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On equivalent martingale measures with bounded densities

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Abstract

We show that for a local martingale S the equivalent martingale measures with bounded densities are norm-dense in the set of equivalent martingale measures.

1 Main results

We show that for a local martingale S, the equivalent martingale measures with bounded densities are dense (for the total variation topology) in the set of all equivalent martingale measures. To our knowledge, despite several attempts, a complete proof of this natural and useful property announced in [8] is still not available. In contrast to the approach of [12] trying to use rather delicate duality results for semimartingales, our arguments are based on a direct approximation and rely upon the same technique as in [6] and [10], which goes back to the proof of the no-arbitrage criteria in [4]. They work without any changes for a slightly more general type of processes, namely, for σ-martingales and σ-supermartingales (see Section 5).

Let $S = (S_t)_{t\in\mathbf{R}_+}$ be a d-dimensional semimartingale defined on a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t), P)$. We assume that the initial σ-algebra $\mathcal{F}_0$ is trivial.

Recall that S is a σ-martingale (notation: $S \in \Sigma_m(P)$) if there is a predictable process G with values in $]0,1]$ such that the integral $G\cdot S$ is a local martingale (in this case one can do better and find $\tilde{G}$ for which $\tilde{G}\cdot S$ is a martingale). This concept, extending the notion of local martingale, was introduced in [1] and investigated in details in [5]. Its importance for mathematical finance was discovered in [3]. In our analysis we shall use a simple characterization of σ-martingales suggested in [10].

Let $\mathcal{Q}^\sigma := \{Q \sim P : \ S \in \Sigma_m(Q)\}$ and let $\mathcal{Q}^\sigma_b := \{Q \in \mathcal{Q}^\sigma : \ dQ/dP \in L^\infty\}$.

Theorem 1.1 *If $S \in \Sigma_m(P)$, then the set $\mathcal{Q}^\sigma_b$ is dense in $\mathcal{Q}^\sigma$ for the total variation topology.*

Note that the total variation distance on $\mathcal{Q}^\sigma$ coincides with the L^1-distance between densities.

If S is a local martingale with respect to P then S remains a local martingale with respect to all measures in $\mathcal{Q}^\sigma_b$ (see n.**2** of Section 2); hence we have

Corollary 1.2 *If $S \in \mathcal{M}_{loc}(P)$ then $\mathcal{Q}^\sigma_b$ is dense in $\mathcal{Q} := \{Q \sim P : \ S \in \mathcal{M}_{loc}(Q)\}$.*

If $P, Q \in \mathcal{Q}^\sigma$ then the measure $Q^\varepsilon := \varepsilon P + (1-\varepsilon)Q$, $\varepsilon \in]0,1[$, is also in $\mathcal{Q}^\sigma$ and

$$\varepsilon \le dQ^\varepsilon/dP \le \varepsilon + (1-\varepsilon)dQ/dP.$$

Since Q^ε converges to P as $\varepsilon \to 0$, we can replace $\mathcal{Q}^\sigma_b$ in the above statements by

$$\mathcal{Q}^\sigma_{bb} := \{Q \in \mathcal{Q}^\sigma : \ dQ/dP \in L^\infty, \ dP/dQ \in L^\infty\}.$$

In the general case, when the reference probability P is arbitrary, it is often asked, whether the equivalent martingale measures with densities satisfying a certain integrability condition (e.g. with bounded entropy), are dense in the set of all equivalent martingale measures. The following corollary of Theorem 1.1 provides an answer which is useful in various applications.

Let $\varphi :]0,\infty[\to \mathbf{R}_+$ be a measurable function and let

$$\mathcal{Q}_\varphi := \{Q \in \mathcal{Q}^\sigma : \ E\varphi(dQ/dP) < \infty\}.$$

Corollary 1.3 *Assume that φ is such that for every $c, x > 0$*

$$\varphi(cx) \le r_1(c)\varphi(x) + r_2(c)(x+1),$$

where r_i are increasing positive functions. If the set $\mathcal{Q}^\sigma_\varphi$ is not empty, then it is dense in $\mathcal{Q}^\sigma$.

Proof. Let $\tilde P \in \mathcal{Q}^\sigma_\varphi$. Take an arbitrary measure $Q \in \mathcal{Q}^\sigma$. By the above theorem there exists a sequence $Q^n \in \mathcal{Q}^\sigma$ converging to Q such that each density $dQ^n/d\tilde P$ is bounded by a constant c_n. We have:

$$E\varphi\left(\frac{dQ^n}{dP}\right) = E\varphi\left(\frac{dQ^n}{d\tilde P}\frac{d\tilde P}{dP}\right) \le r_1(c_n)E\varphi\left(\frac{d\tilde P}{dP}\right) + 2r_2(c_n) < \infty.$$

Hence $Q^n \in \mathcal{Q}^\sigma_\varphi$ and the result follows. □

Typically, one takes: $\varphi(x) = x^p$, $p > 0$, or $\varphi(x) = x(\ln x)^+$. In particular, if non-empty, the set $\mathcal{Q}^\sigma_{x\ln x}$ of σ-martingale measures with finite entropy is dense in $\mathcal{Q}^\sigma$.

Corollary 1.4 *Assume that $\mathcal{Q}^\sigma_\varphi \ne \emptyset$ where the function φ satisfies the hypothesis of Corollary 1.3. Let ξ be a random variable bounded from below. Then*

$$\sup_{Q \in \mathcal{Q}^\sigma} E^Q\xi = \sup_{Q \in \mathcal{Q}^\sigma_\varphi} E^Q\xi. \tag{1}$$

Proof. It is sufficient to check that for any $Q \in \mathcal{Q}^\sigma$ we have the inequality

$$E^Q\xi \le \sup_{Q \in \mathcal{Q}^\sigma_\varphi} E^Q\xi. \tag{2}$$

In virtue of Corollary 1.3 there is a sequence $R_n \in \mathcal{Q}^\sigma_\varphi$ converging to Q. Hence for every $m \in \mathbf{N}$

$$E^Q(\xi \wedge m) = \lim_{n\to\infty} E^{R_n}(\xi \wedge m) \le \sup_{Q \in \mathcal{Q}^\sigma_\varphi} E^Q\xi.$$

The inequality (2) follows by monotone convergence. □

For $\varphi(x) = x(\ln x)^+$ the assertions of Corollaries 1.3 and 1.4 coincide with those of Lemma 7 and Corollary 12 of [2], which were proved under the assumption of continuity of all martingales. The present extension allows us to remove this restrictive hypothesis also in Proposition 11 of [2] on risk-averse asymptotics in a problem of exponential utility maximization.

2 Preliminaries from stochastic calculus

1. Before the proof we recall notations and basic facts about the canonical decomposition and the Girsanov theorem for semimartingales (see [9] for details).

Let (B, C, ν) be the triplet of predictable characteristics of an n-dimensional semimartingale X corresponding to the truncation function $h(x) := xI_{\{|x|\le 1\}}$. Let $\bar{h} := x - h$. Then X can be written in the so-called canonical form

$$X = X^c + h * (\mu - \nu) + \bar{h} * \mu + B,$$

which is nothing but a generalization of the Lévy representation for processes with independent increments. Recall that ν is the compensator of the jump measure μ of X. The process $\bar{h} * \mu$ represents the sum of "large" jumps. The remaining part of X is a special semimartingale which can be uniquely decomposed into a continuous local martingale X^c, a purely discontinuous local martingale $h * (\mu - \nu)$ (of compensated jumps), and a predictable process of bounded variation B. The matrix-valued process $C = \langle X^c \rangle$ is the quadratic variation of the continuous martingale components.

For each ω the measure $\nu(\omega, dt, dx)$ on the product space can be disintegrated, i.e. represented as

$$\nu(\omega, dt, dx) = dA_t(\omega)K_{\omega,t}(dx).$$

The predictable characteristics being defined up to P-null sets, there is enough freedom to do this in a measurable way. One can always work out a "good" version of the triplet, assuming without loss of generality that ν is of the above form where A is a predictable increasing càdlàg process while $K_{\omega,t}(dx)$ is a transition kernel from $(\Omega \times \mathbf{R}_+, \tilde{\mathcal{P}})$ into $(\mathbf{R}^n, \mathcal{B}^n)$ with $K(\{0\}) = 0$ and

$$\int (|x|^2 \wedge 1) K_{\omega,t}(dx) < \infty.$$

Moreover, A can be chosen to ensure the following properties (see [9], II.2.9):

$$B = b \cdot A, \quad C = c \cdot A,$$

where b and c are predictable;

if $\Delta A_t(\omega) > 0$ then $\Delta A_t(\omega) K_{\omega,t}(\mathbf{R}^n) \le 1$ and $b_t(\omega) = \int h(x) K_{\omega,t}(dx)$.

Let $m(d\omega, dt) := P(d\omega)dA_t(\omega)$. The notations $\tilde{\mathcal{P}} := \mathcal{P} \otimes \mathcal{B}^n$ and $a_t := \nu(\{t\}, \mathbf{R}^n)$ are standard. We write $K_{\omega,t}(Y)$ instead of $\int Y(x)K_{\omega,t}(dx)$ and omit often ω, t. Using this abbreviation we put $\theta := K(|x|^2 \wedge |x|)$.

2. A semimartingale X is a local martingale if and only if the following two conditions hold:

(a) $(|x|^2 \wedge |x|) * \nu_t < \infty$ for all t;

(b) $B + \bar{h} * \nu = 0$.

The corresponding characterization of σ-martingales is as follows:

$X \in \Sigma_m(P)$ (with $1/G := 1 + \theta$) $\Leftrightarrow$ $\theta < \infty$ and $b + K(\bar{h}) = 0$ m-a.e.

Since the process $(|x|^2 \wedge 1) * \nu$ is always finite, (a) holds if and only if the process $|\bar{h}| * \nu$ is finite (i.e. S is locally integrable). The condition (b) (which means that $-B$ is the compensator of large jumps of X) can be rewritten as

$$(b + K(\bar{h})) \cdot A = 0.$$

This makes clear the difference between a local martingale and a σ-martingale: the compensation property on the level of intensities holds for both but for the latter the integral $\bar{h} * \nu$ may not be defined. If $X \in \Sigma_m$ is locally integrable then $X \in \mathcal{M}_{loc}$.

3. Let $P^0 \sim P$ and let Z^0 be the density process of P^0 with respect to P. The general Girsanov theorem [9], III.3.24, in connection with [9], III.5.7, provides the existence of a predictable $\mathbf{R}^n$-valued process β^0 and a strictly positive $\tilde{\mathcal{P}}$-measurable function $Y^0 = Y^0(\omega, t, x)$ such that

$$H_\infty(\beta^0, Y^0) := \beta^{0*}c\beta^0 \cdot A_\infty + (1 - \sqrt{Y^0})^2 * \nu_\infty + \sum_{s \geq 0}(\sqrt{1 - a_s} - \sqrt{1 - \widehat{Y}^0_s})^2 < \infty,$$

$\{0 < a < 1\} = \{0 < \widehat{Y}^0 < 1\}$, $\{a = 1\} = \{\widehat{Y}^0 = 1\}$ where

$$\widehat{Y}^0 := K(Y^0)\Delta A,$$

and the triplet of predictable characteristics (B^0, C^0, ν^0) under P^0 has the form:

$$B^0 = B + c\beta^0 \cdot A + K(h(Y^0 - 1)) \cdot A,$$

$$C^0 = C, \quad \nu^0 = Y^0 \nu.$$

The function Y^0 can be calculated as a kind of conditional expectation:

$$Y^0 = M^P_\mu(Z^0/Z^0_- | \tilde{\mathcal{P}}),$$

where M^P_μ means the average with respect to $\mu(\omega, dt, dx)P(dw)$. The process β is any predictable solution of the (vector) equation

$$c\beta \cdot A = (1/Z^0) \cdot \langle Z^c, X^c \rangle.$$

In virtue of the above criteria, $X \in \Sigma(P^0)$ if and only if

$$K((|x|^2 \wedge |x|)Y^0) < \infty, \quad m\text{-a.s.} \tag{3}$$

$$c\beta^0 + K(x(Y^0 - 1)) = 0 \quad m\text{-a.s.} \tag{4}$$

4. Let P^1 be another measure equivalent to P. It is well-known that for any pair of probability measures there exists a predictable increasing process $h(P^0, P^1)$ starting from zero such that for any stopping time τ

$$||P^0_\tau - P^1_\tau|| \leq 4\sqrt{E^0 h_\tau(P^0, P^1)}.$$

In our case this so-called Hellinger process has the following structure, see [11].

Let the martingales Z^i be the density processes of P^i with respect to P. Then

$$\begin{aligned} h_t(P^0, P^1) &= \frac{1}{8}\langle M^{0c} - M^{1c}\rangle_t + \frac{1}{2}K((\sqrt{Y^1} - \sqrt{Y^0})^2) \cdot A_t \\ &+ \frac{1}{2}\sum_{s \leq t}(\sqrt{1 - \widehat{Y}^1_s} - \sqrt{1 - \widehat{Y}^0_s})^2. \end{aligned}$$

where $M^i := (1/Z^i_-) \cdot Z^i$ (hence $M^{ic} = (1/Z^i_-) \cdot Z^{ic}$).

3 Proof of Theorem 1.1

Let $P^0 \in \mathcal{Q}^\sigma$ and let Z^0 be the density process of the measure P^0 with respect to P. We shall work with the semimartingale $X = (S, Z^0)$ taking values in $\mathbf{R}^n$ where $n = d + 1$.

We shall denote by π the natural projection on the first d coordinates.

The semimartingale X can be represented as follows:

$$\begin{aligned} S &= S_0 + S^c + \pi(x) I_{\{|x|\le 1\}} * (\mu - \nu) + \pi(x) I_{\{|x|>1\}} * \mu + \pi(B), \\ Z^0 &= 1 + Z^{0c} + x^{d+1} * (\mu - \nu). \end{aligned}$$

Since $S \in \Sigma_m(P)$ and Z^0 is a martingale we have m-a.e. that $K(|x|^2 \wedge |x|) < \infty$ and

$$K(x I_{\{|x|>1\}}) = -b.$$

The idea of the proof is to approximate P^0 by a measure P^1 with bounded Y^1 close enough to Y^0 and preserving the "compensation" property (4).

Since $S \in \Sigma_m(P^0)$ we can add to the usual integrability conditions

$$\theta^0 := K((|x|^2 \wedge 1) Y^0) < \infty, \quad m\text{-a.e.},$$

$$K((1 - \sqrt{Y^0})^2) < \infty, \quad m\text{-a.e.},$$

the following one:

$$K((|\pi(x)|^2 \wedge |\pi(x)|) Y^0) < \infty, \quad m\text{-a.e.}$$

Remark 3.1 The integrability of functions $|x|^2 \wedge |x|$, $(|x|^2 \wedge |x|)Y^0$, $(1 - \sqrt{Y^0})^2$ with respect to K implies the integrability of $|x||Y^0 - 1|$. Moreover, if K is finite, then Y^0 is also integrable. We leave this easy exercise to the reader.

Let $\mathbf{Y}$ be the set of functions $Y > 0$, $Y \in C(\bar{\mathbf{R}}^n)$; $\mathbf{Y}$ with its Borel σ-algebra $\mathcal{Y}$ is a Lusin space.

Let $\delta = (\delta_t)$ be a strictly positive predictable process such that $\delta \cdot A_\infty < \varepsilon$.

For every (ω, t) we consider in $\mathbf{Y}$ the convex subsets

$$\begin{aligned} \Gamma^1_{\omega,t} &:= \{Y : K_{\omega,t}((\sqrt{Y} - \sqrt{Y^0(\omega,t)})^2) \le \delta_t(\omega)\}, \\ \Gamma^2_{\omega,t} &:= \{Y : K_{\omega,t}((Y-1)\pi(x)) = K_{\omega,t}((Y^0(\omega,t) - 1)\pi(x))\}, \\ \Gamma^3_{\omega,t} &:= \{Y : I_{\{a_t(\omega)>0\}} K_{\omega,t}(Y) = I_{\{a_t(\omega)>0\}} K_{\omega,t}(Y^0(\omega,t))\}. \end{aligned}$$

Put

$$\Gamma_{\omega,t} := \Gamma^1_{\omega,t} \cap \Gamma^2_{\omega,t} \cap \Gamma^3_{\omega,t}.$$

In virtue of Lemma 4.1, these subsets are non-empty m-a.e. and hence, by the measurable selection theorem, there is a predictable $\mathbf{Y}$-valued process $Y'(\omega, t, x)$ such that $Y'(\omega, t, .) \in \Gamma_{\omega,t}$ m-a.e. Being continuous in the variable x, the function

$$(\omega, t, x) \mapsto Y'(\omega, t, x)$$

is $\tilde{\mathcal{P}}$-measurable.

Take $\tau_N := \inf\{t: \ H_t \geq N\}$, where

$$H_t := \frac{1}{8}\langle M^{0c}\rangle_t + \frac{1}{2}K(1-\sqrt{Y^0})^2 \cdot A_t + \frac{1}{2}\sum_{s\leq t}(\sqrt{1-a_s} - \sqrt{1-\hat{Y}^0_s})^2.$$

Then $H_{\tau_N} \leq N+1$. Let

$$Y^r(\omega,t,x) := Y'(\omega,t,x)I_{\{||Y'(\omega,t,.)||\leq r\}} + I_{\{||Y'(\omega,t,.)||>r\}}$$

where $||.||$ is the uniform norm. Choose $r = r_N$ large enough to ensure the inequality

$$E^0 I_{\{||Y'||>r\}} \cdot H_{\tau_N} < \varepsilon.$$

Define the process Z (depending on N) as the solution of the linear equation

$$Z = 1 + ZI_{\{||Y'||\leq r\}} \cdot M^{0c} + Z_-\left(Y^r - 1 + \frac{\hat{Y}^r - 1}{1-a}1_{\{a<1\}}\right) * (\mu - \nu).$$

It is a strictly positive local martingale which is locally bounded. Let us consider the localizing sequence $\tau_{N,n} := \inf\{t: \ Z_t \geq n\}$. We put $\sigma_{N,n} := \tau_N \wedge \tau_{N,n}$ and define the probability measure $P^1 := Z_{\sigma_{N,n}}P$ equivalent to P. It is obvious that $S \in \Sigma_m(P^1)$.

Choose n and N large enough to have

$$||P^0 - P^0_{\sigma_{N,n}}|| = E|Z^0_\infty - Z^0_{\sigma_{N,n}}| \leq \varepsilon.$$

Since

$$||P^0 - P^1|| \leq ||P^0 - P^0_{\sigma_{N,n}}|| + ||P^0_{\sigma_{N,n}} - P^1|| \leq \varepsilon + 4\sqrt{E^0 h_{\sigma_{N,n}}(P^1,P^0)}$$

and

$$E^0 h_{\sigma_{N,n}}(P^1,P^0) \leq 2\varepsilon,$$

we conclude that P^0 can be approximated by measures from $\mathcal{Q}^\sigma$ with bounded densities. □

4 Key lemma

Let K be a measure on $\mathbf{R}^n$ such that $K(\{0\}) = 0$ and

$$K(|x|^2 \wedge |x|) < \infty. \tag{5}$$

We introduce the set U of strictly positive Borel functions on $\mathbf{R}^n$ such that:

$$K((|x|^2 \wedge 1)Y) < \infty, \tag{6}$$

$$K((|\pi(x)|^2 \wedge |\pi(x)|)Y) < \infty, \tag{7}$$

$$K((1-\sqrt{Y})^2) < \infty. \tag{8}$$

Let $U_0 := U \cap C(\bar{\mathbf{R}}^n)$.

Lemma 4.1 *Let $Y^0 \in U$ and $\delta > 0$. Then there exists $Y \in U_0$ with*

$$K((\sqrt{Y} - \sqrt{Y^0})^2) < \delta \tag{9}$$

and such that

$$K((Y-1)\pi(x)) = K((Y^0-1)\pi(x)). \tag{10}$$

Moreover, if K is finite, Y can be chosen to satisfy the equality $K(Y) = K(Y^0)$.

Proof. Let G be the set of $Y \in U$ satisfying (9) and let $G_0 := G \cap C(\bar{\mathbf{R}}^n)$. Let $\phi: \mathbf{R}^n \to \mathbf{R}^p$ be a Borel function such that

$$K(|1-Y||\phi|) < \infty \quad \forall Y \in U. \tag{11}$$

We show that there exists $Y \in G_0$ such that

$$K((Y-1)\phi) = K((Y^0-1)\phi). \tag{12}$$

This claim implies the assertion of the lemma because the integrability property (11) holds for $\phi = \pi$ and, in the case of finite K, for the mapping $\phi: \mathbf{R}^n \to \mathbf{R}^n$ with $\phi^i(x) = x^i$, $i \le n-1$, $\phi^n(x) = 1$, see Remark 3.1.

Notice that (12) is a system of linear equations and we can exclude from consideration any equation which is a linear combination of the others. So, we may assume without loss of generality that the components of the function $(Y^0-1)\phi$ are linearly independent as elements of the vector space $L^1(K)$.

Since G and G_0 are convex, their images under the affine mapping

$$\Phi: Y \mapsto K((Y-Y^0)\phi)$$

are convex sets in $\mathbf{R}^p$. Let V be the linear hull of $\Phi(G)$. Our claim can be reformulated as follows: $0 \in \Phi(G_0)$. In fact, we have even $0 \in \operatorname{ri}\Phi(G_0)$ (we use the standard notation for the relative interior of a convex set). To prove this assertion we check that:

(*a*) $0 \in \operatorname{ri}\Phi(G)$;

(*b*) $\Phi(G_0)$ is a dense subset of $\Phi(G)$ and hence $\operatorname{ri}\Phi(G_0) = \operatorname{ri}\Phi(G)$.

(The last equality is due to the following simple fact: if A and B are convex sets in $\mathbf{R}^p$ and B is dense in A then their relative interiors coincide, see, e.g., Proposition III.2.1.8 in [7].)

Assume that $0 \notin \operatorname{ri}\Phi(G)$. Then there exists a non-zero linear functional l on V such that

$$\langle l, K((Y-Y^0))\phi)\rangle \ge 0 \quad \forall Y \in G.$$

Take a strictly positive (K-a.e.) Borel function $f < Y^0$ such that $K(f) < \delta$ (the requirements are met by the function $f = cgY^0$ where

$$g = \frac{|x|^2 \wedge |1|}{1 + |x|^2 \wedge |1|}$$

and $c > 0$ is sufficiently small). For every Borel set A the functions $Y := Y^0 \pm fI_A$ belong to G because

$$(\sqrt{Y} - \sqrt{Y^0})^2 \le |Y - Y^0| = fI_A.$$

It follows that $\pm K(\langle l,\phi\rangle f I_A)\geq 0$. Hence $\langle l,\phi\rangle=0$ K-a.e. This implies that $\langle l,(Y^0-1)\phi\rangle=0$ K-a.e. in contradiction with the assumed linear independence.

It remains to check (b). First of all, we observe that (11) implies that

$$K(|Y^1-Y^2||\phi|)<\infty \qquad Y^1,Y^2\in U.$$

Using this property for Y^0 and Y^0+cg we infer that for every $\varepsilon>0$

$$K(I_{\{|x|\geq\varepsilon\}}|\phi|)<\infty. \tag{13}$$

Recall that for any finite measure on $\bar{\mathbf{R}}^n$ an integrable function y can be approximated in L^1 by a sequence $y^k\in C(\bar{\mathbf{R}}^n)$; if $y\geq 0$ then y^k can be chosen strictly positive (replace y^k by $y^k\vee 0+1/n$).

Fix $Y\in G$. For any $r>0$ the measure $K_r:=(1+|\phi|)I_{\{|x|\geq r\}}K$ is finite and $Y\in L^1(K_r)$. In virtue of the above remarks there is $y_r\in C(\bar{\mathbf{R}}^n)$ such that $y^r>0$ and

$$||Y-y_r||_{L^1(K_r)}\leq r.$$

Since $Y\in L^1(K_{r/2})$, there exists $r_0=r_0(r)$ such that $r/2<r_0<r$ and

$$K(I_{\{r_0<|x|<r\}}(1+|\phi|)Y)\leq r.$$

Let $\tilde{y}_r$ be a continuous function on $\{x:\ r_0\leq|x|\leq r\}$, equal to 1 on $\{x:\ |x|=r_0\}$, coinciding with y_r on $\{x:\ |x|=r\}$, and such that

$$K(I_{\{r_0<|x|<r\}}(1+|\phi|)\tilde{y}_r)\leq r.$$

Define

$$Y_r:=I_{\{|x|\leq r_0\}}+\tilde{y}_r I_{\{r_0<|x|<r\}}+I_{\{|x|>r\}}y_r.$$

Since

$$\begin{aligned}K((\sqrt{Y_r}-\sqrt{Y})^2) &\leq K(I_{\{|x|\leq r_0\}}(1-\sqrt{Y})^2)+K(I_{\{r_0<|x|<r\}}(\tilde{y}_r+Y))\\ &\quad +K(I_{\{|x|\geq r\}}(\sqrt{y_r}-\sqrt{Y})^2)\to 0,\quad r\to 0,\end{aligned}$$

we have

$$\lim_{r\to 0}K((\sqrt{Y_r}-\sqrt{Y^0})^2)<\delta.$$

Thus, for sufficiently small r the function Y_r belongs to G_0.

At last,

$$\begin{aligned}|\Phi(Y_r)-\Phi(Y)| &\leq K(I_{\{|x|\leq r_0\}}|1-Y||\phi|)+K(I_{\{r_0<|x|<r\}}(\tilde{y}_r+Y)|\phi|)\\ &\quad +K(I_{\{|x|\geq r\}}|y_r-Y||\phi|)\to 0,\quad r\to 0,\end{aligned}$$

and the result follows. □

5 Final comments

We say that a semimartingale S is a σ-supermartingale (notation: $S \in \Sigma_{sup}(P)$) if there is a predictable process G with values in $]0,1]$ such that the process $G \cdot S$ is a supermartingale. Obviously, $S \in \Sigma_{sup}(P)$ if and only if $K(|x|^2 \wedge |x|) < \infty$ and $b + K(\bar{h}) \leq 0$ m-a.e.

Let $\tilde{\mathcal{Q}}^\sigma := \{Q \sim P : S \in \Sigma_{sup}(Q)\}$ and let $\tilde{\mathcal{Q}}^\sigma_b := \{Q \in \mathcal{Q}^\sigma : dQ/dP \in L^\infty\}$.

Theorem 5.1 *If $S \in \Sigma_{sup}(P)$, then the set $\tilde{\mathcal{Q}}^\sigma_b$ is dense in $\tilde{\mathcal{Q}}^\sigma$.*

Our proof of Theorem 1.1 remains literally the same for this result as well. The formulations of the corollaries can be also extended in a similar way. One can observe that Theorem 1.1 follows from Theorem 5.1 because $S \in \Sigma_m(P)$ if and only if $(S, -S) \in \Sigma_{sup}(P)$.

Let us associate with S the set $\mathcal{X}$ of processes bounded from below which can be represented as stochastic integrals $H \cdot S$. We denote by $\mathcal{Q}^{sup}$ the set of probability measures $Q \sim P$ such that every $V \in \mathcal{X}$ is a Q-supermartingale.

Theorem 1.1 implies the following assertion announced in [12].

Theorem 5.2 *Assume that $S \in \Sigma_m(P)$. Then $\mathcal{Q}^\sigma_b$ is a dense subset of $\mathcal{Q}^{sup}$.*

Indeed, if $Q \in \mathcal{Q}^{sup}$ then $E_Q V_\infty \leq 0$ for every $V \in \mathcal{X}$. Thus Q belongs to the set of equivalent separating measures containing $\mathcal{Q}^\sigma$ as a dense subset, see [3] and [10].

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