\right)\mathcal{E}(N).$$

If X and Y are G-valued semimartingales then

$$\mathcal{L}(XY) = \int_0^\cdot Ad(Y_s^{-1}) \circ d\mathcal{L}(X)_s + \mathcal{L}(Y).$$

To conclude this section let us introduce a notion of integrability for group-valued random variables.

Definition 1.11 *Let $p \in]0,+\infty[$. A global coordinate system Φ on G being fixed, a G-valued random variable L is said to be p-integrable with respect to Φ if the coordinate vector $\Phi(L)$ is made of p-integrable random variables.*

We denote by $\mathbb{L}^p_\Phi(\Omega, \mathcal{F}; G)$ the set of all $\mathcal{F}$-mesurable, G-valued random variables which are p-integrable with respect to Φ. In the same vein, a G-valued process $X = (X_t; 0 \leq t \leq 1)$ will be said *p-integrable with respect to Φ* if for all $t \in [0,1]$, $\Phi(X_t)$ is p-integrable.

2 G-martingale with prescribed terminal value

2.1 Example: the Heisenberg group

Denote by H the 3-dimensional Heisenberg group, that is to say the group of matrices

$$g = \begin{pmatrix} 1 & x^1 & x^3 \\ 0 & 1 & x^2 \\ 0 & 0 & 1 \end{pmatrix}$$

with $(x^1, x^2, x^3) \in \mathbb{R}^3$. We call $(x^k)_{k=1,2,3}$ the "natural" coordinate system and will refer to the (0)- or (−)-connection when speaking about H-martingales.

The Lie algebra T_eH associated to H is the algebra of upper-trigonal 3×3-matrices with zero on the diagonal. A basis for it is (H_1, H_2, H_3) with

$$H_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; H_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The normal coordinate system Φ with respect to the basis (H_1, H_2, H_3) of T_eH is given by

$$\Phi \begin{pmatrix} 1 & x^1 & x^3 \\ 0 & 1 & x^2 \\ 0 & 0 & 1 \end{pmatrix} = (x^1, x^2, x^3 - \frac{1}{2}x^1x^2).$$

The group H is not a compact manifold and the problem of finding a H-martingale with prescribed terminal value cannot be treated with the method of [11] or [3]. The result proposed in [4] th.5.2 for noncompact manifolds should be used but quite heavy conditions are to be verified. At the opposite, the problem is solved here with very little material. We give hereafter two different proofs of this result in order to exhibit two different ways of solving the general problem.

Proposition 2.1 *Let L belongs to $\mathbb{L}^2_\Phi(\Omega, \mathcal{F}; H)$. There exists a unique square integrable H-martingale $X = (X_t; 0 \leq t \leq 1)$ such that $X_1 = L$. It is given at time t ($t \in [0,1]$) by its coordinates:*

$$\begin{aligned} X_t^i &= E(L^i/\mathcal{F}_t) \text{ for } i = 1,2 \\ X_t^3 &= E(L^3/\mathcal{F}_t) - \frac{1}{2}(E(L^1L^2/\mathcal{F}_t) - E(L^1/\mathcal{F}_t)E(L^2/\mathcal{F}_t)). \end{aligned}$$

First proof: Use the (0)-connection on H and normal coordinates Φ. The computation of $(D_iD_j - D_jD_i)\Phi^k$ proves that $Hess^{(0)}\Phi = 0$. Hence H is a (Γ)-group and the solution follows immediately by looking for a H-valued square integrable semimartingale X such that $\Phi(X) = (X^1, X^2, X^3 - \frac{1}{2}X^1X^2)$ is a martingale with terminal value $(L^1, L^2, L^3 - \frac{1}{2}L^1L^2)$.

Second proof: Use the (–)-connection on H and the natural coordinate system. With lemma 1.2, you get the Christoffel symbols: $\forall g \in G$,

$$\Gamma^k_{ij}(g) = 0 \text{ for } (i,j,k) \neq (1,2,3) \; ; \; \Gamma^3_{12}(g) = -1$$

and the solution is given by looking for a H-valued square integrable semimartingale X such that $(X^1, X^2, X^3 - \frac{1}{2}\langle X^1, X^2\rangle)$ is a martingale with terminal value $(L^1, L^2, L^3 - \frac{1}{2}\langle X^1, X^2\rangle_1)$. Note that this solution coincides with the first one since $X^1X^2 - \langle X^1, X^2\rangle$ is a martingale. □

A natural question arises: why does this work? The existence of a global coordinate system clearly makes things more simple, but this is not crucial. Actually two arguments are to be considered.

1. In the first proof the main argument relies on hypothesis (Γ); H-martingales are then well known vector martingales.
2. In the second proof, some Christoffel symbols are vanishing and (4) becomes a triangular system; it is then easy to obtain an explicit solution for it.

The next two subsections are devoted to groups where one of these two situations is realized: Lie groups with hypothesis (Γ) for situation 1, nilpotent Lie groups for situation 2.

2.2 Existence and uniqueness; case of a (Γ)-group

Let G be a Lie group with (0)-connection where hypothesis (Γ) is satisfied with respect to a coordinate system Φ.

The following proposition is a straightforward consequence of the nature of the G-martingales; uniqueness of X in the set of G-valued integrable with respect to Φ martingales comes from the uniqueness of $\Phi(X)$ in the set of d-dimensional integrable martingales.

Proposition 2.2 *If G is a Lie group satisfying hypothesis (Γ) then for all integrable $\mathcal{F}_1$-measurable G-valued random variable L there exists a unique integrable G-martingale X with terminal value L; for all $t \in [0,1]$, X_t is given by its coordinates:*

$$\Phi(X_t) = E(\Phi(L)/\mathcal{F}_t).$$

Moreover note that if L is square integrable then X will be a square integrable martingale.

2.3 Existence and uniqueness; case of a nilpotent Lie group

Let G a simply connected finite-dimensional nilpotent Lie group, then G can be considered as a subspace of $GL(\mathbb{R}^n)$, and $\mathcal{G}$ denotes its Lie algebra. Using [7] page 269 the exponential map *exp* is a regular application from $\mathcal{G}$ to G. Moreover, Engel's theorem ([7] page 169) gives the existence of a basis of $\mathbb{R}^n$ such that any $X \in \mathcal{G}$ is expressed by a matrix with zeros on and below the diagonal. So, the case to be studied is this one of G the set of matrices expressed by a matrix with zeros below the diagonal and 1 on the diagonal. The dimension of G and $\mathcal{G}$ in the set of $(n \times n)$ matrices is $\frac{n(n-1)}{2}$, that is to say the cardinal of $\Lambda = \{(i,j), 1 \leq i < j \leq n\}$.

A basis of $\mathcal{G}$ is done with $(H_\alpha, \alpha \in \Lambda)$ where

$$(H_\alpha)^\beta = \delta_\alpha(\beta), \forall \beta \in \Lambda, \forall \alpha \in \Lambda,$$

and a natural global coordinate system is associated:

$$\Phi^\alpha(x) = x^\alpha, \forall x \in G, \forall \alpha \in \Lambda.$$

The set Λ is ordered by

$$\alpha = (i,j) < \beta = (k,l) \text{ if and only if } j < l, \text{ or, if } j = l, i > k.$$

With such an order, the system (4) is a triangular system which coefficients are described in the next lemma.

Lemma 2.3 *For all $x \in G, \lambda \in \Lambda$, the matrix $\Gamma_{\lambda .}^{\cdot}(x) = \{(\Gamma_{\lambda\alpha}^{\beta}(x)); \alpha, \beta \in \Lambda\}$ of the (-)-connection Christoffel symbols can be expressed as a block-diagonal matrix, with $n-1$ blocks $(\Gamma_{\lambda .}^{\cdot}(x))^j, j = 1, \cdots, n-1$. The $(\Gamma_{\lambda .}^{\cdot}(x))^j$ is a j-upper trigonal matrix with coefficients being $j-2$ degree polynoms, depending only on the x-elements $(x^{k,l}, 1 \leq k < l \leq j)$.*

Proof: We define $n-1$ "blocks" in the indices set Λ as following:

$$\Lambda_j = \{(j, j+1), \cdots, (1, j+1)\}, j = 1, \cdots, n-1.$$

In the set of $n(n-1)/2 \times n(n-1)/2$-matrices, we define a set of matrices $\mathcal{D}$ with zeros except in the diagonal blocks $M^1, \cdots, M^{n-1}$, the size of M^j being j and each block matrix is with zeros below the diagonal and 1 on the diagonal. Such a set is clearly a subgroup.

For all $x \in G$, we denote as $\mathcal{H}(x)$ the matrix defined by $\{(\tilde{H}_\alpha \Phi^\beta(x)), \alpha \in \Lambda, \beta \in \Lambda\}$. This matrix satisfies $\tilde{H}_\alpha \Phi^\beta(x) = 0$ if $\alpha > \beta$ and $\tilde{H}_\alpha \Phi^\beta(x) = 1$ if $\alpha = \beta$. Such a matrix $\mathcal{H}(x)$ belongs to the subgroup $\mathcal{D}$ defined above. We remark that the j-th block is the restriction of transposed matrix x to $\mathbb{R}^j$ denoted as x^j, so that

$$\mathcal{H}(x)^j = x^j.$$

Obviously, $(\mathcal{H}(x)^{-1})^j = (\mathcal{H}(x)^j)^{-1}$ and its elements are $(j-1)$-degree polynoms with respect to $(x^{ik}, 1 \leq i < k \leq j)$. Actually the degree of those polynoms is less or equal to $j-2$, except for the coefficient in the upper-right corner which is $j-1$.

Now, let $D_\lambda(\mathcal{H}(x)) \in \mathcal{D}$ for $\lambda \in \Lambda$, the derivative of $x^\lambda \mapsto \mathcal{H}(x)$. For $\lambda = (a,b)$ with $1 \leq a < b \leq j$, the j-th block of this matrix is given by:

$$(D_{ab}(\mathcal{H}(x))^j)_{\alpha,\beta} = \delta_{(b,j+1)(a,j+1)}(\alpha, \beta). \tag{6}$$

Then we apply lemma 1.2

$$\Gamma_{\lambda .}^{\cdot}(x) = -(\mathcal{H}(x))^{-1} \times D_\lambda \mathcal{H}(x)$$

This equation lives in the set $\mathcal{D}$, so we can solve it for each block, $\forall j = 1, \cdots, n-1$:

$$(\Gamma^{\cdot}_{\lambda \cdot}(x))^j = -(\mathcal{H}(x)^j)^{-1} \times D_\lambda \mathcal{H}(x)^j$$

Note that the coefficients of the j-th block $(\Gamma^{\cdot}_{\lambda \cdot}(x))^j$ are polynoms, with respect to $(x^{ik}, 1 \leq i < k \leq j)$. The degree of all of them is less or equal to $j-2$, even for the coefficients in the first line (computing them involves the $(j-1)$-degree polynom situated in the upper-right corner of $(\mathcal{H}(x)^j)^{-1}$, but the last line of $D_\lambda \mathcal{H}(x)^j$ vanishes). This proves the lemma. □

Remark that we can express $(\Gamma^{\cdot}_{\lambda \cdot}(x))^j$ more precisely. Using multiplication matrix rules, yields:

$$\Gamma^{(i,j+1)}_{(a,b)(l,j+1)}(x) = -(\mathcal{H}(x)^j)^{-1}_{l,k} \delta_{k,i}(b,a) = -(x^j)^{-1}_{l,b} \delta_i(a),$$

so these terms are usually 0 except

$$\Gamma^{(i,j+1)}_{(i,b)(l,j+1)}(x) = -(x^j)^{-1}_{b,l},$$

for all $1 \leq i < b \leq l \leq j$.

Once the Christoffel symbols of the (–)-connection has been computed, the structure of the G-martingales is known and the prescribed terminal value problem can be solved.

Note that in our next theorem, neither is the group G a compact manifold (as in [3] or [11]), nor are the processes required to stay in a compact convex subset of G (as in [2]). Moreover an effective construction of the solution is given.

Theorem 2.4 *Let G be a simply connected nilpotent Lie group equipped with natural coordinate system Φ and suppose the dimension of G is $\frac{n(n-1)}{2}$. For any $L \in \mathbb{L}^{n-1}_{\Phi}(\Omega, \mathcal{F}_1; G)$, there exists a unique integrable martingale X taking its values in G such that $X_1 = L$.*

More precisely, the coordinates of X can be computed recursively by

$$X^{i,j+1}_t = E[L^{i,j+1} + \frac{1}{2} \sum_{i+1 \leq b \leq l \leq j} \int_t^1 \Gamma^{(i,j+1)}_{(i,b)(l,j+1)}(X_s) d\langle X^{i,b}, X^{l,j+1}\rangle_s / \mathcal{F}_t].$$

Proof: We work with the (-)-connection and, using lemma 1.5, we have to solve the BSDE:

$$\forall \gamma \in \Lambda, X^\gamma + \frac{1}{2} \Gamma^\gamma_{\alpha\beta}(X).\langle X^\alpha, X^\beta \rangle \in \mathcal{M}, X_1 = L.$$

For the first block, the one-dimensional equation is:

$$X^{1,2} + \frac{1}{2}\int_0^{\cdot} \Gamma_{\alpha\beta}^{(1,2)}(X_s)d\langle X^\alpha, X^\beta\rangle_s \in \mathcal{M} \; ; \; X_1^{1,2} = L^{1,2}.$$

But $\Gamma_{\alpha\beta}^{(1,2)}$ is identically 0, and so $X^{1,2}$ is uniquely defined by:

$$X_t^{1,2} = E[L^{1,2}/\mathcal{F}_t].$$

We suppose all elements of vector X are known up to $(j-1)$-th block and solve the j-th block: for $i = j, j-1, \ldots, 2, 1$,

$$X^{i,j+1} + \frac{1}{2}\int_0^{\cdot} \Gamma_{(i,b)(l,j+1)}^{(i,j+1)}(X_s)d\langle X^{(i,b)}, X^{(l,j+1)}\rangle_s \in \mathcal{M} \; ; \; X_1^{i,j+1} = L^{i,j+1},$$

the sum running over the set of indices $\{(b,l); 1 \le i < b \le l \le j\}$. So lemma 2.3 proves that this system is solvable recursively from $i = j$ to $i = 1$. □

Let us mention that the computation in the previous proof makes the martingale solution X square integrable as soon as the terminal value L is $2(n-1)$-integrable.

3 BSDE

Let G be a finite dimensional Lie group, $\mathcal{G}$ its associated Lie algebra and k an integer. We denote by $\mathcal{L}(\mathbb{R}^k, \mathcal{G})$ the space of all linear maps from $\mathbb{R}^k$ to $\mathcal{G}$ and by $(e_j)_{1\le j\le k}$ the canonical basis of $\mathbb{R}^k$.

The aim of this section is to solve in the two cases, nilpotent Lie group or hypothesis (Γ), the BSDE:

$$X_1 = \xi \; ; \; dX_t = X_t \circ (F(t, X_t, Z_t)dt + Z_t \, dW_t) \tag{7}$$

where

- $(W_t; 0 \le t \le 1)$ is a k-dimensional Brownian motion,
- $(\mathcal{F}_t; 0 \le t \le 1)$ is its natural filtration,
- ξ is a $\mathcal{F}_1$-adapted G-valued random variable,
- $F : \Omega \times [0,1] \times G \times \mathcal{L}(\mathbb{R}^k, \mathcal{G}) \to \mathcal{G}$ is $(\mathcal{F}_t; 0 \le t \le 1)$-progressively measurable.

By a **solution of (7)** we mean a pair (X, Z) of square integrable (with respect to some global coordinate system Φ) $(\mathcal{F}_t)_t$-adapted processes with values in $G \times \mathcal{L}(\mathbb{R}^k, \mathcal{G})$ such that: $\forall f \in \mathcal{C}^\infty(G)$

$$f(\xi) = f(X_t) + \int_t^1 (\tilde{F}(s, X_s, Z_s).f)(X_s)ds + \int_t^1 (\widetilde{Z_s(e_j)}.f)(X_s) \circ dW_s^j.$$

If Φ is any global coordinate mapping, solving (7) is equivalent to solving in $\mathbb{R}^d$, identifying any $x \in G$ with $y = \Phi(x) \in \mathbb{R}^d$,

$$Y_1 = \Phi(\xi) = \Phi(Y_t) + \int_t^1 (\tilde{F}(s, Y_s, Z_s).\Phi)(Y_s)ds + \int_t^1 (\widetilde{Z_s(e_j)}.\Phi)(Y_s) \circ dW_s^j. \quad (8)$$

This equation is not a standard one; the standard assumptions are not satisfied (see for instance Pardoux and Peng [9]) as uniform Lipschitz properties, because of the multiplicative form of the coefficients. Nevertheless, we follow the proof scheme of [9].

The first step consists in taking the drift term F identically zero. Via the classical martingale representation property and proposition 1.9, equation (7) is solved by finding a G-martingale X with terminal value ξ; this has been treated in previous section, proposition 2.2 and theorem 2.4. The following subsections are devoted to BSDE with non-vanishing drift term.

3.1 BSDE with drift depending only on time: existence and uniqueness.

Proposition 3.1 *Let ξ a $\mathcal{F}_1$-adapted random variable taking its values in a Lie group G and f an $\mathcal{F}$-adapted bounded process taking its values in the Lie algebra $\mathcal{G}$. Assume that ξ satisfies the following property :*

the G-valued random variable L defined by $L = \xi \ (\mathcal{E}_1(\int_0^{\cdot} f_s ds))^{-1}$ is such that there exists a unique square integrable G-martingale X with terminal value L.

Then the BSDE

$$dY_t = Y_t \circ (f_t dt + Z_t \, dW_t) \ ; \ Y_1 = \xi \quad (9)$$

admits a unique solution (Y, Z).

Remark: The assumption is satisfied, for example, if G is a $\frac{n(n-1)}{2}$-dimensional nilpotent simply connected Lie group with natural coordinates Φ and L belongs to $\mathbb{L}_\Phi^{2(n-1)}(\Omega, \mathcal{F}_1; G)$; or if G satisfies (Γ) hypothesis relatively to some coordinate system Φ and L belongs to $\mathbb{L}_\Phi^2(\Omega, \mathcal{F}_1; G)$ (see section 2).

Proof: The hypothesis implies the existence of a pair (Y', Z') of square integrable processes such that

$$dY'_t = Y'_t \circ (Z'_t\, dW_t); Y'_1 = L$$

Let now $Z_t = Ad^{-1}(\mathcal{E}_t(\int_0^\cdot f_s ds)).Z'_t$ and $\overline{Y_t} = Y'_0\mathcal{E}_t(\int_0^\cdot f_s ds + \int_0^\cdot Z_u dW_u)$.
The proposition 1.10 yields $\overline{Y_t} = Y'_0\mathcal{E}_t\left(\int_0^\cdot Ad(\mathcal{E}_s(N)) \circ Z_s dW_s\right)\mathcal{E}_t(N)$ where $N = \int_0^\cdot f_s ds$. As N has finite variation, the Stratonovich integral in the stochastic exponential of $\overline{Y_t}$ is actually an Itô integral and $\overline{Y_t} = \mathcal{E}_t\left(\int_0^\cdot Z'_s dW_s\right)\mathcal{E}_t(N)$. Then

$$\begin{aligned}\overline{Y_1} &= \mathcal{E}_1\left(\int_0^\cdot Z'_s dW_s\right)\mathcal{E}_1(N) \\ &= Y'_1\, \mathcal{E}_1(N) = L\, \mathcal{E}_1(\int_0^\cdot f_s ds) = \xi\end{aligned}$$

and $(\bar{Y}, Z)$ is a solution to (9).

To prove uniqueness, let (Y^1, Z^1) and (Y^2, Z^2) be two solutions of (9). Then:

$$\xi = Y_1^i = Y_0^i\mathcal{E}_1(\int_0^\cdot f_s ds + \int_0^\cdot Z_u^i dW_u), i = 1, 2.$$

Using once more proposition 1.10 and the finite variation property of N,

$$\begin{aligned}Y_0^1\mathcal{E}_1\left(\int_0^\cdot Ad(\mathcal{E}_s(N)).Z_s^1 dW_s\right) &= Y_0^2\mathcal{E}_1\left(\int_0^\cdot Ad(\mathcal{E}_s(N)).Z_s^2 dW_s\right) \\ &= \xi(\mathcal{E}_1(\int_0^\cdot f_s ds))^{-1} = L.\end{aligned}$$

But, according to the hypothesis, there exists a unique G-martingale X such that $X_1 = L$, that is to say their stochastic logarithms are equal:

$$\int_0^t Ad(\mathcal{E}_s(N)).Z_s^1 dW_s = \int_0^t Ad(\mathcal{E}_s(N)).Z_s^2 dW_s, dt \otimes d\mathbb{P}\ a.s.$$

The operator $Ad(\mathcal{E}_s(N))$ is invertible, so $Z^1 = Z^2$ and by the way, $Y^1 = Y^2$. □

3.2 BSDE with bounded drift F: case of a (Γ)-group

We suppose now that G satisfies hypothesis (Γ) and that the drift F satisfies, relatively to the map Φ of hypothesis (Γ), the following uniform Lipschitz property:

$$\exists C, \forall g, g' \in G, \forall X, X' \in \mathcal{L}(\mathbb{R}^k, \mathcal{G}), \forall s \in [0,1]:$$

$$|(\tilde{F}(s,g,X).\Phi)(g) - (\tilde{F}(s,g',X').\Phi)(g')| \le C\|(\tilde{X}\Phi)(g) - (\tilde{X}'\Phi)(g')\| \quad (10)$$

where

$$\|(\tilde{X}\Phi)(g) - (\tilde{X}'\Phi)(g')\|^2 = \sum_{1\leq j\leq k} |(\widetilde{X(e_j)}.\Phi)(g) - (\widetilde{X'(e_j)}.\Phi)(g')|^2.$$

Notice that this hypothesis replaces [9] (3.2.ii) when solving equation (1.1) from [9]. In the following, the mention *integrable* for a G-valued random variable or semimartingale X will refer to $\Phi(X)$.

Let $\xi \in \mathbb{L}^2_\Phi(\Omega, \mathcal{F}_1; G)$. Define recursively the sequence of processes (X^n, Z^n) taking their values in $G \times \mathcal{L}(\mathbb{R}^k, \mathcal{G})$:

$$X_t^0 = e \in G \; ; \; Z_t^0 = 0.$$
$$dX_t^{n+1} = X_t^{n+1} \circ (F(t, X_t^n, Z_t^n)dt + Z_t^{n+1}dW_t) \; ; \; X_1^{n+1} = \xi. \tag{11}$$

With the hypothesis that F is uniformly bounded, we put for all $n \geq 0$:

$$f_t^n = F(t, X_t^n, Z_t^n)$$

and following the previous section 3.1, for all $n \geq 1, (X_t^n, Z_t^n)$ exists since $L^{n-1} = \xi\; (\mathcal{E}_1(\int_0^\cdot f_s^{n-1}ds))^{-1} \in \mathbb{L}^2_\Phi(\Omega, \mathcal{F}_1; G)$ and proposition 3.1 works for a $(\Gamma)-$group.

Proposition 3.2 *Assume the Lie group G satisfies the hypothesis (Γ) with coordinates Φ and that the drift F is uniformly bounded and satisfies (10). Then, for every $\xi \in \mathbb{L}^2_\Phi(\Omega, \mathcal{F}_1; G)$, equation (7) admits one and only one solution. It is obtained as the limit in $\mathbb{L}^2_\Phi(\Omega \times [0,1])$ of the sequence defined by (11), and also as the almost sure limit of a subsequence of (11).*

Remark: Actually the assumption on the uniform boundness of F can be omitted if one only wants a uniqueness result for equation (7). This will appear in the following proof.

Proof

Uniqueness: Assume there exist two solutions $(X^i, Z^i), i = 1, 2$, of equation (7). Itô formula yields, for $i = 1, 2$:

$$\Phi(X_t^i) + \int_t^1 (\widetilde{Z_s(e_j)}.\Phi)(X_s^i) \circ dW_s^j = \Phi(\xi) - \int_t^1 (\tilde{F}(s, X_s^i, Z_s^i).\Phi)(X_s^i)ds.$$

Remember that hypothesis (Γ) implies that $Hess(\Phi) = 0$, so (lemma 1.8) the Stratonovich stochastic integral above is only a Itô stochastic integral. Using Pardoux-Peng's proof principle [9], we compute the expectation of the square difference between both solutions:

$$E[|\Phi(X_t^1) - \Phi(X_t^2)|^2 + \textstyle\int_t^1 (\|\widetilde{Z}_s^1.\Phi)(X_s^1) - \widetilde{Z}_s^2.\Phi)(X_s^2)\|^2 ds] =$$
$$-2E[\textstyle\int_t^1 (F(s, X_s^1, Z_s^1).\Phi(X_s^1) - F(s, X_s^2, Z_s^2).\Phi(X_s^2)).(\Phi(X_s^1) - \Phi(X_s^2))ds].$$

A majorant of this last term (as a double product) is, using hypothesis (10):

$$\begin{aligned} 2C\|\tilde{Z}_s^1\Phi(X_s^1) - \tilde{Z}_s^2\Phi(X_s^2)\| &\times \|\Phi(X_s^1) - \Phi(X_s^2)\| \\ &\leq \frac{1}{2}\|\tilde{Z}_s^1\Phi(X_s^1) - \tilde{Z}_s^2\Phi(X_s^2)\|^2 + 2C^2\|\Phi(X_s^1) - \Phi(X_s^2 s)\|^2. \end{aligned}$$

So, one has:

$$\begin{aligned} E[|\Phi(X_t^1) - \Phi(X_t^2)|^2 &+ \frac{1}{2}\int_t^1 |\tilde{Z}_s^1\Phi(X_s^1) - \tilde{Z}_s^2\Phi(X_s^2)\|^2 ds] \\ &\leq 2C^2 \int_t^1 E[\|\Phi(X_s^1) - \Phi(X_s^2)\|^2 ds]. \end{aligned}$$

Gronwall lemma concludes that $\Phi(X_t^1) - \Phi(X_t^2) = 0$, $dt \otimes d\mathbb{P}$ almost surely, that is to say $X^1 = X^2$. Moreover, equaling the martingale part of both stochastic logarithms $\mathcal{L}(X^1)$ and $\mathcal{L}(X^2)$ gives $\int_0^t Z_s^1\, dW_s = \int_0^t Z_s^2\, dW_s$, $dt \otimes d\mathbb{P}$ almost surely. Hence $Z^1 = Z^2$ and thus the uniqueness is proved.

Existence: Itô formula yields, for any element of the sequence:

$$\Phi(X_t^n) + \int_t^1 (\widetilde{Z_s^n(e_j)}.\Phi)(X_s^n) \circ dW_s^j = \Phi(\xi) - \int_t^1 (\tilde{F}(s, X_s^{n-1}, Z_s^{n-1}).\Phi)(X_s^{n-1}) ds.$$

Following the same arguments as for uniqueness to compute the expectation of the square difference between two elements of the sequence, we get:

$$\begin{aligned} E[|\Phi(X_t^n) - \Phi(X_t^{n-1})|^2 &+ \int_t^1 \|(\widetilde{Z_s^n}.\Phi)(X_s^n) - (\widetilde{Z_s^{n-1}}.\Phi)(X_s^{n-1})\|^2 ds] \\ &\leq 2C^2 \int_t^1 E[\|\Phi(X_s^n) - \Phi(X_s^{n-1})\|^2 ds] \\ &+ \frac{1}{2}\int_t^1 E[\|(\widetilde{Z_s^{n-1}}.\Phi)(X_s^{n-1}) - (\widetilde{Z_s^{n-2}}.\Phi)(X_s^{n-2})\|^2 ds]. \end{aligned}$$

Let $u_n(t) = \int_t^1 E[\|\Phi(X_s^n) - \Phi(X_s^{n-1})\|^2 ds]$ and
$v_n(t) = E[\int_t^1 \|(\widetilde{Z_s^n}.\Phi)(X_s^n) - (\widetilde{Z_s^{n-1}}.\Phi)(X_s^{n-1})\|^2 ds]$. We summarize the inequality above with:

$$-u_n'(t) + v_n(t) \leq 2C^2 u_n(t) + \frac{1}{2}v_{n-1}(t), \forall t \in [0,1],$$

which is equivalent to

$$-(e^{2C^2 t}u_n(t))' + e^{2C^2 t}v_n(t) \leq \frac{1}{2}e^{2C^2 t}v_{n-1}(t), \forall t \in [0,1].$$

Integrating this inequality between t and 1, we get:

$$u_n(t) + \int_t^1 e^{2C^2(s-t)}v_n(s)ds \leq \frac{1}{2}\int_t^1 e^{2C^2(s-t)}v_{n-1}(s)ds.$$

Taking $t = 0$, recursively get

$$\int_t^1 e^{2C^2(s-t)} v_n(s)ds \leq K(\frac{1}{2})^n \text{ where } K = 2\int_0^1 e^{2C^2(s)} v_1(s)ds.$$

Moreover, $u_n(0)$ is less than the sum, so $u_n(0) \leq K(\frac{1}{2})^n$. Hence $\sum_n \sqrt{u_n(0)}$ is a convergent series; in particular, the Cauchy difference

$$||\Phi(X^n) - \Phi(X^p)||_{L^2(\Omega\times[0,1]} \leq \sum_{i=p+1}^{n} \sqrt{u_i(0)}$$

goes to 0 when n and p go to infinity: so $\Phi(X^n)$ converges in $\mathbb{L}^2(\Omega \times [0,1]; \mathbb{R}^d)$ to a process Y, and so X^n converges in $\mathbb{L}^2_\Phi(\Omega \times [0,1]; G)$ to $\Phi^{-1}(Y) = X$.

Similarly, the sequence of processes $((\widetilde{Z^n_s}.\Phi)(X^n))_n$ admits a limit in $\mathbb{L}^2(\Omega \times [0,1]; \mathbb{R}^{d\times d})$ denoted as U. So, there exists a subsequence $(X_{n_k}, (\widetilde{Z^{n_k}_s}.\Phi)(X_{n_k}))$ converging almost surely to (X, U). We use the fact that $\forall n, Z_n(e_j)$ is a process taking its values in $\mathcal{G}$: it is a linear combination of H_α, basis of $\mathcal{G}$: $\widetilde{Z_n(e_j)} = a^{j,\alpha}_n \tilde{H}_\alpha$ and for the subsequence (n_k) and almost surely on $\Omega \times [0,1]$, $(\widetilde{Z^{n_k}_s}.\Phi)(X_{n_k}) = a^{j,\alpha}_{n_k}(\tilde{H}_\alpha.\Phi)(X_{n_k})$. This sequence converges almost surely to U; the matrices $(\tilde{H}_\alpha.\Phi)(X_{n_k})$ are invertible and the application $\tilde{H}_\alpha.\Phi$ is continuous; thus,

$$a^{j,\alpha}_{n_k} = (\widetilde{Z^{n_k}_s}.\Phi)(X_{n_k})((\tilde{H}_\alpha.\Phi)(X_{n_k}))^{-1}$$

and converges almost surely to $U.(\tilde{H}_\alpha.\Phi)(X)$ denoted as Z. So, the subsequence (X_{n_k}, Z_{n_k}) converges almost surely in $\Omega \times [0,1]$ to (X, Z). Finally we check that (X, Z) is a solution to (7). □

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