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Canonical Lift and Exit Law of the Fundamental Diffusion Associated with a Kleinian Group

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Abstract

Let Γ be a geometrically finite Kleinian group, relative to the hyperbolic space $\mathbb{H} = \mathbb{H}^{d+1}$, and let δ denote the Hausdorff dimension of its limit set. Denote by Φ the eigenfunction of the hyperbolic Laplacian Δ , associated with its first eigenvalue $2\lambda_0 = \delta(\delta - d)$, and by Z_t^Φ the associated diffusion on \mathbb{H} , whose generator is $\frac{1}{2}\Delta^\Phi := \frac{1}{2}\Phi^{-1}\Delta\Phi - \lambda_0$. We give a simple construction of Z_t^Φ through its canonical lift to the frame bundle $\mathcal{O}\mathbb{H}$, that allows to determine directly its asymptotic behavior.

Keywords : diffusion process, hyperbolic space, Patterson measure.

AMS-classification 2000 : 60 J 60, 37 D 40, 58 J 65.

1 Introduction

Consider the hyperbolic space $\mathbb{H} = \mathbb{H}^{d+1}$, endowed with some geometrically finite Kleinian group Γ . The Hausdorff dimension $\delta \in [0, d]$ of its limit set (see [P], [Su1] or [Su2]) plays a fundamental role. When δ is larger than $d/2$, $\delta(\delta - d)$ is the highest eigenvalue of the Laplacian on a fundamental domain. The associated eigenstate Φ plays an important role in the study of the quotient $\Gamma \backslash \mathbb{H}$ and of its geodesic flow. The corresponding fundamental diffusion Z_t^Φ , which we call " Φ -diffusion", is then also a natural object and tool in this framework : see [Su1], [E-F-LJ-1], [E-F-LJ-2].

Now Φ is classically represented as the mass of the celebrated Patterson measure, and thus makes sense also when it is not square-integrable. So that the Φ -diffusion Z_t^Φ on \mathbb{H} can naturally be considered for all values of δ . It is ergodic on $\Gamma \backslash \mathbb{H}$ if and only if $\delta > d/2$.

The aim of this article is to give a simple construction of Z_t^Φ , from which can be immediately deduced the asymptotic behaviour of Z_t^Φ on \mathbb{H} , that exhibits an interesting dichotomy : whereas the almost sure limit point $Z_\infty^\Phi \in \partial\mathbb{H}$ has a singular law when $\delta \geq d/2$, namely the normalized Patterson measure (which appears thus as an harmonic measure), it happens to have an absolutely continuous (explicit) law when $\delta < d/2$.

We note however that this asymptotic behaviour of Z_t^Φ could also be deduced from the general theory of Martin boundary, see remark 7 below. Our method starts from the Brownian motion on \mathbb{H} modified by a constant drift, then uses the group action to define a diffusion on the stable leaves of the orthonormal frame bundle $\mathcal{O}\mathbb{H}$, whose projection onto \mathbb{H} will be the Φ -diffusion. This method could likely work in the general case of a symmetric space of non compact type and rank one.

2 Notations and basic data

Let \mathbb{H} denote the hyperbolic space \mathbb{H}^{d+1} , with boundary $\partial\mathbb{H}$, unitary tangent bundle $T^1\mathbb{H}$, orthonormal frame bundle $\mathcal{O}\mathbb{H}$, Riemannian area dV , and (hyperbolic) Laplacian Δ .

Given (z, z', u) in $\mathbb{H} \times \mathbb{H} \times \partial\mathbb{H}$, denote by $\log[B_u(z, z')]$ the Busemann function, that is to say the algebraic hyperbolic distance, on any geodesic ending at u , from the stable horocycle $H(z, u)$ determined by z to the stable horocycle $H(z', u)$. In the Poincaré half-space model, we have $B_u(z, z') = p(z', u)/p(z, u)$, $p(z, u)$ denoting the Poisson kernel : $p(z, u) = \text{Im}(z) \times |z - u|^{-2}$ if $u \neq \infty$ and $p(z, \infty) = \text{Im}(z)$. We have the cocycle property : $B_u(z, z'') = B_u(z, z') \times B_u(z', z'')$.

Let Γ be a discrete (non-elementary) group of Möbius isometries of \mathbb{H} , that we suppose geometrically finite. Let $\Lambda = \Lambda(\Gamma)$ denote its limit set, with Hausdorff dimension say δ . Recall that δ is also the critical convergence exponent of the Poincaré series relative to Γ ; (see for example ([Su2], Theorem 1)). Obviously $\delta \leq d$.

Let $\{\mu_z \mid z \in \mathbb{H}\}$ denote the family of Patterson (finite) measures on Λ associated with Γ . It can be defined, up to a multiplicative constant (that we definitively fix), as the only family of measures on Λ satisfying the following geometric "conformal density" property :

$$d\mu_{z'}(u) = B_u^\delta(z, z') d\mu_z(u) \quad \text{for any } z, z' \text{ in } \mathbb{H}$$

together with the invariance property by the group Γ , in the sense that :

$$\gamma^* \mu_z = \mu_{\gamma z} \quad \text{for any } \gamma \text{ in } \Gamma \text{ and } z \text{ in } \mathbb{H},$$

with the convention $\gamma^* \mu := \mu \circ \gamma^{-1}$.

See for example ([P], Lecture 2), [Su2], or ([Ni], Sections 3.4 and 4.7).

Set
$$\Phi(z) := \int d\mu_z = \mu_z(\partial\mathbb{H}) = \mu_z(\Lambda), \quad \text{and} \quad \lambda_0 := \delta(\delta - d)/2.$$

This is a function on \mathbb{H} that verifies $\Delta\Phi = 2\lambda_0\Phi$. See ([P], theorem 1 page 301).

Note that for every γ in Γ we have $\Phi(\gamma z) = \gamma^* \mu_z(\partial\mathbb{H}) = \mu_z(\gamma^{-1}(\partial\mathbb{H})) = \Phi(z)$.

Moreover, when $\delta > d/2$ then Φ is square-integrable with respect to dV on the fundamental domains of $\Gamma \backslash \mathbb{H}$, and it is the fundamental eigenstate on $\Gamma \backslash \mathbb{H}$.

See ([P], Theorem 1 page 301), or ([P-S], page 177).

Note that a consequence is that the volume of $\Gamma \backslash \mathbb{H}$ is finite if and only if $\delta = d$.

Let π denote the canonical projection from $T^1\mathbb{H}$ onto \mathbb{H} , π_1 denote the canonical projection from $\mathcal{O}\mathbb{H}$ onto $T^1\mathbb{H}$, and $\pi_2 = \pi \circ \pi_1$ denote the canonical projection from $\mathcal{O}\mathbb{H}$ onto \mathbb{H} .

We shall use on the unitary tangent bundle $T^1\mathbb{H}$ the two following systems of coordinates :

- firstly, $(z, u) \in \mathbb{H} \times \partial\mathbb{H}$, the geodesic running from z to u determining the unitary tangent vector at the base point z ; this identifies $T^1\mathbb{H}$ with $\mathbb{H} \times \partial\mathbb{H}$;
- secondly, given a reference point $z_0 \in \mathbb{H}$, the point (z, u) of $T^1\mathbb{H}$ (just defined above) can be represented by the triple $(u, v, s) \in \partial\mathbb{H} \times \partial\mathbb{H} \times \mathbb{R}$, where
 - v is the starting point of the geodesic ending at u and running through z ;
 - s is the algebraic hyperbolic distance from z to the orthogonal projection z_1 of z_0 onto the geodesic \overrightarrow{vu} .

The $PSO(d+1, 1)$ model for both $\mathcal{O}\mathbb{H}$ and the Möbius isometries of \mathbb{H} allows to identify $T^1\mathbb{H}$ with $PSO(d+1, 1)/SO_d$ and \mathbb{H} with $PSO(d+1, 1)/SO_{d+1}$, and to use on $\mathcal{O}\mathbb{H}$ the coordinates system $(z, u, r) \in \mathbb{H} \times \partial\mathbb{H} \times SO_d$.

Denote by $dist(\zeta, uv)$ the hyperbolic distance from $\zeta \in \mathbb{H}$ to the geodesic \overrightarrow{vu} . The following well-known identity is valid for any ζ in \mathbb{H} , any distinct u, v in $\partial\mathbb{H}$, and any z on the geodesic \overrightarrow{vu} running from v to u .

$$(*) \quad \text{ch}^2(dist(\zeta, uv)) = B_u(\zeta, z) B_v(\zeta, z) .$$

(Indeed, since this is an intrinsic formula, we may consider the half-space model with $u = \infty$ and $v = 0$. Denoting then by (X, Y) the Euclidean coordinates of ζ in this model, and by $(0, y)$ those of z , it is elementary that $B_u(\zeta, z) = y/Y$, $B_v(\zeta, z) = (|X|^2 + Y^2)/(yY)$, and, using the classical formula for the distance (see [P]), that $\text{ch}^2(dist(\zeta, uv)) = \text{ch}^2(dist(\zeta, (0, |\zeta|))) = (|X|^2 + Y^2)^2/(Y|\zeta|)^2 = 1 + |X|^2/Y^2 = B_u(\zeta, z)B_v(\zeta, z)$.)

The Liouville measure $\tilde{\lambda}$ on $T^1\mathbb{H}$ can be expressed for any reference point z_0 , by :

$$d\tilde{\lambda}(u, v, s) = \text{ch}^{2d}(dist(z_0, uv)) d\mu_{z_0}^h(u) d\mu_{z_0}^h(u) ds ,$$

where μ_z^h denotes the harmonic measure at z . Recall that we have in the half-space model :

$$d\mu_z^h(u) = p^d(z, u) du .$$

Note that the above geometric property holds for harmonic measures, by changing δ into d : $d\mu_{z'}^h(u) = B_u^d(z, z') d\mu_z^h(u)$ for any z, z' in \mathbb{H} .

This and the identity $(*)$ show the irrelevance of the reference point z_0 in the expression of the Liouville measure $\tilde{\lambda}$ above. As can be verified by a direct elementary computation, the expression of $\tilde{\lambda}$ in the (z, u) coordinates is : $d\tilde{\lambda}(z, u) = d\mu_z^h(u) dV(z)$.

$\tilde{\lambda}$ is naturally lifted to the Liouville measure λ' on $\mathcal{O}\mathbb{H}$, by taking λ' uniform on each fibre SO_d . λ' is finite on a fundamental domain of $\Gamma \backslash \mathcal{O}\mathbb{H}$ only when $\delta = d$.

Observe from the two expressions above for the Liouville measure the following formula

$$(1) \quad \int F dV = \int F \circ \pi(u, v, s) B_v^d(z_0, \pi(u, v, s)) d\mu_{z_0}^h(v) ds ,$$

valid for any $u \in \partial\mathbb{H}$, any $z_0 \in \mathbb{H}$, and any test function F on \mathbb{H} .

Let θ_t and θ_x^+ denote respectively the geodesic and the positive horocycle flows on the orthonormal frame bundle \mathcal{OH} . Moreover for any $z = (x, y) \in \mathbb{R}^d \times \mathbb{R}_+^*$, set

$$T_z := \theta_x^+ \theta_{Log y} .$$

Observe the following important classical relation :

$$(2) \quad T_{(x,y)} T_{z'} = T_{(x,0)+yz'} .$$

This means in particular that the set $\{T_z | z \in \mathbb{R}^d \times \mathbb{R}_+^*\}$ constitutes a group, isomorphic to a subgroup of the affine group of \mathbb{R}^d .

In the $PSO(d+1, 1)$ model, the Liouville measure λ' is the Haar measure, and the flows are expressed by right multiplication by some matrices of $PSO(d+1, 1)$. Thus the Liouville measure λ' is invariant by the horocycle and geodesic flows.

We can decompose the horocycle flow according to the canonical basis of \mathbb{R}^d : so for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, we set : $\theta_x^+ = \theta_{x_1}^1 \dots \theta_{x_d}^d$.

Let us introduce then the Lie derivatives : for any smooth function F on \mathcal{OH} , any ξ in \mathcal{OH} , $1 \leq j \leq d$ we set :

$$(3) \quad \mathcal{L}_0 F(\xi) := \frac{d_o}{dt} F(\xi \theta_t) , \quad \mathcal{L}_j F(\xi) := \frac{d_o}{dt} F(\xi \theta_t^j) .$$

$\frac{d_o}{dt}$ means and will mean the derivative at $t = 0$ with respect to t .

We immediately see that : (4) $[\mathcal{L}_0, \mathcal{L}_j] = \mathcal{L}_j$, $[\mathcal{L}_j, \mathcal{L}_{j'}] = 0$, and

$$(5) \quad \mathcal{L}_0 F(\xi T_{(x,y)}) = y \frac{\partial}{\partial y} F(\xi T_{(x,y)}) , \quad \mathcal{L}_j F(\xi T_{(x,y)}) = y \frac{\partial}{\partial x_j} F(\xi T_{(x,y)}) .$$

Note that since the flows act on the right hand side, while Γ acts on the left hand side, these two operations commute.

Note that while the geodesic flow still makes sense on $T^1\mathbb{H}$, the horocycle flow makes sense only on \mathcal{OH} .

By identifying Möbius isometries of \mathbb{H} and orthonormal frames on \mathbb{H} , we deduce the following relations : $\pi_2(\xi T_z) = \xi(z)$, and $\pi_1(\xi T_z) = \xi(\vec{z})$, for any $\xi \in \mathcal{OH}$ and $z \in \mathbb{H}$, $\vec{z} \in T^1\mathbb{H}$ denoting the line element based at z and pointing at ∞ .

Let us call "Φ-diffusion" and denote by Z_t^Φ the diffusion on \mathbb{H} associated to the fundamental state Φ , that is to say having infinitesimal generator

$$\frac{1}{2} \Delta^\Phi := \frac{1}{2\Phi} \Delta \circ \Phi - \lambda_0 .$$

This diffusion was already considered by Sullivan in [Su1] and for $d = 1$ in [E-F-LJ-1,2].

3 An intrinsic measure on $T^1\mathbb{H}$

We introduce an intrinsic measure ν on $T^1\mathbb{H}$, which was already used for $d = 1$ in [E-F-LJ-1,2]. Its interest is to be smooth along the stable leaves and quasi-invariant under the geodesic and positive horocycle flows, and to be an invariant measure for two dual diffusions on $\mathcal{O}\mathbb{H}$, which are both projected by π_2 onto the Φ -diffusion.

Definition 1 Let $\tilde{\nu}$ be the measure on $T^1\mathbb{H}$ defined by :

$$d\tilde{\nu}(z, u) = \Phi(z) d\mu_z(u) dV(z) .$$

Denote by ν' the unique measure on $\mathcal{O}\mathbb{H}$ which has marginal $\tilde{\nu}$ on $T^1\mathbb{H}$ and whose conditional laws on the fibres are the normalized Haar measure on $SO_d \equiv \mathcal{O}\mathbb{H}/T^1\mathbb{H}$. Set also $dV^\Phi(z) := \Phi^2(z) dV(z)$.

Remark 1 Observe that the Γ -invariance of Φ , \tilde{V} and the geometric property of (μ_z) imply the Γ -invariance of $\tilde{\nu}$ (and then of ν'). Observe also that by definition of Φ we have $\pi_2^*\nu' = \pi^*\tilde{\nu} = V^\Phi$, and then that $\tilde{\nu}$ and ν' are finite above a fundamental domain relating to Γ if and only if $\delta > d/2$, with mass $\|\Phi\|_2^2$.

Remark 2 In the finite volume case, we have $\delta = d$, Φ constant, $d\mu(u)$ is proportional to the uniform measure du , and then our measure $\tilde{\nu}$ is proportional to the Liouville measure $\tilde{\lambda}$.

Proposition 1 The measure ν' is quasi-invariant under the geodesic and positive horocycle flows :

$$\frac{d(T_z^* \nu')}{d\nu'}(\xi) = y^{d-\delta} \times \frac{\Phi \circ \pi_2(\xi T_z^{-1})}{\Phi \circ \pi_2(\xi)} \quad \text{for any } \xi \in \mathcal{O}\mathbb{H} \text{ and } z = (x, y) \in \mathbb{R}^d \times \mathbb{R}_+^* .$$

Note that this quasi-invariance property is what remains from the invariance of the Liouville measure λ' under the flows, in the finite volume case. The proof was already given for $d = 1$ in [E-F-LJ-2]. We write it now for any d and for selfcontainedness.

Proof Let us use the invariance of the Liouville measure and of the coordinate u under the flows, and the expression of the Liouville measure in the coordinates system $\xi = \xi(z, u, r) \in \mathcal{O}\mathbb{H}$. We get for any $\zeta \in \mathbb{R}^d \times \mathbb{R}_+^*$ and any test functions H on $\partial\mathbb{H}$ and G on $\mathcal{O}\mathbb{H}$:

$$\int G(\xi(z, u, r)T_\zeta) H(u) d\mu_z^h(u) dV(z) dr = \int G(\xi(z, u, r)) H(u) d\mu_z^h(u) dV(z) dr .$$

Thus we obtain for any $u \in \partial\mathbb{H}$, $z_0 \in \mathbb{H}$, and any test function G on $\mathcal{O}\mathbb{H}$:

$$\int G(\xi(z, u, r)T_\zeta) B_u^d(z_0, z) dV(z) dr = \int G(\xi(z, u, r)) B_u^d(z_0, z) dV(z) dr .$$

Whence using the definition 1 of $\tilde{\nu}$, a reference point $z_0 \in \mathbb{H}$, the geometric property of (μ_z) , and the (z, u, r) -coordinates on $\mathcal{O}\mathbb{H}$, we get :

$$\int G(\xi T_\zeta) d\nu'(\xi) = \int G(\xi(z, u, r)T_\zeta) \Phi(z) B_u^\delta(z_0, z) d\mu_{z_0}(u) dV(z) dr =$$

$$\begin{aligned}
& \int G(\xi(z, u, r)) \Phi \circ \pi_2(\xi(z, u, r) T_\zeta^{-1}) B_u^{\delta-d}(z_0, \pi_2(\xi(z, u, r) T_\zeta^{-1})) B_u^d(z_0, z) d\mu_{z_0}(u) dV(z) dr \\
&= \int G(\xi) \times \frac{\Phi \circ \pi_2(\xi T_\zeta^{-1})}{\Phi \circ \pi_2(\xi)} \times \frac{B_u^{\delta-d}(z_0, \pi_2(\xi T_\zeta^{-1}))}{B_u^{\delta-d}(z_0, \pi_2(\xi))} d\nu'(\xi) \\
&= \int G(\xi) \times \frac{\Phi \circ \pi_2(\xi T_\zeta^{-1})}{\Phi \circ \pi_2(\xi)} \times B_{\xi\theta_\infty}^{\delta-d}(\pi_2(\xi), \pi_2(\xi T_\zeta^{-1})) d\nu'(\xi).
\end{aligned}$$

The result follows, since writing $\zeta = (x, y)$, we clearly have by the definition of B_u :

$$B_{\xi\theta_\infty}^{\delta-d}(\pi_2(\xi), \pi_2(\xi T_\zeta^{-1})) = B_{\xi\theta_\infty}^{d-\delta}(\pi_2(\xi\theta_{-\log y}), \pi_2(\xi)) = y^{d-\delta}. \quad \diamond$$

4 Diffusions on \mathcal{IH} and on \mathcal{OH}

Let us from now on identify \mathcal{IH} with its Poincaré half-space model $\mathbb{R}^d \times \mathbb{R}_+^*$, and denote by $z = (x, y)$ the current point. Recall that

$$\Delta = y^2 \times \left(\frac{\partial^2}{\partial y^2} + \frac{1-d}{y} \times \frac{\partial}{\partial y} + \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2} \right).$$

4.1 The diffusions Z_t^δ , ξ_t^δ and Z_t^Φ

Let (w_t, W_t) denote a Brownian motion on $\mathbb{R} \times \mathbb{R}^d$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Set

$$y_t := \exp[w_t + (\delta - d/2)t], \quad x_t := \int_0^t y_s dW_s, \quad Z_t^\delta := (x_t, y_t) \in \mathcal{IH}.$$

For all δ , Z_t^δ is the diffusion on \mathcal{IH} starting from $e_o := (0, 1)$, with invariant measure $y^{2\delta-2d} dx dy$, and generator

$$\frac{1}{2} \Delta^\delta := \frac{1}{2} \Delta + \delta y \frac{\partial}{\partial y} = \frac{y^2}{2} \left(\frac{\partial^2}{\partial y^2} + \frac{2\delta + 1 - d}{y} \times \frac{\partial}{\partial y} + \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2} \right).$$

Similarly, denote by $Z_t^b = (x_t^b, y_t^b)$ the analogous process with δ replaced by $b \in [0, d]$. In particular, Z_t^0 is the Brownian motion on \mathcal{IH} .

Recall the classical identification between \mathcal{OH} and the set of Möbius isometries of \mathcal{IH} , and that in this identification we have for any $\xi \in \mathcal{OH}$ and any $z \in \mathcal{IH}$: $\xi(z) = \pi_2(\xi T_z)$.

In particular, we see that $\pi_2(\xi T_{Z_t^0}) = \xi(Z_t^0)$ is a Brownian motion on \mathcal{IH} , started from $\pi_2(\xi)$, for any $\xi \in \mathcal{OH}$. As a consequence, denoting by P_t the Brownian semi-group on \mathcal{IH} , we have $E(f \circ \pi_2(\xi T_{Z_t^0})) = P_t f(\pi_2(\xi))$.

Observe then that $T_{Z_t^0}$ is a right Brownian motion on a subgroup of the affine group of \mathbb{R}^d . Indeed, for any $b \in [0, d]$,

$$T_{Z_t^b}^{-1} T_{Z_{t+s}^b} = T_{\left(\frac{x_{t+s}^b - x_t^b}{y_t^b}, \frac{y_{t+s}^b}{y_t^b} \right)} = T_{Z_t^b} \circ \Theta_t$$

is independent of the sub- σ -field \mathcal{F}_t generated by the coordinates until time t .

Definition 2 For any $\xi \in \mathcal{O}H$, set $\xi_t^\delta := \xi T_{Z_t^\delta}$.

Set $\Delta^\Phi := \Phi^{-1} \Delta \circ \Phi - 2\lambda_0 = \Delta + 2(\nabla \log \Phi) \cdot \nabla$.

Denote by Z_t^Φ and call " Φ -diffusion" the diffusion on $\mathcal{O}H$ with generator $\frac{1}{2} \Delta^\Phi$.

By the preceding observation, ξ_t^δ is a diffusion on $\mathcal{O}H$, starting from ξ .

From (5) we get $\Delta^\delta[F(\xi T_z)] = (D^\delta F)(\xi T_z)$, where

$$D^\delta := \sum_{j=0}^d \mathcal{L}_j^2 + (2\delta - d) \mathcal{L}_0 = D^0 + 2\delta \mathcal{L}_0.$$

Then the generator of the diffusion ξ_t^δ is $\frac{1}{2} D^\delta$.

Moreover, note that the Φ -diffusion is symmetrical and has invariant measure V^Φ .

Note also that it happens to be the diffusion already considered in [Su1].

In the finite volume case $\delta = d$, this is just the Brownian motion.

Remark 3 We have for any test-function F on $\mathcal{O}H$:

$$D^0(F \circ \pi_2)(\xi T) = \Delta[F \circ \pi_2(\xi T)] = \Delta(F \circ \xi) = (\Delta F) \circ \xi = (\Delta F) \circ \pi_2(\xi T), \quad \text{whence} \\ D^0(F \circ \pi_2) = (\Delta F) \circ \pi_2.$$

4.2 ν' as an invariant measure

We deduce now from the quasi-invariance property of ν' an adjonction property for ν' , and thus its invariance with respect to two diffusions on $\mathcal{O}H$.

Proposition 2 We have for all δ and all test functions F, G on $\mathcal{O}H$:

$$\int (D^\delta F) G d\nu' = \int F (D^\Phi G) d\nu', \quad \text{where}$$

$$D^\Phi := \sum_{j=0}^d \mathcal{L}_j^2 - d \mathcal{L}_0 + 2 \sum_{j=0}^d (\mathcal{L}_j \log \Phi \circ \pi_2) \mathcal{L}_j = (\Phi \circ \pi_2)^{-1} D^0 \circ (\Phi \circ \pi_2) - 2\lambda_0.$$

Proof We deduce directly from proposition 1 the infinitesimal expression of the quasi-invariance of ν' : we have for $j \in \{0, \dots, d\}$:

$$\int \mathcal{L}_j F d\nu' = - \int F \times (\mathcal{L}_j \log \Phi \circ \pi_2) d\nu' + 1_{\{j=0\}} (d - \delta) \int F d\nu'.$$

This implies immediately (writing Φ for $\Phi \circ \pi_2$):

$$\mathcal{L}_j^* = -\mathcal{L}_j - \mathcal{L}_j(\log \Phi) + 1_{\{j=0\}} (d - \delta),$$

the adjoint being relating to ν' ; whence (using that $D^0 \Phi = (\Delta \Phi) \circ \pi_2 = 2\lambda_0 \Phi$):

$$(D^\delta)^* = D^\Phi + (d - \delta)\delta - d \mathcal{L}_0 \log \Phi + \sum_{j=0}^d ((\mathcal{L}_j \log \Phi)^2 + \mathcal{L}_j^2 \log \Phi) \\ = D^\Phi + \Phi^{-1} (D^0 \Phi - 2\lambda_0 \Phi) = D^\Phi. \quad \diamond$$

Corollary 1 For $\xi \in \mathcal{O}\mathbb{H}$ and each δ we have :

- (i) under \mathbb{P} , the diffusion ξ_t^δ admits the invariant measure ν' ;
- (ii) under $\nu' \otimes \mathbb{P}$, ξ_t^δ extends to a stationary diffusion defined for all real t , and ξ_{-t}^δ is the stationary diffusion associated with the infinitesimal generator $\frac{1}{2} D^\Phi$, say ξ_t^Φ .

Remark 4 (i) We see from remark 3 and from the h -process form of D^Φ in proposition 2 above that we have for any test-function F on $\mathcal{O}\mathbb{H}$:

$$D^\Phi(F \circ \pi_2) = (\Delta^\Phi F) \circ \pi_2 .$$

(ii) The h -process form of D^Φ shows that ξ_t^Φ can be defined by the following formula, where $0 = t_0 < \dots < t_n$, F_0, \dots, F_n are test-functions on $\mathcal{O}\mathbb{H}$, and $\xi_t^0 = \xi T_{Z_t^0}$:

$$\int \int \prod_{j=0}^n F_j(\xi_{t_j}^\Phi) d\nu'(\xi) d\mathbb{P} = \int \int \frac{e^{-\lambda_0 t_n}}{\Phi \circ \pi_2(\xi)} \times \Phi \circ \pi_2(\xi_{t_n}^0) \times \prod_{j=0}^n F_j(\xi_{t_j}^0) d\nu'(\xi) d\mathbb{P} .$$

(iii) Recall that the identification between \mathbb{H} and the half-space $\mathbb{R}^d \times \mathbb{R}_+^*$, respectively between $T^1\mathbb{H}$ and $PSO(d+1, 1)/SO_d$, amounts to fixing one point of \mathbb{H} to be e_o , respectively of $T^1\mathbb{H}$ to be \vec{e}_o , and similarly between $\mathcal{O}\mathbb{H}$ and $PSO(d+1, 1)$. Note that although the process Z_t^δ depends on this identification, the diffusions Z_t^Φ , ξ_t^δ and ξ_t^Φ do not and are intrinsic.

By means of the coordinates (z, u) , we have for each $z \in \mathbb{H}$ an identification between $\partial\mathbb{H}$ and $T_z^1\mathbb{H}$. This allows to consider the Patterson measure μ_z as a measure on $T_z^1\mathbb{H}$, and then to localize the measure ν' as follows :

Definition 3 For each $z \in \mathbb{H}$, set $\tilde{\nu}_z := \delta_z \otimes \mu_z / \Phi(z)$. This is a probability measure on $T^1\mathbb{H}$, concentrated on $T_z^1\mathbb{H}$.

Denote also by ν'_z the probability measure on $\mathcal{O}\mathbb{H}$ concentrated on $\pi_2^{-1}(z)$, uniform on each fibre $\pi_1^{-1} \circ \pi_1(\xi)$, and projected on $\tilde{\nu}_z$ by π_1 .

So that we have the disintegration : $\nu' = \nu'_z dV^\Phi(z)$ on $\mathcal{O}\mathbb{H}$.

4.3 $\pi_2(\xi_t^\delta)$ is the Φ -diffusion

We obtain here the Φ -diffusion by projection of the diffusion ξ_t^δ . We begin with the stationary case.

Proposition 3 Under $\nu' \otimes \mathbb{P}$, the projection $\pi_2(\xi_t^\delta)$ of the stationary diffusion ξ_t^δ on $\mathcal{O}\mathbb{H}$ is the stationary Φ -diffusion.

Proof Consider $0 = t_0 < t_1 < \dots < t_n$ and test-functions f_0, \dots, f_n on \mathbb{H} . Using corollary (1, i, ii) and remark (4, ii), we have :

$$\begin{aligned} \int \int \prod_{j=0}^n f_j \circ \pi_2(\xi_{t_j}^\delta) d\nu'(\xi) d\mathbb{P} &= \int \int \prod_{j=0}^n f_j \circ \pi_2(\xi_{t_j - t_n}^\delta) d\nu'(\xi) d\mathbb{P} \\ &= \int \int e^{-\lambda_0 t_n} \times \frac{\Phi \circ \pi_2(\xi_{t_n}^0)}{\Phi \circ \pi_2(\xi_0^0)} \times \prod_{j=0}^n f_j \circ \pi_2(\xi_{t_n - t_j}^0) d\nu'(\xi) d\mathbb{P} \end{aligned}$$

$$= \int \int e^{-\lambda_0 t_n} \times \frac{\Phi(\xi(Z_{t_n}^0))}{\Phi \circ \pi_2(\xi)} \times \prod_{j=0}^n f_j(\xi(Z_{t_n-t_j}^0)) d\pi_2^* \nu'(\xi) d\mathbb{P}$$

(since $\pi_2(\xi T_z) = \xi(z)$ and since $\xi(Z_t^0)$ is a Brownian motion starting from $\xi(e_o) = \pi_2(\xi)$, ξ being an isometry)

$$= \mathbb{E}_{V^\Phi} \left(\prod_{j=0}^n f_j(Z_{t_n-t_j}^\Phi) \right)$$

(since $\pi_2^* \nu' = V^\Phi$ and by the h -process form of Δ^Φ in definition 2)

$$= \mathbb{E}_{V^\Phi} \left(\prod_{j=0}^n f_j(Z_{t_j}^\Phi) \right)$$

by symmetry and invariance of Z^Φ with respect to V^Φ . \diamond

We can localize this result as follows.

Corollary 2 *Under the probability law $\nu'_z \otimes \mathbb{P}$, the projection $\pi_2(\xi_t^\delta)$ on \mathbb{H} of the diffusion ξ_t^δ on $\mathcal{O}\mathbb{H}$ is the Φ -diffusion Z_t^Φ starting from z .*

Proof Let us apply the preceding proposition with $f_0(z) := 1_{\{dist(z,z') < \epsilon\}}$. We get

$$\int_{\{dist(z,z') < \epsilon\}} \int \mathbb{E} \left(\prod_{j=1}^n f_j \circ \pi_2(\xi_{t_j}^\delta) \right) d\nu'_z(\xi) dV^\Phi(z) = \int_{\{dist(z,z') < \epsilon\}} \mathbb{E}_z \left(\prod_{j=1}^n f_j(Z_{t_j}^\Phi) \right) dV^\Phi(z).$$

Now the semi-group P_t^Φ of the Φ -diffusion is clearly Fellerian, and then the map $z \mapsto \mathbb{E}_z \left(\prod_{j=1}^n f_j(Z_{t_j}^\Phi) \right)$ is continuous. Thus the result will follow from the continuity of the map $z \mapsto \int \mathbb{E} \left(\prod_{j=1}^n f_j \circ \pi_2(\xi_{t_j}^\delta) \right) d\nu'_z(\xi)$, for compactly supported continuous functions f_j .

Now $(\xi, z) \mapsto f_j \circ \pi_2(\xi T_z)$ is (uniformly) continuous on $T^1\mathbb{H} \times \mathbb{H}$, and then

$\xi = (z, u, r) \mapsto F(\xi) := \mathbb{E} \left(\prod_{j=1}^n f_j \circ \pi_2(\xi_{t_j}^\delta) \right)$ is continuous and bounded on $\mathcal{O}\mathbb{H}$.

Thus

$$\left| \int F(\xi) d\nu'_z(\xi) - \int F(\xi) d\nu'_{z'}(\xi) \right| = \left| \int \left(\frac{F(z, u, r) B_u^\delta(z', z)}{\Phi(z)} - \frac{F(z', u, r)}{\Phi(z')} \right) d\mu_{z'}(u) dr \right|$$

goes to zero as $z \rightarrow z'$ by dominated convergence, since $B_\cdot(z', z)$ is uniformly bounded for bounded $dist(z, z')$. \diamond

This allows to deduce immediately the asymptotic behaviour of the Φ -diffusion in the case $\delta \geq d/2$.

Corollary 3 *The Φ -diffusion Z_t^Φ starting from $z \in \mathbb{H}$ converges almost surely as $t \rightarrow \infty$ to a random point belonging to $\partial\mathbb{H}$, whose law is $\mu_z/\Phi(z)$.*

Proof It is clear from the definition that y_t , and then Z_t^δ , goes almost surely to ∞ with t when $\delta > d/2$. When $\delta = d/2$, setting $r_t := (|x_t|^2 + y_t^2)^{1/2}$, we see from the expression of the generator $\frac{1}{2}\Delta^{d/2}$ that $r_t = B^{d+2}(Y_t)$, where $B^{d+2}(t)$ is some $(d+2)$ -dimensional Bessel process and $Y_t := \int_0^t y_s^2 ds = \int_0^t e^{2w_s} ds$, which both go almost surely to ∞ with t , thereby showing that r_t , and then $Z_t^{d/2}$, also does.

Now corollary 2 above and the convergence of Z_t^δ to ∞ imply the convergence of $\pi_2(\xi_t^\delta)$ to $\xi(\infty) = u$, whose law under $\nu'_z \otimes \mathbb{P}$ is $\mu_z/\Phi(z)$, by definition of ν'_z . \diamond

Remark 5 It is however false that $\pi_2(\xi_t^\delta)$ has under \mathbb{P} the law of the Φ -diffusion. Indeed, as observed above for corollary 3, $\pi_2(\xi_t^\delta)$ is conditioned to exit at $\xi(\infty)$.

5 Exit measure of the Φ -diffusion if $\delta < d/2$

For this whole section, we consider the only case : $\delta \in [0, d/2[$.

Lemma 1 $\int_{\partial\mathbb{H}} \text{ch}^{2\delta}(\text{dist}(z, uw)) d\mu_z^h(w)$ does not depend on $(z, u) \in \mathbb{H} \times \partial\mathbb{H}$, but only on d and δ (and equals $\pi^{d/2} \times \frac{\Gamma(d/2 - \delta)}{\Gamma(d - \delta)}$).

Proof Indeed, the left hand side of this formula being clearly intrinsic, it is sufficient to work with the model of the ball centered at z . Then the harmonic measure μ_z^h is uniform, which in turn implies the independence with respect to u .

This independence is sufficient for our future purpose, however let us now perform the computation of the constant, interesting for the intertwining formula in remark 6 below. Taking polar coordinates in $\partial\mathbb{H} \equiv \mathbb{S}^d$, with leading angle the one between u and w , say α , we have $\text{ch}(\text{dist}(z, uw)) = 2/|u - w| = (\sin \alpha/2)^{-1}$ and thus we get:

$$\begin{aligned} \int_{\partial\mathbb{H}} \text{ch}^{2\delta}(\text{dist}(z, uw)) d\mu_z^h(w) &= \int_{\mathbb{S}^d} (\sin \alpha/2)^{-2\delta} d\mu_{\mathcal{O}}^h \\ &= |\mathbb{S}^{d-1}| \times \int_0^\pi (\sin \alpha/2)^{-2\delta} (\sin \alpha)^{d-1} d\alpha \times |\mathbb{S}^d|^{-1} \times |\mu^h| \\ &= \frac{2\pi^{d/2}}{\Gamma(d/2)} \times 2^d \int_0^\infty (1+r^2)^{\delta-d} r^{d-2\delta-1} dr \times |\mathbb{S}^d|^{-1} \times \int_{\mathbb{R}^d} p^d(z, v) dv \\ &= \frac{2\pi^{d/2}}{\Gamma(d/2)} \times 2^{d-1} \int_0^1 s^{d/2-1} (1-s)^{d/2-\delta-1} ds \times |\mathbb{S}^d|^{-1} \times \int_{\mathbb{R}^d} (1+|w|^2)^{-d} dw \\ &= 2^d \pi^{d/2} \times \frac{\Gamma(d/2 - \delta)}{\Gamma(d - \delta)} \times |\mathbb{S}^d|^{-1} \times |\mathbb{S}^{d-1}| \times \frac{\Gamma(d/2)^2}{2\Gamma(d)} = \pi^{d/2} \times \frac{\Gamma(d/2 - \delta)}{\Gamma(d - \delta)}. \quad \diamond \end{aligned}$$

Remark 6 Another form of the formula of the preceding lemma is the following intertwining formula between $p^{d-\delta}$ and p^δ , which expresses the unextremal kernel p^δ as a mean of the extremal kernels $p^{d-\delta}$: For all $(z, u) \in \mathbb{H} \times \mathbb{R}^d$ we have

$$\int_{\mathbb{R}^d} p^{d-\delta}(z, w) |u - w|^{-2\delta} dw = \pi^{d/2} \times \frac{\Gamma(d/2 - \delta)}{\Gamma(d - \delta)} \times p^\delta(z, u).$$

Indeed, to see this it is sufficient to apply the formula (*), to express the harmonic measure and the Busemann function by means of the Poisson kernel, and to observe that for ζ at the top of the geodesic \overline{uw} we clearly have : $p(\zeta, u)p(\zeta, w) = |u - w|^{-2}$. A complicated proof of this intertwining formula was written in ([M], epilogue). M. Babillot and J.P. Otal already knew a simple proof for it. See also ([G], p. 386).

Definition 4 Set for $w \in \partial\mathbb{H}$ et $z \in \mathbb{H}$:

$$q(z, w) := \int_{\partial\mathbb{H}} \text{ch}^{2\delta}(\text{dist}(z, uw)) d\mu_z(u), \quad d\check{\mu}_z(w) := \pi^{-d/2} \frac{\Gamma(d - \delta)}{\Gamma(d/2 - \delta)} q(z, w) d\mu_z^h(w),$$

and on $T^1\mathbb{H}$:

$$\check{\nu}(dz, du) := \Phi(z) \check{\mu}_z(du) dV(z), \quad \text{and} \quad \check{\nu}_z := \delta_z \otimes \check{\mu}_z / \Phi(z).$$

Denote by $\check{\nu}'$ (respectively $\check{\nu}'_z$) the measure on $\mathcal{O}\mathbb{H}$ projected onto $\check{\nu}$ (respectively $\check{\nu}_z$) by π_1 and uniform on each fibre $\pi_1^{-1} \circ \pi_1(\xi)$.

Lemma 2 (i) For all $\gamma \in \Gamma$, $z \in \mathbb{H}$ and $w \in \partial\mathbb{H}$, we have $q(\gamma z, \gamma w) = q(z, w)$.

(ii) $\check{\mu}_z(\partial\mathbb{H}) = \Phi(z)$, and then $\check{\nu}_z$ is a probability measure.

(iii) The family of measures $(\check{\mu}_z)$ satisfies the invariance property and the geometrical property of exponent $(d - \delta)$:

$$\gamma^* \check{\mu}_z = \check{\mu}_{\gamma z} \quad \text{for all } \gamma \in \Gamma, \quad \text{and} \quad d\check{\mu}_{z'}(u) = B_u^{d-\delta}(z, z') d\check{\mu}_z(u) \quad \text{for all } z, z' \in \mathbb{H}.$$

(iv) $\check{\nu}(dz, du) = \check{\nu}_z(du) V^\Phi(dz)$ has an infinite mass on a fundamental domain, and $\pi^* \check{\nu} = V^\Phi$.

Proof (i) follows directly from the invariance property of (μ_z) ;

(ii) follows directly from Lemma 1 ;

(iii) The invariance formula follows from the invariance property of the harmonic measure and from (i) above, and the second formula follows from the formula (*) and from the cocycle property of the Busemann function ;

(iv) is straightforward. \diamond

As the proof of the quasi-invariance formula for ν' (see proposition 1) uses only the geometrical property of (μ_z) , we deduce the quasi-invariance formula for $\check{\nu}$ below (using the above lemma (2, iii)) merely by changing δ into $d - \delta$.

Proposition 4 The measure $\check{\nu}'$ is quasi-invariant under the geodesic and positive horocycle flows :

$$\frac{d(T_z^* \check{\nu}')}{d\check{\nu}'}(\xi) = y^\delta \times \frac{\Phi \circ \pi_2(\xi T_z^{-1})}{\Phi \circ \pi_2(\xi)} \quad \text{for all } \xi \in \mathcal{O}\mathbb{H} \text{ and } z = (x, y) \in \mathbb{H}.$$

As for Proposition 2, we deduce :

Corollary 4 For all test-functions F, G we have :

$$(i) \quad \int \mathcal{L}_j F d\check{\nu}' = - \int F \times (\mathcal{L}_j \log \Phi \circ \pi_2) d\check{\nu}' + 1_{\{j=0\}} \delta \int F d\check{\nu}';$$

$$(ii) \quad \int (D^{d-\delta} F) G d\check{\nu}' = \int F (D^\Phi G) d\check{\nu}'.$$

As for Corollary 1, we deduce immediately :

Corollary 5 For all $\delta < d/2$ we have :

- (i) under \mathbb{P} , $\xi_t^{d-\delta}$ is the diffusion on $\mathcal{O}\mathbb{H}$ starting from ξ and with generator $\frac{1}{2}D^{d-\delta}$; it admits the invariant measure $\check{\nu}'$;
- (ii) under $\check{\nu}' \otimes \mathbb{P}$, $\xi_t^{d-\delta}$ extends to a stationary diffusion defined for all real t , and $\xi_{-t}^{d-\delta}$ equals the stationary diffusion ξ_t^Φ on $\mathcal{O}\mathbb{H}$ (with generator $\frac{1}{2}D^\Phi$).

We have also the analogue of Proposition 3, using Proposition 4 instead of Proposition 1. Indeed it is sufficient to adapt the proof of proposition 3, changing everywhere δ into $d - \delta$ and ν' into $\check{\nu}'$, and noticing that $\pi_2^* \check{\nu}' = V^\Phi$.

Proposition 5 For all $\delta < d/2$, under $\check{\nu}' \otimes \mathbb{P}$, the projection $\pi_2(\xi_t^{d-\delta})$ of the stationary diffusion $\xi_t^{d-\delta}$ on $\mathcal{O}\mathbb{H}$ is the stationary Φ -diffusion.

Using this proposition 5 instead of Proposition 3, we get (by the same argument, merely adapted by changing respectively δ into $d - \delta$, ν'_z into $\check{\nu}'_z$, and $\mu_{z'}$ into $\check{\mu}_{z'}$) the analogue of Corollary 2.

Corollary 6 Under the probability law $\check{\nu}'_z \otimes \mathbb{P}$, the projection $\pi_2(\xi_t^{d-\delta})$ on \mathbb{H} of the diffusion $\xi_t^{d-\delta}$ on $\mathcal{O}\mathbb{H}$ is the Φ -diffusion Z_t^Φ starting from z .

This allows to deduce the asymptotic behaviour of the Φ -diffusion in the case $\delta < d/2$. The proof is the same as for Corollary 3 (when $\delta > d/2$).

Corollary 7 When $\delta < d/2$, the Φ -diffusion Z_t^Φ starting from $z \in \mathbb{H}$ converges almost surely as $t \rightarrow \infty$ to a random point of $\partial\mathbb{H}$, whose law is $\check{\mu}_z/\Phi(z)$.

Remark 7 The general theory of Martin boundary for rank one symmetric spaces can be applied to yield Corollaries 3 and 7.

Let us briefly outline the main steps of such a proof.

a) Characterization of the Martin boundary of $(\mathbb{H}, \Delta - \lambda I)$ as $\partial\mathbb{H}$, and of its Martin kernels as $B_u(z, z')^\delta$ when $\delta \geq d/2$.

This work has been done for example in Proposition 3.2 of [L-MG-T] and in [G].

b) Application of an extended version of the Fatou-Naïm-Doob theorem, as the one given in Theorem 3.1 of [A].

c) Use of the intertwining formula (see our remark 6), as we used above our lemma 1, for the case $\delta < d/2$, in which the $B_u(z, z')^\delta$ are not extremal (see also [G], p. 386).

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REFERENCES

[A] ANCONA A. *Théorie du potentiel sur les graphes et les variétés.*

Ecole d'été de probabilités de Saint-Flour XVIII, Lecture Notes 1427, Springer 1990.

- [E] ENRIQUEZ N. Thèse de l'Université Paris Sud , 3^{ème} partie,
91405 Orsay, Septembre 1995.
- [E-F-LJ-1] ENRIQUEZ N., FRANCHI J., LE JAN Y. *Enroulements des géodésiques sous la mesure de Patterson-Sullivan.*
C.R.A.S. Paris, tome 326, Série 1, 723-726, 1998.
- [E-F-LJ-2] ENRIQUEZ N., FRANCHI J., LE JAN Y. *Stable windings on hyperbolic surfaces.* Prépublication, September 1999.
- [E-LJ] ENRIQUEZ N., LE JAN Y. *Statistic of the winding of geodesics on a Riemann surface with finite area and constant negative curvature.*
Rev. Mat. Iberoamericana, Vol. 13, 2, 377-401, 1997.
- [F] FRANCHI J. *Asymptotic singular homology of a complete hyperbolic 3-manifold of Finite Volume.* Proc. London Math. Soc. (3) n° 79, 451-480, 1999.
- [G] GUIVARC'H Y. *Sur la représentation intégrale des fonctions harmoniques et des fonctions propres positives dans un espace riemannien symétrique.*
Bull. Sci. Math., 2^e série, n° 108, 373-392, 1984.
- [L] LEDRAPPIER F. *Harmonic 1-forms on the stable foliation.*
Bol. Soc. Bras. Math. 25, n° 2, 121-138, 1994.
- [LJ1] LE JAN Y. *The central limit theorem for the geodesic flow on non compact manifolds of constant negative curvature.* Duke Math. J. 74 n° 1, 159-175, 1994.
- [LJ2] LE JAN Y. *Free energy for Brownian and geodesic homology.*
Probab. Theory Rel. Fields 102, 57-61, 1995.
- [L-MG-T] LYONS T.J., MAC GIBBON K.B., TAYLOR J.C. *Projection theorems for hitting probabilities and a theorem of Littlewood.*
Journal of Functional Analysis n° 59, 470-489, 1984.
- [M] MANDOUVALOS N. *Scattering operator, Eisenstein series, inner product formula and Maass-Selberg relations for Kleinian groups.*
Memoirs of the A.M.S. vol 78, n° 400, 1989.
- [Ni] NICHOLLS P.J. *The ergodic theory of discrete groups.*
London Math. Society, Lecture Note Series n° 143, Cambridge University Press, 1989.
- [P] PATTERSON S.J. *Lectures on measures on limit sets of Kleinian groups.*
Analytical and geometrical aspects of hyperbolic space, D. Epstein editor, 281-323,
London Math. Society, Lecture Note Series n° 111, Cambridge University Press, 1987.
- [P-S] PHILLIPS R.S., SARNAK P. *The Laplacean for domains in hyperbolic space and limit sets of Kleinian groups.* Acta Math. 155, 173-241, 1985.
- [S-V] STRATMAN B., VELANI S.L. *The Patterson measure for geometrically finite groups with parabolic elements, new and old.*
Proc. London Math. Soc. (3) n° 71, 197-220, 1995.

- [Su1] SULLIVAN D. *The density at infinity of a discrete group of hyperbolic motions.*
Publ. Math. I.H.E.S. n° 50, 171-209, 1979.
- [Su2] SULLIVAN D. *Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups.* Acta Math. 153, 259-277, 1984.

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