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On the martingale problem for super–Brownian motion

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Abstract. The law of super-Brownian motion can be characterized as the solution to a certain martingale problem. We give a new proof of this fact that uses only basic stochastic calculus and some simple facts about weak convergence.

1. Introduction.

The law of super-Brownian motion may be characterized in several ways, one of which is as the solution to a martingale problem. To state this result, we use the following notation. If μ is a measure, we will often write $\mu(f)$ for $\int f\,d\mu$. Let $\mathcal{M}$ be the set of finite Borel measures on $\mathbb{R}^d$ with the topology of weak convergence. The collection of continuous functions from $[0,\infty)$ to $\mathcal{M}$ is denoted $C([0,\infty),\mathcal{M})$. Let C_b^∞ denote the collection of infinitely differentiable functions on $\mathbb{R}^d$ with all derivatives bounded and let C_b^k denote the k times continuously differentiable functions whose first k derivatives are all bounded.

We say that a probability measure $\mathbb{P}$ on $C([0,\infty),\mathcal{M})$ is a solution to the martingale problem for super-Brownian motion started at a finite measure μ if

(a) $\mathbb{P}(X_0=\mu)=1$;

(b) if $f\in C_b^2$ then

$$X_t(f)-\int_0^t X_r(\Delta f/2)\,dr$$

is a continuous martingale with quadratic variation

$$\int_0^t X_r(f^2)\,dr.$$

The following theorem is well-known. See Dawson (1993) for a proof that uses log-Laplace functionals and an associated nonlinear partial differential equation.

Theorem 1.1. *Let $\mu\in\mathcal{M}$. There exists one and only one solution to the martingale problem for super-Brownian motion started at μ.*

The goal of this paper is to give an elementary proof of the uniqueness of the martingale problem for super-Brownian motion. We use only basic stochastic calculus and some simple facts about weak convergence. The proof illustrates a

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basic fact: the existence of a solution to a martingale problem which depends nicely on the initial data in fact gives uniqueness of solutions to the martingale problem. We hope that suitable modifications of this idea will be helpful in establishing new weak uniqueness results for certain interactive measure-valued diffusions.

We discuss existence briefly in Section 2 and prove uniqueness in Section 3.

2. Existence.

Super-Brownian motion may be constructed as the limit of branching diffusions. One way to do this is to let X_t^n be the process constructed as follows. Let the initial configuration of particles be given by a Poisson point process with mean measure $n\mu(\cdot)$ and let X_0^n assign mass $1/n$ to the site of each particle. They move as independent Brownian motions for time $1/n$, at which time each particle either splits into two or dies, each with probability $1/2$ and independently of the other particles. The particles that are now alive move as independent Brownian motions for time $1/n$, at which time each particle either splits into two or dies with equal probability, and so on. X_t^n is the measure that assigns mass $1/n$ to each point at which there is a particle alive at time t. We choose the right continuous version of X_t^n.

We then have the following theorem.

Theorem 2.1.

(a) *For each n the process $X_t^n(1)$ is a martingale in t.*

(b) *Any subsequence of $\{X_t^n\}$ has a further subsequence which converges weakly in the Skorokhod space $D([0,\infty),\mathcal{M})$ to a process taking values in the space $C([0,\infty),\mathcal{M})$.*

(c) *If $\mathbb{P}$ is any subsequential limit point of the laws of X_t^n, then $\mathbb{P}$ is a solution to the martingale problem for super-Brownian motion started at μ.*

For the proof of Theorem 2.1, see Perkins (2000). Of course Theorem 1.1 now shows that in fact $\{X^n\}$ converges weakly to super-Brownian motion but our goal is to give an elementary proof of the uniqueness part of Theorem 1.1 and and the above construction of a solution will play an integral role.

3. Uniqueness.

Let D be the set of points $(x_1,\dots,x_d)$ in $\mathbb{R}^d$ such that $x_1,\dots,x_d$ are rational. Let $E=\{2^{-m} : m \text{ integer }\}$. Define

$$\mathcal{M}_c=\Big\{\sum_{i=1}^{\infty} r_i\delta_{x_i}\in\mathcal{M} : r_i\in E, x_i\in D\Big\},$$

where δ_x is point mass at x. Nothing precludes several of the x_i being equal. Then $\mathcal{M}_c$ is a countable dense subset of $\mathcal{M}$.

For each measure μ we can construct a solution to the martingale problem started at μ by means of tightness as in Section 2. By using tightness and a diagonalization procedure, we can find a subsequence $\{n'\}$ of $\{2^k\}$ such that the laws of X_t^n along this subsequence converge for every starting measure μ in $\mathcal{M}_c$. Let us call the limit law $\mathbb{P}^\mu$; note that so far this is only defined for $\mu \in \mathcal{M}_c$. Let $\mathbb{P}^\mu_n$ denote the law of X^n_t when μ is our limiting initial measure.

Now let f be a nonnegative function in C_b^∞, and for $x \in D$, let

$$g_s(x) = \mathbb{E}^{\delta_x} e^{(-X_s(f))}, \qquad h_s(x) = \log g_s(x) \tag{3.1}$$

$$g_s^n(x) = \mathbb{E}^{\delta_x} e^{(-X_s^n(f))}, \qquad h_s^n(x) = \log g_s^n(x).$$

We next use translation invariance to show the log-Laplace functional h_s is smooth.

Lemma 3.1. *There exists c_1 such that for all $x \in D$ and all natural numbers n,*
(a) $1 \geq g_s(x), g_s^n(x) \geq c_1$;
(b) $0 \geq h_s(x), h_s^n(x) \geq \log c_1$; and
(c) $g_s(x)$ is uniformly continuous on D.

Proof. By Theorem 2.1(a), $X_t^n(1)$ is a martingale for each n, so

$$\mathbb{E}\, X_t^n(1) = \mathbb{E}\, X_0^n(1) = \mu(1),$$

and by Fatou's lemma, $\mathbb{E}\, X_t(1) \leq \mu(1)$. So for $x \in D$,

$$\mathbb{P}^{\delta_x}(X_t(f) \geq 2\|f\|_\infty) \leq \mathbb{P}^{\delta_x}(X_t(1) \geq 2) \leq 1/2.$$

With probability at least 1/2 we have that $X_t(f) \leq 2\|f\|_\infty$, hence $\exp(-X_t(f)) \geq \exp(-2\|f\|_\infty)$. Therefore $g_s(x) \geq (1/2)\exp(-2\|f\|_\infty)$, and as the same argument holds for g_n, this gives (a). (b) follows immediately.

If we set $f_x(y) = f(y+x)$, clearly the $\mathbb{P}^{\delta_x}$ law of $X_s(f)$ is the same as the $\mathbb{P}^{\delta_0}$ law of $X_s(f_x)$ because this is the case for the approximating X^n's. So $g_s(x) = \mathbb{E}^{\delta_0} e^{-X_s(f_x)}$. Since $X_s(f_x) = \int f_x(y)\, X_s(dy) = \int f(y+x)\, X_s(dy)$ is continuously differentiable in x, then so is $g_s(x)$ and also $h_s(x)$. The latter uses the integrability of $X_s(1)$ to allow differentiation under the integral sign. □

Remark 3.2. We may thus extend the definition of g_s and h_s to all of $\mathbb{R}^d$ as functions in C_b^1. From the proof of part (c), we also see that for each s, $\{h_s^n : n \in \mathbb{N}\}$ are equicontinuous–in fact uniformly Lipschitz continuous. Note that if $\mathbb{P}$ is any solution to the martingale problem starting at μ, then an elementary Fatou argument shows

$$\mathbb{E}\, X_t(1)^2 \leq \mu(1)^2 + t\mu(1). \tag{3.2}$$

This allows us to differentiate twice in the proof of part (c) to see that g_s and h_s are in C_b^2.

Next we use the multiplicative property of super-Brownian motion.

Proposition 3.3. *(a)* $\mu \mapsto \mathbb{E}^{\mu} e^{-X_s(f)}$ *is a uniformly continuous function on the set* $\mathcal{M}_c(N) \equiv \{\mu \in \mathcal{M}_c : \mu(1) \leq N\}$ *and* $\mu \mapsto \mathbb{E}^{\mu} e^{-X_s^n(f)}$ *is equicontinuous in* n *on* $\mathcal{M}_c(N)$.

(b)
$$\mathbb{E}^{\mu} e^{-X_s(f)} = e^{\mu(h_s)}, \qquad \mu \in \mathcal{M}_c.$$

Proof. If $\mu, \nu \in \mathcal{M}$, then the law of X^n under $\mathbb{P}^{\mu+\nu}$ is the same as the law of the sum of two independent copies of X^n, one governed by $\mathbb{P}^{\mu}$ and the other governed by $\mathbb{P}^{\nu}$; this follows by the way the X^n were constructed and the fact that the particles move independently of each other. So

$$\mathbb{E}^{\mu+\nu} e^{-X_s^n(f)} = \mathbb{E}^{\mu} e^{-X_s^n(f)} \mathbb{E}^{\nu} e^{-X_s^n(f)}. \tag{3.3}$$

In particular, if $r_i = 2^{-m}$,

$$\mathbb{E}^{\delta_{x_i}} e^{-X_s^n(f)} = \left[\mathbb{E}^{r_i\delta_{x_i}} e^{-X_s^n(f)}\right]^{2^m},$$

or

$$\mathbb{E}^{r_i\delta_{x_i}} e^{-X_s^n(f)} = \exp(2^{-m} \log g_s^n(x_i)) = \exp\Big(\int h_s^n(x)\, r_i\delta_{x_i}(dx)\Big).$$

Therefore

$$\mathbb{E}^{\mu} e^{-X_s^n(f)} = \exp\Big(\int h_s^n(x)\, \mu(dx)\Big) = e^{\mu(h_s^n)}, \quad \mu \in \mathcal{M}_c. \tag{3.4}$$

Let $n \to \infty$ along $\{n'\}$ in (3.4) to prove (b) (use the bounds in Lemma 3.1). As $h_s \in C_b^1$ and $\{h_s^n\}$ is uniformly Lipschitz by Remark 3.2, (a) is then clear from (b) and (3.4). □

Let $\mathcal{M}(N) = \{\mu \in \mathcal{M} : \mu(1) \leq N\}$. What we have so far is sufficient to prove the Markov property for $(\mathbb{P}^{\mu}, X_t)$.

Proposition 3.4. *(a) For all* $\mu \in \mathcal{M}$, $\mathbb{P}^{\mu}_{n'}$ *converges weakly, say to* $\mathbb{P}^{\mu}$.
- *(b) If* G *is bounded and continuous on* $\mathcal{M}$, *then for each* $N > 0$ *the function* $\mu \mapsto \mathbb{E}^{\mu} G(X_t)$ *is uniformly continuous on* $\mathcal{M}(N)$.
- *(c) If* G *is bounded and continuous on* $\mathcal{M}$ *and* $s, t \geq 0$, *then*

$$\mathbb{E}^{\mu} \mathbb{E}^{X_t} G(X_s) = \mathbb{E}^{\mu} G(X_{s+t}).$$

Proof. Let G be a bounded and continuous function on $\mathcal{M}$. We first show that for each t and N, $\mu \mapsto \mathbb{E}^{\mu} G(X_t^n)$ is equicontinuous on $\mathcal{M}(N)$.

If $f \geq 0$ is in C_b^{∞}, then the equicontinuity of $\mu \mapsto \mathbb{E}^{\mu} e^{-X_t^n(f)}$ on $\mathcal{M}_c(N)$, established in Proposition 3.3, and the continuity on $\mathcal{M}(N)$ for each fixed n (which is trivial) show that $\mu \mapsto \mathbb{E}^{\mu} e^{-X_t^n(f)}$ is equicontinuous on $\mathcal{M}(N)$. Letting $f = \sum_{i=1}^m \beta_i f_i$, where each f_i is nonnegative and in C_b^{∞}, we deduce that the map

$\mu \mapsto \mathbb{E}^{\mu} e^{-\sum_i \beta_i X_t^n(f_i)}$ is equicontinuous on $\mathcal{M}(N)$. Linear combinations of expressions such as $\exp(-\sum_i \beta_i X_t^n(f_i))$ are dense in the set of continuous functions of X_t^n, hence $\mathbb{E}^{\mu} G(X_t^n)$ is equicontinuous over $\mathcal{M}(N)$.

Let $s < t$ both be multiples of $1/2^m$ for some m. Define $P_t^n G(\mu)$ to be $\mathbb{E}^{\mu} G(X_t^n)$. By independence, the X_t^n have the Markov property at times t that are multiples of $1/n$, so

$$\mathbb{E}^{\mu} G_1(X_s^n) G_2(X_t^n) = \mathbb{E}^{\mu}(G_1(P_{t-s}^n G_2))(X_s^n) \tag{3.5}$$

if n is a multiple of 2^m. By what we just proved, $P_{t-s}^n G_2$ is equicontinuous, and it follows that $\mu \mapsto \mathbb{E}^{\mu} G_1(X_s^n) G_2(X_t^n)$ is equicontinuous over $\mathcal{M}(N)$ if n is a multiple of 2^m. Repeating, if $s_1 < \cdots < s_\ell$ are all multiples of 2^{-m} for some m and $G_1, \ldots, G_\ell$ are bounded and continuous functions, then $\mu \mapsto \mathbb{E}^{\mu} G_1(X_{s_1}^n) \cdots G_\ell(X_{s_\ell}^n)$ is equicontinuous over $\mathcal{M}(N)$.

We now prove (a), (b), and (c). In view of Theorem 2.1(b), the processes $X_t^{n'}$ are tight, and it suffices to show that if $G_1, \ldots, G_\ell$ are bounded and continuous and $s_1 < \cdots < s_\ell$ are dyadic rationals, then $\mathbb{E}^{\mu} G_1(X_{s_1}^{n'}) \cdots G_\ell(X_{s_\ell}^{n'})$ converges. We have convergence for $\mu \in \mathcal{M}_c$, and the convergence for arbitrary $\mu \in \mathcal{M}$ follows by the equicontinuity. This proves (a). (b) follows easily from (a) and the equicontinuity. Finally, the equicontinuity, the convergence for each μ, and (3.5) imply (c). □

Now define

$$A(h_s, \mu) = \mu(\Delta h_s/2 + h_s^2/2),$$

$$B(h, \mu) = \int_0^\infty e^{-\lambda s} e^{\mu(h_s)} [\lambda - A(h_s, \mu)]\, ds.$$

$e^{\mu(f)} A(f, \mu)$ is essentially the infinitesimal generator of $\mu \mapsto e^{\mu(f)}$, $f \in C_b^2$, $f \le 0$, and B is related to a resolvent. The key step is the following.

Proposition 3.5. *Let $\mu \in \mathcal{M}$ and let $\mathbb{P}$ be any solution to the martingale problem started at μ. Then*

$$\mathbb{E} \int_0^\infty e^{-\lambda r} B(h, X_r)\, dr = \int_0^\infty e^{-\lambda s} e^{\mu(h_s)}\, ds.$$

Proof. By Itô's formula and the fact that $h_s \in C_b^2$ (Remark 3.2),

$$\begin{aligned} e^{X_t(h_s)} &= e^{X_0(h_s)} + \int_0^t e^{X_r(h_s)}\, d(X_r(h_s)) + \frac{1}{2} \int_0^t e^{X_r(h_s)} d\langle X(h_s), X(h_s)\rangle_r \\ &= e^{X_0(h_s)} + \text{ martingale } + \int_0^t e^{X_r(h_s)} X_r(\Delta h_s/2)\, dr \\ &\quad + \frac{1}{2} \int_0^t e^{X_r(h_s)} X_r(h_s^2)\, dr \\ &= e^{X_0(h_s)} + \text{ martingale } + \int_0^t e^{X_r(h_s)} A(h_s, X_r)\, dr. \end{aligned}$$

We now take expectations with respect to $\mathbb{P}$. Since $h_s \le 0$, then $e^{X_r(h_s)} \le 1$. The quadratic variation of the martingale term is $\int_0^t e^{2X_r(h_s)} X_r(h_s)^2\,dr$. Since $|h_s|$ and $|\Delta h_s|$ are bounded by Remark 3.2, in view of (3.2) we can take expectations. We thus have

$$\mathbb{E}\, e^{X_t(h_s)} = e^{\mu(h_s)} + \mathbb{E}\int_0^t e^{X_r(h_s)} A(h_s, X_r)\,dr.$$

We now multiply by $e^{-\lambda s}$ and integrate over s from 0 to ∞ to obtain

$$\mathbb{E}\int_0^\infty e^{-\lambda s} e^{X_t(h_s)}\,ds = \int_0^\infty e^{-\lambda s} e^{\mu(h_s)}\,ds + \mathbb{E}\int_0^\infty \int_0^t e^{-\lambda s} e^{X_r(h_s)} A(h_s, X_r)\,dr\,ds.$$

Finally, we multiply both sides by $e^{-\lambda t}$ and integrate over t from 0 to ∞. Hence

$$\begin{aligned}
\mathbb{E}&\int_0^\infty e^{-\lambda t}\Big\{\int_0^\infty e^{-\lambda s} e^{X_t(h_s)}\,ds\Big\}\,dt \\
&= \frac{1}{\lambda}\int_0^\infty e^{-\lambda s} e^{\mu(h_s)}\,ds + \mathbb{E}\int_0^\infty \int_0^t e^{-\lambda t}\int_0^\infty e^{-\lambda s} e^{X_r(h_s)} A(h_s, X_r)\,ds\,dr\,dt \\
&= \frac{1}{\lambda}\int_0^\infty e^{-\lambda s} e^{\mu(h_s)}\,ds \\
&\qquad + \mathbb{E}\int_0^\infty \Big(\int_r^\infty e^{-\lambda t}\,dt\Big)\int_0^\infty e^{-\lambda s} e^{X_r(h_s)} A(h_s, X_r)\,ds\,dr.
\end{aligned}$$

Therefore

$$\int_0^\infty e^{-\lambda s} e^{\mu(h_s)}\,ds = \mathbb{E}\int_0^\infty e^{-\lambda r}\Big\{\int_0^\infty e^{-\lambda s} e^{X_r(h_s)}[\lambda - A(h_s, X_r)]\,ds\Big\}\,dr,$$

which is the desired result. □

The other important proposition is

Proposition 3.6. *For all $\mu \in \mathcal{M}$*

$$B(h,\mu) = e^{-\mu(f)}.$$

Proof. The measure $\mathbb{P}^\mu$ is a solution to the martingale problem started at μ, so by Proposition 3.5,

$$\mathbb{E}^\mu \int_0^\infty e^{-\lambda r} B(h, X_r)\,dr = \int_0^\infty e^{-\lambda s} e^{\mu(h_s)}\,ds.$$

Applying this and Proposition 3.3 with μ replaced by the measure X_t, we have

$$\mathbb{E}^{X_t}\int_0^\infty e^{-\lambda r} B(h, X_r)\,dr = \int_0^\infty e^{-\lambda s} e^{X_t(h_s)}\,ds = \int_0^\infty \mathbb{E}^{X_t} e^{-\lambda s} e^{-X_s(f)}\,ds.$$

Taking expectation with respect to μ and using Proposition 3.4(c), we then obtain

$$\mathbb{E}^\mu \int_0^\infty e^{-\lambda s} e^{-X_{s+t}(f)}\,ds = \mathbb{E}^\mu \int_0^\infty e^{-\lambda r} B(h, X_{r+t})\,dr.$$

Multiplying by $e^{-\lambda t}$, we have

$$\mathbb{E}^{\mu} \int_t^{\infty} e^{-\lambda s} e^{-X_s(f)}\, ds = \mathbb{E}^{\mu} \int_t^{\infty} e^{-\lambda r} B(h, X_r)\, dr. \tag{3.6}$$

Differentiating (3.6) with respect to t and using the continuity of the integrand, we get

$$\mathbb{E}^{\mu} e^{-\lambda t} e^{-X_t(f)} = \mathbb{E}^{\mu} e^{-\lambda t} B(h, X_t)$$

for all t. Letting $t \to 0$ proves the proposition. □

Proof of Theorem 1.1, uniqueness. Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be any two solutions to the martingale problem started at μ, and denote the corresponding expectations by $\mathbb{E}_1$ and $\mathbb{E}_2$. By Propositions 3.5 and 3.6, for each $f \geq 0$ in C_b^{∞},

$$\mathbb{E}_1 \int_0^{\infty} e^{-\lambda r} e^{-X_r(f)}\, dr = \mathbb{E}_1 \int_0^{\infty} e^{-\lambda r} B(h, X_r)\, dr = \int_0^{\infty} e^{-\lambda r} e^{\mu(h_r)}\, dr$$

and similarly with $\mathbb{E}_1$ replaced by $\mathbb{E}_2$. Therefore

$$\mathbb{E}_1 \int_0^{\infty} e^{-\lambda r} e^{-X_r(f)}\, dr = \mathbb{E}_2 \int_0^{\infty} e^{-\lambda r} e^{-X_r(f)}\, dr.$$

By the uniqueness of the Laplace transform and the continuity of $X_r(f)$ in r,

$$\mathbb{E}_1 e^{-X_r(f)} = \mathbb{E}_2 e^{-X_r(f)}$$

for all r.

Let $f_1, \ldots, f_m$ be non-negative functions in C_b^{∞} and let $\beta_1, \ldots, \beta_m$ be positive reals. Letting $f = \sum_{i=1}^m \beta_i f_i$,

$$\mathbb{E}_1 e^{-\sum \beta_i X_r(f_i)} = \mathbb{E}_1 e^{-X_r(f)} = \mathbb{E}_2 e^{-X_r(f)} = \mathbb{E}_2 e^{-\sum \beta_i X_r(f_i)}.$$

By the uniqueness of the Laplace transform, the joint distribution of $(X_r(f_1), \ldots, X_r(f_m))$ is the same under $\mathbb{P}_1$ and $\mathbb{P}_2$. This implies that the distribution of X_r is the same under $\mathbb{P}_1$ and $\mathbb{P}_2$.

The space $\mathcal{M}$ is a separable metric space, hence regular conditional probabilities exist (see Stroock and Varadhan (1979), p. 34). With this comment, we can proceed just as in Stroock and Varadhan (1979), Section 6.2, and conclude that all the finite dimensional distributions of $\{X_t; t \geq 0\}$ are the same under $\mathbb{P}_1$ and $\mathbb{P}_2$. Since X is a continuous process, the law of the process X under $\mathbb{P}_1$ and $\mathbb{P}_2$ are the same. □

Remark 3.7. By Stroock and Varadhan (1979), uniqueness of the martingale problem implies that $(\mathbb{P}^{\mu}, X_t)$ is a strong Markov process.

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