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A discrete approach to the chaotic representation property

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A DISCRETE APPROACH TO THE CHAOTIC REPRESENTATION PROPERTY

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Abstract. — In continuous time, let $(X_t)_{t\geqslant 0}$ be a normal martingale (i.e. a process such that both X_t and $X_t^2 - t$ are martingales). One says that X has the *chaotic representation property* if $L^2\big(\sigma(X)\big)$ is the (direct) Hilbert sum $\bigoplus_{p\in\mathbb{N}} \chi_p(X)$, where $\chi_p(X)$ is the space of all p-fold iterated stochastic integrals

$$\int_{0<t_1<\ldots<t_p} f(t_1,\ldots,t_p)\, dX_{t_1}\ldots dX_{t_p}$$

with f square-integrable ($\chi_p(X)$ is called the p^{th} chaotic space; by convention, $\chi_0(X)$ is the one-dimensional space of deterministic random variables). An open problem is to characterize those processes X.

Instead of working in continuous time, we shall address an analogue of this problem where the time-axis is the set $\mathbb{Z}$ of signed integers; in this setting, we shall give a sufficient (but probably far from necessary) condition for the chaotic representation property to hold.

Notation and preliminaries

We shall use the set $\mathbb{Z}$ of all signed integers as our time-axis; the set of all *finite* subsets of $\mathbb{Z}$ will be denoted by $\mathcal{P}$. For m and n in $\mathbb{Z}$, we shall have to do with the following "intervals":

$$\begin{aligned}]m,n] &= \{k\in\mathbb{Z} \,:\, m<k\leqslant n\}\,; \\ ]n,\infty[&= \{k\in\mathbb{Z} \,:\, n<k\}\,; \\ ]-\infty,n] &= \{k\in\mathbb{Z} \,:\, k\leqslant n\}\,. \\ ]-\infty,n[&= \{k\in\mathbb{Z} \,:\, k<n\}\,. \end{aligned}$$

Given a filtration $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}}$, a process $X = (X_n)_{n\in\mathbb{Z}}$ is *adapted* (respectively *predictable*) if for each n the random variable X_n is $\mathcal{F}_n$-measurable (respectively $\mathcal{F}_{n-1}$-measurable); a *stopping time* is an $\mathcal{F}_\infty$-measurable random variable T with values in $\mathbb{Z}\cup\{+\infty\}$, such that for each $n\in\mathbb{Z}$ the event $\{T=n\}$ (or, for that matter, $\{T\leqslant n\}$) belongs to $\mathcal{F}_n$; notice that the value $-\infty$ is not allowed to stopping times.

An empty sum $\sum_{i\in\varnothing} x_i$ is always null, an empty product $\prod_{i\in\varnothing} x_i$ is always 1.

With $\mathbb{Z}$ as the time-axis, the analogue of a normal martingale is no longer a martingale, but a sort of normalized martingale increment:

DEFINITION. — *On a probability space $(\Omega,\mathcal{A},\mathbb{P})$, let $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}}$ be a filtration. A process $X = (X_n)_{n\in\mathbb{Z}}$ is a* novation *(more precisely: an $\mathcal{F}$-*novation*) if, for each time $n\in\mathbb{Z}$, X_n belongs to $L^2(\mathcal{F}_n)$ and verifies*

(N1) $$\mathbb{E}[X_n|\mathcal{F}_{n-1}] = 0\,;$$

(N2) $$\mathbb{E}[X_n^2|\mathcal{F}_{n-1}] = 1\,.$$

The name 'novation' aims at suggesting that X plays the rôle of an innovation, but the prefix 'in' has been dropped to stress that no independence is required.[1] Condition (N1) says that X_n should be understood as a martingale increment; and (N2) is a normalization hypothesis. The simplest example of a novation is a sequence of independent random variables with mean 0 and variance 1.

PROPOSITION 1 AND DEFINITIONS. — *Let X be a novation on $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F})$; for each $A \in \mathcal{P}$, denote by X_A the product $\prod_{n\in A} X_n$. The set of random variables $\{X_A,\ A \in \mathcal{P}\}$ is orthonormal in $\mathrm{L}^2(\Omega, \mathcal{A}, \mathbb{P})$.*

So this set is an orthonormal basis of some closed subset of $\mathrm{L}^2(\Omega, \mathcal{A}, \mathbb{P})$, called the chaotic space *associated to X, and denoted by $\chi(X)$.*

If $\{X_A,\ A \in \mathcal{P}\}$ is total in $\mathrm{L}^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$, or equivalently if the chaotic space is equal to $\mathrm{L}^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$, one says that X has the chaotic representation property.

The simplest example of a novation with the chaotic representation property is the fair coin-tossing: the X_n are independent and uniformly distributed on $\{-1, 1\}$, and $\mathcal{F}$ is the filtration generated by X.

PROOF OF PROPOSITION 1. — Fix A and B in $\mathcal{P}$. For $n \in \mathbb{Z}$, the formula

$$\mathbb{E}[X_A X_B | \mathcal{F}_n] = \begin{cases} 0 & \text{if } A\cap]n,\infty[\ \neq B\cap]n,\infty[; \\ X_{A\cap]-\infty,n]} X_{B\cap]-\infty,n]} & \text{if } A\cap]n,\infty[\ = B\cap]n,\infty[. \end{cases}$$

is true if n is large enough for $]-\infty, n]$ to contain A and B; and if it holds for some n, it holds for $n-1$ too because X is a novation. So it holds for every $n \in \mathbb{Z}$, and in particular when n is small enough for A and B to be included in $]n, \infty[$. Thus, for such an n,

$$\mathbb{E}[X_A X_B | \mathcal{F}_n] = \begin{cases} 0 & \text{if } A \neq B \\ 1 & \text{if } A = B, \end{cases}$$

and the proposition is proved by taking expectations on both sides. ∎

Here are five necessary conditions for a novation X to have the chaotic representation property.

PROPOSITION 2 AND DEFINITION. — *Let X be a novation defined on some filtered probability space $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F})$. If X has the chaotic representation property, then*

(i) *for each $n \in \mathbb{Z}$, the set $\{X_A,\ A \in \mathcal{P},\ A \subset\ ]-\infty, n]\}$ is an orthonormal basis of $\mathrm{L}^2(\Omega, \mathcal{F}_n, \mathbb{P})$;*

(ii) *the filtration $\mathcal{F}$ is generated by X;*

(iii) *the σ-field $\mathcal{F}_{-\infty}$ $(= \bigcap_n \mathcal{F}_n)$ is degenerate;*

(iv) *for all $n \in \mathbb{Z}$ and $U \in \mathrm{L}^2(\mathcal{F}_n)$, there exist Q and R in $\mathrm{L}^2(\mathcal{F}_{n-1})$ such that $U = Q + RX_n$;*

(v) *for each $U \in \mathrm{L}^2(\mathcal{F}_\infty)$, there exists an $\mathcal{F}$-predictable process $H = (H_n)_{n\in\mathbb{Z}}$ such that*

$$\mathbb{E}\Big[\sum_{n\in\mathbb{Z}} H_n^2\Big] < \infty \qquad \textit{and} \qquad U = \mathbb{E}[U] + \sum_{n\in\mathbb{Z}} H_n X_n\,.$$

When (v) holds, one says that the novation X has the predictable representation property *(with respect to the filtration $\mathcal{F}$).*

1. « Ne crains doncques, Poëte futur, d'innover quelques termes. » (J. du Bellay, La deffence et illustration de la langue françoyse.)

The analogy between this definition and the predictable representation property in continuous time (see for instance [3]) is plain: X_n replaces dX_t and $\sum$ replaces $\int$.

PROOF. — (i) Fix $n \in \mathbb{Z}$. For each $A \in \mathcal{P}$, one has $X_A \perp \mathrm{L}^2(\mathcal{F}_n)$ if A meets $]n,\infty[$ and $X_A \in \mathrm{L}^2(\mathcal{F}_n)$ if A is included in $]-\infty,n]$. Thus $\{X_A,\ A\in\mathcal{P};\ A\subset\,]-\infty,n]\}$ is an orthonormal basis of $\mathrm{L}^2(\mathcal{F}_n)$ (and $\{X_A,\ A\in\mathcal{P};\ A \text{ meets }]n,\infty[\}$ is an orthonormal basis of its orthogonal supplement).

(ii) is a consequence of (i).

(iii) For $A \in \mathcal{P}$ and $A \neq \varnothing$, X_A is orthogonal to $\mathrm{L}^2(\mathcal{F}_n)$ for every $n < \sup A$, and a fortiori to $\mathrm{L}^2(\mathcal{F}_{-\infty})$. So $\mathrm{L}^2(\mathcal{F}_{-\infty})$ is included in the orthogonal supplement to $\{X_A,\ A\in\mathcal{P},\ A\neq\varnothing\}$; as this supplement consists of deterministic random variables, $\mathcal{F}_{-\infty}$ is degenerate.

(iv) We know from (i) that every random variable $U \in \mathrm{L}^2(\mathcal{F}_n)$ admits an L^2-expansion as

$$U = \sum_{\substack{A\in\mathcal{P}\\ A\subset]-\infty,n]}} u_A X_A$$

with $\sum u_A^2 < \infty$. Setting

$$Q = \sum_{\substack{A\in\mathcal{P}\\ A\subset]-\infty,n[}} u_A X_A \qquad\text{and}\qquad R = \sum_{\substack{A\in\mathcal{P}\\ A\subset]-\infty,n[}} u_{A\cup\{n\}} X_A\ ,$$

one has

$$RX_n = \sum_{\substack{A\in\mathcal{P}\\ \sup A=n}} u_A X_A$$

(approximate both sides by finite sums and take limits in L^2), whence (iv).

(v) By the chaotic representation property, each $U \in \mathrm{L}^2(\mathcal{F}_\infty)$ has an expansion

$$U = \sum_{A\in\mathcal{P}} u_A X_A = u_\varnothing + \sum_{n\in\mathbb{Z}} \sum_{\substack{A\in\mathcal{P}\\ A\subset]-\infty,n[}} u_{A\cup\{n\}} X_A X_n\ .$$

Now, the random variable

$$H_n = \sum_{\substack{A\in\mathcal{P}\\ A\subset]-\infty,n[}} u_{A\cup\{n\}} X_A = \sum_{\substack{A\in\mathcal{P}\\ \sup A=n}} u_A X_{A-\{n\}}$$

belongs to $\mathrm{L}^2(\mathcal{F}_{n-1})$ by (i), with squared norm $\mathbb{E}[H_n^2] = \sum_{\substack{A\in\mathcal{P}\\ \sup A=n}} u_A^2$; summing in n gives

$$\mathbb{E}\big[\sum_n H_n^2\big] = \sum_{\substack{A\in\mathcal{P}\\ A\neq\varnothing}} u_A^2 < \infty\ ,$$

and, as $u_\varnothing = \mathbb{E}[UX_\varnothing] = \mathbb{E}[U]$, the formula for U becomes $\mathbb{E}[U] + \sum_n H_n X_n$. ∎

The well-known equivalence between extremality and the predictable representation property (see Theorem (V.4.6) of [3]) becomes completely elementary in our discrete setting; it is recalled in the next proposition. We shall call $\mathcal{L}$ the set of all probability laws on the real line that are carried by two points, and have mean 0 and variance 1. In other words, an element of $\mathcal{L}$ is a probability of the form $p\delta_a + q\delta_b$, with $p > 0$, $q > 0$, $p + q = 1$, $pa + qb = 0$, and $pa^2 + qb^2 = 1$.

LEMMA 1. — *For each $h \in \mathbb{R}$, there is a unique law $\ell(h) \in \mathcal{L}$ supported by the two roots of the quadratic equation $x^2 = 1 + hx$; it gives mass $1/(1+x^2)$ to each root x of the equation. Moreover, this map $\ell : \mathbb{R} \to \mathcal{L}$ is a bijection.*

PROOF. — The roots of $x^2 = 1 + hx$ are two real numbers y and z with product -1, so one is strictly positive and the other strictly negative. There is a unique probability law λ carried by y and z and having mean 0; it weights y with mass $z/(z-y) = 1/(1+y^2)$ and z with $y/(y-z) = 1/(1+z^2)$. And any random variable X with law λ verifies $X^2 = 1 + hX$ and $\mathbb{E}[X] = 0$, whence $\mathbb{E}[X^2] = 1$, so $\lambda \in \mathcal{L}$.

Conversely, any $\lambda \in \mathcal{L}$ is supported by two points, so a random variable X with law λ verifies a quadratic equation $X^2 = hX + k$; taking expectations gives $k = 1$, so ℓ is surjective. ∎

The elements of $\mathcal{L}$ can also be characterized as the centered laws with unit variance that are extremal in the set of all centered laws. They are a fortiori extremal in the smaller set of all centered laws with unit variance, but this is no longer a characterization: the extreme points of the set of all centered laws with unit variance are the centered laws with unit variance carried by two or three points.

PROPOSITION 3. — *Let X be a novation on a filtered probability space $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F})$.*

a) The following three conditions are equivalent:

(i) *there exists a predictable process H such that $X^2 = 1 + HX$;*

(ii) *there exists a predictable process L with values in $\mathcal{L}$ such that, for all Borel f and all $n \in \mathbb{Z}$, $\mathbb{E}[f(X_{n+1})|\mathcal{F}_n] = L_{n+1}(f)$;*

(iii) *there exists a predictable process L with values in $\mathcal{L}$ such that, for all Borel f and all stopping times T, $\mathbb{E}[f(X_{T+1})|\mathcal{F}_T] = L_{T+1}(f)$ on the event $\{T < \infty\}$.*

If X has the predictable representation property, then $\mathcal{F}_{-\infty}$ is degenerate, and the three conditions (i) – (iii) *hold.*

b) If $\mathcal{F}_{-\infty}$ is degenerate, the following five conditions are equivalent:

(iv) *X has the predictable representation property;*

(v) *for each $n \in \mathbb{Z}$, there exists an event Γ such that $\mathcal{F}_{n+1} = \sigma(\mathcal{F}_n, \Gamma)$;*

(vi) *for all $n \in \mathbb{Z}$ and all random variables $U \in \mathrm{L}^2(\mathcal{F}_{n+1})$, one has*

$$U = \mathbb{E}[U|\mathcal{F}_n] + \mathbb{E}[UX_{n+1}|\mathcal{F}_n]\, X_{n+1} \, .$$

(vii) *for all $n \in \mathbb{Z}$ and all $\mathcal{F}_{n+1}$-measurable random variables U, there are two $\mathcal{F}_n$-measurable random variables Q and R such that $U = Q + RX_{n+1}$;*

(viii) *for all stopping times T and all $\mathcal{F}_{T+1}$-measurable random variables U, there are two $\mathcal{F}_T$-measurable random variables Q and R such that $U = Q + RX_{T+1}$ on the event $\{T < \infty\}$.*

c) If $\mathcal{F}_{-\infty}$ is degenerate and if $\mathcal{F}$ is the filtration generated by X, all eight conditions (i) – (viii) *are equivalent to each other, and to the following further two conditions:*

(ix) *for all $n \in \mathbb{Z}$ and all $U \in \mathrm{L}^2(\mathcal{F}_\infty)$, one has $U = \sum\limits_{\substack{A \in \mathcal{P} \\ A \subset]n,\infty[}} \mathbb{E}[UX_A|\mathcal{F}_n]\, X_A$;*

(x) *for all stopping times S and T such that $S \leqslant T$ and all $U \in \mathrm{L}^2(\mathcal{F}_T)$, one has $U = \sum\limits_{A \in \mathcal{P}} \mathbb{1}_{\{A \subset]S,T]\}} \mathbb{E}[UX_A|\mathcal{F}_S]\, X_A$.*

Condition (i) is called a *structure equation*; its analogue in continuous time has the form $d[X,X]_t = dt + H_t\,dX_t$. Conditions (i) to (iii) say that the natural filtration of X is dyadic. Conditions (iv) to (viii) say that the filtration $\mathcal{F}$ itself is dyadic (given the past $\mathcal{F}_{n-1}$, the innovation consists in choosing among two possible values only for X_n, or equivalently in choosing the sign of X_n). But *these conditions* (iv) *to* (viii) *do not imply that $\mathcal{F}$ is generated by X;* they do not even imply that $\mathcal{F}$ is generated by any novation whatsoever (see Vershik's Example 2 in [5]). Conditions (ix) and (x) are the conditional chaotic representation property at times n and S; it is essential here that n and S are not allowed to take the value $-\infty$: when n and S are $-\infty$, these conditions become the (unconditional) chaotic representation property, which is in general strictly stronger than the predictable representation property. The rest of this work will precisely be concerned with the gap between these properties: which hypothesis should be added to the predictable representation property to imply the chaotic representation property? We shall only give a very partial answer.

PROOF OF PROPOSITION 3. — (i) $\Rightarrow$ (ii). Assuming (i), define an $\mathcal{L}$-valued predictable process L by $L_n = \ell(H_n)$, where ℓ is the map defined in Lemma 1. As $X_{n+1}^2 = 1 + H_{n+1}X_{n+1}$, X_{n+1} is a.s. one of the two points of the support of $\ell(H_{n+1})$; as $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = 0$ and $\mathbb{E}[X_{n+1}^2|\mathcal{F}_n] = 1$, the conditional law of X_{n+1} given $\mathcal{F}_n$ must be L_{n+1}. This gives (ii).

(ii) $\Rightarrow$ (iii). Assuming (ii), for each $n \in \mathbb{Z}$ the conclusion holds on the event $\{T=n\}$, so it holds on $\{T<\infty\}$.

(iii) $\Rightarrow$ (ii) is trivial, and to obtain (ii) $\Rightarrow$ (i) it suffices to define the predictable process H by $\ell(H) = L$.

Assuming X has the predictable representation property, for every $U \in \mathrm{L}^2(\mathcal{F}_\infty)$, there is a predictable H such that $\sum_n \mathbb{E}[H_n^2] < \infty$ and $U = \mathbb{E}[U] + \sum_n H_n X_n$; this implies that the (square-integrable) martingale $M_n = \mathbb{E}[U|\mathcal{F}_n]$ is given by $M_n = \mathbb{E}[U] + \sum_{m\leqslant n} H_m X_m$. For $U \in \mathrm{L}^2(\mathcal{F}_{-\infty})$, one has $U = \mathbb{E}[U]$, showing that $\mathcal{F}_{-\infty}$ is degenerate.

We now pass to the equivalence of (iv) – (viii); the end of a), that is, (iv) $\Rightarrow$ (i), will be established later.

(iv) $\Rightarrow$ (vi). We suppose X has the predictable representation property. The martingale argument a few lines above implies that, for every $U \in \mathrm{L}^2(\mathcal{F}_{n+1})$, one has $U = \mathbb{E}[U|\mathcal{F}_n] + H_{n+1}X_{n+1}$. As U and $\mathbb{E}[U|\mathcal{F}_n]$ are in L^2, so is $H_{n+1}X_{n+1}$ too. Multiplying both sides by X_{n+1} and conditioning by $\mathcal{F}_n$ gives $H_{n+1} = \mathbb{E}[UX_{n+1}|\mathcal{F}_n]$, whence (vi).

(vi) $\Rightarrow$ (v). Choosing $U = \mathbb{1}_{\{X_{n+1}=0\}}$ in (vi) yields $\mathbb{1}_{\{X_{n+1}=0\}} = \mathbb{P}[X_{n+1}=0\,|\mathcal{F}_n]$, and shows that $\{X_{n+1}=0\}$ is in $\mathcal{F}_n$. Using (N2), this implies $X_{n+1} \neq 0$ a.s., and, using (N1), $\mathbb{E}[X_{n+1}^+|\mathcal{F}_n] = \mathbb{E}[X_{n+1}^-|\mathcal{F}_n] > 0$ a.s.

Choosing now $U = \mathbb{1}_{\{X_{n+1}\geqslant 0\}}$ in (vi) gives $\mathbb{1}_{\{X_{n+1}\geqslant 0\}} = Q + RX_{n+1}$, with Q and R measurable for $\mathcal{F}_n$ and $R = \mathbb{E}[X_{n+1}^+|\mathcal{F}_n] > 0$. So $X_{n+1} = (\mathbb{1}_{\{X_{n+1}\geqslant 0\}} - Q)/R$, and (vi) becomes

$$\forall U \in \mathrm{L}^2(\mathcal{F}_{n+1}) \qquad U = \mathbb{E}[U|\mathcal{F}_n] + \mathbb{E}[UX_{n+1}|\mathcal{F}_n]\,(\mathbb{1}_{\{X_{n+1}\geqslant 0\}} - Q)/R\,,$$

showing that $\mathcal{F}_{n+1}$ is generated by $\mathcal{F}_n$ and the event $\{X_{n+1} \geqslant 0\}$.

(v) $\Rightarrow$ (vii). Hypothesis (v) implies for each n the existence of two $\mathcal{F}_n$-measurable random variables F and G such that $X_{n+1} = F\,\mathbb{1}_\Gamma + G\,\mathbb{1}_{\Gamma^c}$. Observing that $X_{n+1}^2 - (F+G)X_{n+1} + FG = 0$ and conditioning on $\mathcal{F}_n$, one obtains $FG = -1$. Consequently, $F \neq G$ a.s. and $\mathbb{1}_\Gamma = (X_{n+1}-G)/(F-G)$. Using (v) again, every $\mathcal{F}_{n+1}$-measurable U has the form $V\mathbb{1}_\Gamma + W$, with V and W measurable for $\mathcal{F}_n$; replacing $\mathbb{1}_\Gamma$ by $(X_{n+1}-G)/(F-G)$ shows (vii).

(vii) $\Leftrightarrow$ (viii). Given an $\mathcal{F}_{T+1}$-measurable U, apply (vii) to each $U_n = U\,\mathbb{1}_{\{T=n\}}$.

(vii) and ($\mathcal{F}_{-\infty}$ degenerate) $\Rightarrow$ (iv). Let U be any random variable in $\mathrm{L}^2(\mathcal{F}_\infty)$ and M be the martingale $M_n = \mathbb{E}[U|\mathcal{F}_n]$. When applied to $M_{n+1} - M_n$, (vii) gives $M_{n+1} - M_n = H_{n+1}X_{n+1}$ for some $\mathcal{F}_n$-measurable H_{n+1} (Q vanishes by conditioning on $\mathcal{F}_n$); so one has $M_n - M_m = H_{m+1}X_{m+1} + \ldots + H_nX_n$ for $m < n$. Since $\mathcal{F}_{-\infty}$ is degenerate, M_n tends to $\mathbb{E}[U]$ a.s. and in L^2 when $n \to -\infty$; it also tends to $M_\infty = U$ when $n \to +\infty$, so $U = \mathbb{E}[U] + \sum_{n\in\mathbb{Z}} H_nX_n$. Writing

$$\mathbb{E}[H_{n+1}^2] = \mathbb{E}\big[H_{n+1}^2\,\mathbb{E}[X_{n+1}^2|\mathcal{F}_n]\big] = \mathbb{E}[H_{n+1}^2X_{n+1}^2] = \mathbb{E}\big[(M_{n+1}-M_n)^2\big]\ ,$$

one obtains $\mathbb{E}[H_{m+1}^2 + \ldots + H_n^2] = \mathbb{E}\big[(M_n-M_m)^2\big]$ for $m < n$, giving in the limit $\mathbb{E}\big[\sum_n H_n^2\big] = \mathbb{E}[U^2] - \mathbb{E}[U]^2 < \infty$.

Proposition 3 b) is completely proved; to end proving a), that is, proving (iv) $\Rightarrow$ (i), it suffices to establish (vii) $\Rightarrow$ (i). That is quite easy: Hypothesis (vii) gives $X_{n+1}^2 = Q + RX_{n+1}$, and $Q = 1$ is obtained by conditioning on $\mathcal{F}_n$.

We now start showing c); from here on, we assume $\mathcal{F}$ to be generated by X and $\mathcal{F}_{-\infty}$ to be degenerate.

(i) $\Rightarrow$ (v). From $X_n^2 = 1 + H_nX_n$ it follows that $X_n = f(H_n, \mathbb{1}_{\{X_n\geqslant 0\}})$, where $f(h,0) = \frac{1}{2}(h - \sqrt{h^2+4})$ and $f(h,1) = \frac{1}{2}(h + \sqrt{h^2+4})$. This formula shows that $\mathcal{F}_n$, which we know is generated by $\mathcal{F}_{n-1}$ and X_n, is also generated by $\mathcal{F}_{n-1}$ and the event $\{X_n \geqslant 0\}$.

(vi) $\Rightarrow$ (ix). Iterating (vi), one obtains for all m and n in $\mathbb{Z}$ such that $m \leqslant n$

$$\forall U \in \mathrm{L}^2(\mathcal{F}_\infty) \qquad \mathbb{E}[U|\mathcal{F}_n] = \sum_{A\subset]m,n]} \mathbb{E}[UX_A|\mathcal{F}_m]X_A\ .$$

Indeed, fixing n, it is true when $m = n$ (for the right-hand side consists in one term only, $\mathbb{E}[UX_\varnothing|\mathcal{F}_m]X_\varnothing$); and if it holds for some $m \leqslant n$, one sees that it also holds for $m-1$ by applying (vi) to replace each $\mathbb{E}[UX_A|\mathcal{F}_m]$ by $\mathbb{E}[UX_A|\mathcal{F}_{m-1}] + \mathbb{E}[UX_mX_A|\mathcal{F}_{m-1}]X_m$.

To obtain (ix), it suffices to let n tend to infinity in this formula; convergence takes place in L^2 owing to the following estimate:

$$\begin{aligned}\sum_{A\subset]m,n]} \mathbb{E}[UX_A|\mathcal{F}_m]^2 &= \sum_{A,B\subset]m,n]} \mathbb{E}[UX_A|\mathcal{F}_m]\,\mathbb{E}[UX_B|\mathcal{F}_m]\,\mathbb{E}[X_AX_B|\mathcal{F}_m] \\ &= \mathbb{E}\big[\mathbb{E}[U|\mathcal{F}_n]^2\,\big|\,\mathcal{F}_m\big] \leqslant \mathbb{E}[U^2|\mathcal{F}_m]\ .\end{aligned}$$

(ix) $\Rightarrow$ (x). For $m \leqslant n$ and $U \in \mathrm{L}^2(\mathcal{F}_n)$, $\mathbb{E}[UX_A|\mathcal{F}_n] = 0$ if $\sup A > n$, so (ix) implies $U = \sum_{A\subset]m,n]} \mathbb{E}[UX_A|\mathcal{F}_m]X_A$. Thus,

$$\forall U \in \mathrm{L}^2(\mathcal{F}_\infty) \qquad \mathbb{E}[U|\mathcal{F}_n] = \sum_{A\subset]m,n]} \mathbb{E}[UX_A|\mathcal{F}_m]X_A\ .$$

If S is a stopping time and if $m \leqslant n$,

$$\mathbb{1}_{\{S=m\}}\,\mathbb{E}[U|\mathcal{F}_n] = \mathbb{1}_{\{S=m\}} \sum_{A\subset]m,n]} \mathbb{E}[UX_A|\mathcal{F}_m]X_A$$

$$= \sum_{A\in\mathcal{P}} \mathbb{1}_{\{A\subset]S,n]\}}\,\mathbb{1}_{\{S=m\}}\,\mathbb{E}[UX_A|\mathcal{F}_S]X_A\ ;$$

summing in m gives

$$(*)\qquad \mathbb{E}[U|\mathcal{F}_n] = \sum_{A\in\mathcal{P}} \mathbb{1}_{\{A\subset]S,n]\}}\,\mathbb{E}[UX_A|\mathcal{F}_S]X_A \qquad \text{on the event } \{S\leqslant n\}.$$

On the complementary event $\{S>n\}$, the right-hand side is just $\mathbb{E}[U|\mathcal{F}_S]$; taken together, these two results can be rewritten

$$\mathbb{E}[U|\mathcal{F}_{S\vee n}] = \sum_{A\in\mathcal{P}} \mathbb{1}_{\{A\subset]S,n]\}}\,\mathbb{E}[UX_A|\mathcal{F}_S]X_A\ .$$

Letting now n tend to ∞, this yields, for all $U \in \mathrm{L}^2(\mathcal{F}_\infty)$,

$$U = \sum_{A\in\mathcal{P}} \mathbb{1}_{\{A\subset]S,\infty[\}}\,\mathbb{E}[UX_A|\mathcal{F}_S]X_A\ .$$

Given a stopping time $T \geqslant S$ and a $U \in \mathrm{L}^2(\mathcal{F}_T)$, we have to show that

$$U = \sum_{A\in\mathcal{P}} \mathbb{1}_{\{A\subset]S,T]\}}\,\mathbb{E}[UX_A|\mathcal{F}_S]X_A\ ;$$

it suffices to verify that the difference between the right-hand sides of these two formulas vanishes:

$$¿\qquad \sum_{A\in\mathcal{P}} \mathbb{1}_{\{A\subset]S,\infty[\}}\,\mathbb{1}_{\{A\text{ meets }]T,\infty[\}}\,\mathbb{E}[UX_A|\mathcal{F}_S]X_A = 0 \qquad ?$$

Saying that A meets $]T,\infty[$ amounts to saying that $\sup A > T$ (with the convention $\sup\varnothing = -\infty$). This sum can be rewritten

$$\sum_{n\in\mathbb{Z}} \mathbb{1}_{\{T\leqslant n\}} \sum_{\substack{A\in\mathcal{P}\\ \sup A=n+1}} \mathbb{1}_{\{A\subset]S,\infty[\}}\,\mathbb{E}[UX_A|\mathcal{F}_S]X_A$$

$$= \sum_{n\in\mathbb{Z}} \mathbb{1}_{\{T\leqslant n\}} \sum_{B\in\mathcal{P}} \mathbb{1}_{\{B\subset]S,n]\}}\mathbb{E}[UX_{B\cup\{n+1\}}|\mathcal{F}_S]X_{B\cup\{n+1\}}$$

$$= \sum_{n\in\mathbb{Z}} \mathbb{1}_{\{T\leqslant n\}}\,X_{n+1} \sum_{B\in\mathcal{P}} \mathbb{1}_{\{B\subset]S,n]\}}\mathbb{E}[(UX_{n+1})X_B|\mathcal{F}_S]X_B\ .$$

Now, on the event $\{T\leqslant n\}$, one has a fortiori $S \leqslant n$, so we may use $(*)$ to transform the sum over B into $\mathbb{E}[UX_{n+1}|\mathcal{F}_n]$, yielding $\sum_n \mathbb{1}_{\{T\leqslant n\}}\,X_{n+1}\,\mathbb{E}[UX_{n+1}|\mathcal{F}_n]$. Taking into account that $\{T\leqslant n\}$ belongs to $\mathcal{F}_n$ and that $U\mathbb{1}_{\{T\leqslant n\}}$ is $\mathcal{F}_n$-measurable (because U is $\mathcal{F}_T$-measurable), the conclusion is obtained by writing

$$\mathbb{1}_{\{T\leqslant n\}}\,\mathbb{E}[UX_{n+1}|\mathcal{F}_n] = \mathbb{E}[U\mathbb{1}_{\{T\leqslant n\}}X_{n+1}|\mathcal{F}_n] = U\mathbb{1}_{\{T\leqslant n\}}\,\mathbb{E}[X_{n+1}|\mathcal{F}_n] = 0\ .$$

(x) $\Rightarrow$ (vi) is trivial by taking $S=n$ and $T=n+1$. ∎

From now on, *we suppose given a novation X and its natural filtration $\mathcal{F}$, and we assume that X enjoys the predictable representation property with respect to $\mathcal{F}$;* so all ten conditions (i) – (x) of Proposition 3 hold. They do not imply the chaotic representation property (see [2] for a counterexample); the question is to find additional conditions that are sufficient for the chaotic representation property to hold. Observe that the problem depends only on the law of the process X; so the conditions we are looking for are conditions on the law of X.

Two simple instances of chaotic representation property

The simplest case is when the X_n are independent; by Proposition 3 (ii), the law of each X_n belongs to $\mathcal{L}$, and an easy dimension argument gives the chaotic representation property:

PROPOSITION 4. — *If the novation X consists of independent random variables X_n, each with law in $\mathcal{L}$, the chaotic representation property holds.*

PROOF. — Let $(A_p)_{p\in\mathbb{N}}$ be an increasing sequence in $\mathcal{P}$ with limit $\bigcup_p A_p = \mathbb{Z}$. By martingale convergence, any random variable in $\mathrm{L}^2(\mathcal{F}_\infty)$ can be approximated by its projection on $\mathrm{L}^2\big(\sigma(X_n, n\in A_p)\big)$; so it suffices to establish that for fixed $A \in \mathcal{P}$, the space $S_A = \mathrm{L}^2\big(\sigma(X_n, n\in A)\big)$ is included in the chaotic space $\chi(X)$. Each X_n takes two values, hence the random vector $(X_n)_{n\in A}$ takes $2^{|A|}$ values, and S_A has dimension $2^{|A|}$. But the subspace S'_A of S_A with orthonormal basis $\{X_B, B\subset A\}$ also has dimension $2^{|A|}$; thus $S_A = S'_A$, whence $S_A \subset \chi(X)$. ∎

Another case is when the novation X is deterministic in some neighbourhood of $-\infty$; before giving a precise statement (Proposition 5), we prove an auxiliary lemma saying that the chaotic representation property needs to be checked near $-\infty$ only. Recall that stopping times are not allowed to assume the value $-\infty$.

LEMMA 2. — *Let T be a stopping time. If $\mathrm{L}^2(\mathcal{F}_T) \subset \chi(X)$, the chaotic representation property holds.*

PROOF. — By replacing T with $T \wedge 0$, we may suppose $T < +\infty$ a.s. To prove the lemma, it suffices to show

$$(*) \qquad \mathrm{L}^2(\mathcal{F}_T) \subset \chi(X) \quad \Rightarrow \quad \mathrm{L}^2(\mathcal{F}_{T+1}) \subset \chi(X)\,;$$

for this implies first $\mathrm{L}^2(\mathcal{F}_{T+p}) \subset \chi(X)$ for each $p \in \mathbb{N}$, and then $\mathrm{L}^2(\mathcal{F}_\infty) \subset \chi(X)$ since $\bigcup_{p\in\mathbb{N}} \mathrm{L}^2(\mathcal{F}_{T+p})$ is dense in $\mathrm{L}^2(\mathcal{F}_\infty)$ by martingale convergence.

To show $(*)$, take any $U \in \mathrm{L}^2(\mathcal{F}_{T+1})$ and apply Condition (viii) of Proposition 3 to $U - \mathbb{E}[U|\mathcal{F}_T]$; this yields $U = \mathbb{E}[U|\mathcal{F}_T] + KX_{T+1}$ for some $\mathcal{F}_T$-measurable K. As

$$\mathbb{E}[U^2|\mathcal{F}_T] = \mathbb{E}[U|\mathcal{F}_T]^2 + K^2\mathbb{E}[X_{T+1}^2|\mathcal{F}_T] = \mathbb{E}[U|\mathcal{F}_T]^2 + K^2\,,$$

K belongs to $\mathrm{L}^2(\mathcal{F}_T)$. Observing that $U = \mathbb{E}[U|\mathcal{F}_T] + \sum_n K\mathbb{1}_{\{T=n\}}X_{n+1}$ and that $\mathbb{E}[U|\mathcal{F}_T] \in \chi(X)$, it suffices to verify that $K\mathbb{1}_{\{T=n\}}X_{n+1}$ is in $\chi(X)$. But we know $K\mathbb{1}_{\{T=n\}}$ to be in $\mathrm{L}^2(\mathcal{F}_T)\cap\mathrm{L}^2(\mathcal{F}_n) \subset \chi(X)\cap\mathrm{L}^2(\mathcal{F}_n)$; so it has an expansion of the form $\sum_{\substack{A\in\mathcal{P}\\ A\subset]-\infty,n]}} u_A X_A$, and its product with X_{n+1} is in $\chi(X)$ too, with chaotic expansion

$$\sum_{\substack{A\in\mathcal{P}\\ \sup A=n+1}} u_{A\setminus\{n+1\}} X_A\,. \qquad ∎$$

PROPOSITION 5. — *The following five conditions are equivalent:*

(i) *there exist a random variable S with values in $\mathbb{Z}\cup\{+\infty\}$ and a predictable process $Y = (Y_n)_{n\in\mathbb{Z}}$ such that $X = Y$ on the random interval $]\!]-\infty, S[\![$;*

(ii) *there exist a stopping time T such that $\mathbb{P}[T \geqslant n] > 0$ for all $n \in \mathbb{Z}$ and a deterministic process $y = (y_n)_{n\in\mathbb{Z}}$ such that $X = y$ on $]\!]-\infty, T[\![$, $X_T = -1/y_T$ on $\{T<\infty\}$, and*

$$\mathbb{P}\big[X_n = y_n \,\big|\, T \geqslant n\big] = \frac{1}{1+y_n^2} \qquad \mathbb{P}\big[X_n = -1/y_n \,\big|\, T \geqslant n\big] = \frac{y_n^2}{1+y_n^2}\,;$$

(iii) *the predictable process* H *of Proposition 3* (i) *verifies* $\sum_{n\leqslant 0} \frac{1}{1+H_n^2} < \infty$ *a.s.;*

(iv) *the series* $\sum_{n\leqslant 0} X_n^2$ *converges a.s.;*

(v) *with probability* 1, $|X_n| < 1$ *for all* n *small enough.*

When these conditions are met, the σ*-field* $\mathcal{F}_T$ *is generated by* T, *and the chaotic representation property holds.*

For an analogue of this statement in continuous time, see Théorème 5 of [1] and Théorème 5.3.6 of Taviot [4].

Convergence of the series in (iv) holds a.s. but not in L^1, for $\mathbb{E}[X_n^2] = 1$.

PROOF OF PROPOSITION 5. — (i) $\Rightarrow$ (ii). Fix n such that $\mathbb{P}[S>n] > 0$. We shall first see by induction that for every $m \leqslant n$ there exists an $\mathcal{F}_m$-measurable random variable Z_m such that $X_n = Z_m$ on the event $\{S>n\}$. This is true for $m = n$ with $Z_n = X_n$. Supposing it to hold for some $m \leqslant n$, there exists a Borel function f such that $Z_m = f(\ldots, X_{m-2}, X_{m-1}, X_m)$; so on $\{S>n\}$, $X_m = f(\ldots, X_{m-2}, X_{m-1}, Y_m)$, and it holds for $m-1$ too, with $Z_{m-1} = f(\ldots, X_{m-2}, X_{m-1}, Y_m)$ being $\mathcal{F}_{m-1}$-measurable because Y is predictable.

Consequently, $\mathbb{E}[X_n \mathbb{1}_{\{S>n\}} | \mathcal{F}_m] = \mathbb{E}[Z_m \mathbb{1}_{\{S>n\}} | \mathcal{F}_m] = Z_m \,\mathbb{P}[S>n\,|\mathcal{F}_m]$, and

$$\mathbb{1}_{\{S>n\}}\,\mathbb{E}[X_n \mathbb{1}_{\{S>n\}} | \mathcal{F}_m] = \mathbb{1}_{\{S>n\}}\, X_n\, \mathbb{P}[S>n\,|\mathcal{F}_m]\ .$$

Letting m tend to $-\infty$, we get $\mathbb{1}_{\{S>n\}}\,\mathbb{E}[X_n \mathbb{1}_{\{S>n\}}] = \mathbb{1}_{\{S>n\}}\, X_n\, \mathbb{P}[S>n]$, showing that X_n is the constant $x_n = \mathbb{E}[X_n | S>n]$ on the event $\{S>n\}$. Unfixing n, we obtain that X agrees with some deterministic process x on the interval $]\!]-\infty, S[\![$.

By Condition (i) of Proposition 3, there is a predictable process H such that $X^2 = 1 + HX$; so $\Phi = \frac{1}{2}(H+\sqrt{H^2+4})$ and $\Psi = \frac{1}{2}(H-\sqrt{H^2+4})$ are two predictable processes such that $\Phi\Psi = -1$ and that for each m, X_m is a.s. equal to Φ_m or Ψ_m. There are two Borel functions ϕ_m and ψ_m such that $\Phi_m = \phi_m(\ldots, X_{m-2}, X_{m-1})$ and $\Psi_m = \psi_m(\ldots, X_{m-2}, X_{m-1})$.

Fix again n such that $\mathbb{P}[S>n] > 0$. Define a deterministic process $y = (y_m)_{m\in\mathbb{Z}}$ by $y_m = x_m$ if $m \leqslant n$ and (inductively) by $y_m = \phi_m(\ldots, y_{m-2}, y_{m-1})$ if $m > n$. For $m \leqslant n$, $\mathbb{P}[X_k = y_k\ \forall k \leqslant m] \geqslant \mathbb{P}[S>n] > 0$; putting $E_m = \{X_k = y_k\ \forall k < m\}$, one has $\mathbb{P}[X_m = y_m | E_m] > 0$. But the conditional law of X_m given E_m is the law in $\mathcal{L}$ supported by the two points $\phi_m(\ldots, y_{m-2}, y_{m-1})$ and $\psi_m(\ldots, y_{m-2}, y_{m-1})$. So y_m is one of these two points and $-1/y_m$ is the other one; this holds for $m > n$ too by the very definition of y_m.

Put $T = \inf\{m : X_m \neq y_m\}$. As T is minorated by $S \wedge n$, it does not take the value $-\infty$, and T is a stopping time. On the event $\{T \geqslant m\}$, X and y agree up to time $m-1$, and X_m takes the two values y_m and $-1/y_m$ with respective probabilities given by Lemma 1:

$$\mathbb{P}[X_m = y_m | T \geqslant m] = 1/(1+y_m^2) \quad \text{and} \quad \mathbb{P}[X_m = -1/y_m | T \geqslant m] = y_m^2/(1+y_m^2)\ .$$

On $\{T=m\}$, one has furthermore $X_m \neq y_m$, whence $X_m = -1/y_m$, and $X_T = -1/y_T$ on $\{T<\infty\}$. Last, for each $m \in \mathbb{Z}$, the essential supremum of T cannot be m since $\mathbb{P}[T>m | T \geqslant m] = \mathbb{P}[X_m = y_m | T \geqslant m] = 1/(1+y_m^2) > 0$; thus T is not bounded above.

(ii) $\Rightarrow$ (iv). If (ii) holds, iterating the relation $\mathbb{P}[T>n|T>n-1] = \dfrac{1}{1+y_n^2}$ gives for $n<0$

$$\mathbb{P}[T>0|T>n] = \prod_{m\in]n,0]} \frac{1}{1+y_m^2}$$

and, taking the limit when $n\to-\infty$,

$$\mathbb{P}[T>0] = \prod_{m\leqslant 0} \frac{1}{1+y_m^2}\,.$$

The left-hand side being strictly positive by hypothesis, the infinite product must converge, and $\sum_{n\leqslant 0} y_n^2 < \infty$. As $X_n = y_n$ for all $n<T$, $\sum_{n\leqslant 0} X_n^2 < \infty$ a.s.

(iv) $\Rightarrow$ (iii). According to the structure equation $X^2 = 1 + HX$, the process X never vanishes and $H = X - 1/X$. Hence $1/(1+H^2) = X^2/(1-X^2+X^4) \leqslant \frac{4}{3}X^2$, and if the series $\sum_{n\leqslant 0} X_n^2$ is a.s. convergent, so is also $\sum_{n\leqslant 0} 1/(1+H_n^2)$.

(iii) $\Rightarrow$ (i). One of the roots of the structure equation $x^2 = 1 + H_n x$ satisfied by X_n is $\Phi_n = \frac{1}{2}(\operatorname{sgn} H_n)\big(|H_n| + \sqrt{H_n^2+4}\big)$, with the convention $\operatorname{sgn} 0 = 1$. Notice that the predictable process Φ verifies $|\Phi_n| > |H_n|$, so the series $\sum_{n\leqslant 0} 1/(1+\Phi_n^2)$ is a.s. convergent. Set

$$T = \inf\Big\{n\in\mathbb{Z} \,:\, \sum_{m\leqslant n} \frac{1}{1+\Phi_m^2} \geqslant 1\Big\}\,.$$

Because the series is convergent, $T > -\infty$ a.s. As Φ is predictable, T is a predictable stopping time (i.e. $T-1$ is a stopping time) and the event $\{n<T\}$ is in $\mathcal{F}_{n-1}$. By Proposition 3 (ii) and Lemma 1, $\mathbb{P}[X_n{=}\Phi_n|\mathcal{F}_{n-1}] = 1/(1+\Phi_n^2)$, so one can write

$$\mathbb{E}\Big[\sum_{n<T} \mathbb{1}_{\{X_n=\Phi_n\}}\Big] = \mathbb{E}\Big[\sum_{n} \mathbb{1}_{\{n<T\}}\mathbb{P}[X_n=\Phi_n|\mathcal{F}_{n-1}]\Big] = \mathbb{E}\Big[\sum_{n<T} \frac{1}{1+\Phi_n^2}\Big] \leqslant 1\,.$$

Consequently, the sum $\sum_{n<T} \mathbb{1}_{\{X_n=\Phi_n\}}$ is a.s. finite; so, with probability 1, for all but finitely many $n\leqslant 0$, X_n is the other root $-1/\Phi_n$ of the equation, and (i) holds with $Y = -1/\Phi$.

(iv) $\Rightarrow$ (v) is trivial.

(v) $\Rightarrow$ (i). The two roots of the structure equation $x^2 = 1 + H_n x$ satisfied by X_n are $\Phi_n = \frac{1}{2}(\operatorname{sgn} H_n)\big(|H_n| + \sqrt{H_n^2+4}\big)$ and $-1/\Phi_n$; they verify $|\Phi_n| \geqslant 1$ and $|-1/\Phi_n| \leqslant 1$. So X is equal to the predictable process $Y = -1/\Phi$ on the random set $\{|X|<1\}$. By hypothesis, this random set contains a random interval $]\!]-\infty, S[\![$ with $S > -\infty$ a.s., so (i) holds.

(ii) $\Rightarrow$ ($\sigma(T) = \mathcal{F}_T$). Supposing (ii) to hold, let U be any $\mathcal{F}_T$-measurable random variable. There is for each $n\in\mathbb{Z}\cup\{+\infty\}$ a Borel function u_n such that

$$\begin{aligned} U &= U\mathbb{1}_{\{T=\infty\}} + \sum_{n\in\mathbb{Z}} U\mathbb{1}_{\{T=n\}} \\ &= u_\infty(X_m,\ m\in\mathbb{Z})\mathbb{1}_{\{T=\infty\}} + \sum_{n\in\mathbb{Z}} u_n(\ldots, X_{n-2}, X_{n-1}, X_n)\mathbb{1}_{\{T=n\}} \\ &= u_\infty(y_m,\ m\in\mathbb{Z})\mathbb{1}_{\{T=\infty\}} + \sum_{n\in\mathbb{Z}} u_n(\ldots, y_{n-2}, y_{n-1}, -1/y_n)\mathbb{1}_{\{T=n\}}\ ; \end{aligned}$$

since y is deterministic, U is $\sigma(T)$-measurable.

(ii) $\Rightarrow$ (chaotic representation property). Fix $n \in \mathbb{Z}$. For any $A \in \mathcal{P}$, one has

$$\mathbb{E}[X_A \mathbb{1}_{\{T=n\}}] = \begin{cases} y_A \, \mathbb{P}[T=n] & \text{if } A \subset \,]-\infty, n[; \\ y_{A\setminus\{n\}} \, \dfrac{(-1)}{y_n} \, \mathbb{P}[T=n] & \text{if } \sup A = n; \\ 0 & \text{if } \sup A > n. \end{cases}$$

Squaring and summing over A yields

$$\begin{aligned} \sum_{A\in\mathcal{P}} \big(\mathbb{E}[X_A \mathbb{1}_{\{T=n\}}]\big)^2 &= \mathbb{P}[T=n]^2 \sum_{\substack{A\in\mathcal{P}\\ A\subset]-\infty,n[}} \Big(y_A^2 + y_A^2 \frac{1}{y_n^2}\Big) \\ &= \mathbb{P}[T=n]^2 \Big(1 + \frac{1}{y_n^2}\Big) \sum_{\substack{A\in\mathcal{P}\\ A\subset]-\infty,n[}} y_A^2 \\ &= \mathbb{P}[T=n]^2 \Big(1 + \frac{1}{y_n^2}\Big) \prod_{m<n} \big(1 + y_m^2\big) \, . \end{aligned}$$

Now, by induction on $k < n$, $\mathbb{P}[T=n|T>k] = \dfrac{y_n^2}{1+y_n^2} \displaystyle\prod_{k<m<n} \frac{1}{1+y_m^2}$, so, in the limit when $k \to -\infty$,

$$\frac{y_n^2}{1+y_n^2} \prod_{m<n} \frac{1}{1+y_m^2} = \mathbb{P}[T=n] \, ,$$

and the above sum becomes

$$\sum_{A\in\mathcal{P}} \big(\mathbb{E}[X_A \mathbb{1}_{\{T=n\}}]\big)^2 = \mathbb{P}[T=n]^2 \frac{1}{\mathbb{P}[T=n]} = \mathbb{P}[T=n] = \mathbb{E}[(\mathbb{1}_{T=n})^2] \, .$$

The left-hand side is the squared L^2-norm of the orthogonal projection of $\mathbb{1}_{T=n}$ on the chaotic space $\chi(X)$; the right-hand side is the squared L^2-norm of $\mathbb{1}_{T=n}$ itself. Their being equal shows that $\mathbb{1}_{T=n}$ belongs to $\chi(X)$, and, n being arbitrary, that $\mathrm{L}^2(\sigma(T)) \subset \chi(X)$. We have seen above that $\sigma(T) = \mathcal{F}_T$; so $\mathrm{L}^2(\mathcal{F}_T) \subset \chi(X)$, and the chaotic representation property holds by Lemma 2. ∎

Another, less simple, case of chaotic representation property

Recall the context: X is a novation, $\mathcal{F}$ is its natural filtration, and all ten conditions of Proposition 3 are in force; in particular, by condition (v), $\mathcal{F}$ is dyadic and by condition (iv) $\mathcal{F}_{-\infty}$ is degenerate. In this section, we shall work in a narrower setting: we shall further suppose that $\mathcal{F}$ is generated by a process taking values in a two-point space (the set $\{-1, 1\}$ will be convenient). Example 2 of Vershik [5] shows that this additional hypothesis is not a consequence of the other assumptions.

LEMMA 3. — *a) Let ε be a process with values in $\{-1, 1\}$; call $\mathcal{E}$ the natural filtration of ε and suppose* $\mathrm{Var}[\varepsilon_n|\mathcal{E}_{n-1}] > 0$ *a.s. for each $n \in \mathbb{Z}$. There exists a unique $\mathcal{E}$-novation X^ε such that* $\operatorname{sgn} X^\varepsilon = \varepsilon$*; moreover, X^ε has the same natural filtration $\mathcal{E}$ as ε.*

b) Suppose given a filtration $\mathcal{F}$ and an $\mathcal{F}$-novation X; put $\varepsilon = \operatorname{sgn} X$ (with for instance $\operatorname{sgn} 0 = 1$*). The following two conditions are equivalent:*

(i) *both processes X and ε generate the same filtration;*

(ii) *the novation X^ε defined in a) is equal to X.*

REMARKS. — a) In Lemma 3 b), since X is an $\mathcal{F}$-novation, $\operatorname{Var}[\varepsilon_n|\mathcal{F}_{n-1}] > 0$, and a fortiori $\operatorname{Var}[\varepsilon_n|\mathcal{E}_{n-1}] > 0$ where $\mathcal{E}$ is the filtration generated by ε. Hence the process X^ε in condition (ii) is well defined.

b) If the time-axis is $\{n\in\mathbb{Z} : n \leqslant 0\}$ instead of $\mathbb{Z}$, conditions (i) and (ii) in b) are also equivalent to the seemingly weaker condition:

(i′) *the processes X and ε generate the same σ-field.*

To check this, calling $\mathcal{F}$ (respectively $\mathcal{E}$) the natural filtration of X (respectively ε), it suffices to verify that *if $\mathcal{F}_n = \mathcal{E}_n$, then $\mathcal{F}_{n-1} = \mathcal{E}_{n-1}$*. Supposing $\mathcal{F}_n = \mathcal{E}_n$, $X_n = U\mathbb{1}_{\{\varepsilon_n=1\}} + V\mathbb{1}_{\{\varepsilon_n=-1\}}$ for some $\mathcal{E}_{n-1}$-measurable U and V. This implies $(X_n-U)(X_n-V) = 0$; expanding and conditioning on $\mathcal{F}_{n-1}$ gives $UV = -1$; in particular, $U \neq V$ a.s., and $\{X_n=U\} = \{\varepsilon_n=1\}$, $\{X_n=V\} = \{\varepsilon_n=-1\}$. Now if W is any $\mathcal{F}_{n-1}$-measurable random variable, it is also $\mathcal{E}_n$-measurable, so $W = Q\mathbb{1}_{\{\varepsilon_n=1\}} + R\mathbb{1}_{\{\varepsilon_n=-1\}}$ with Q and R measurable for $\mathcal{E}_{n-1}$. This can be rewritten as $W = AX_n + B$, where $Ax + B$ is the $\mathcal{E}_{n-1}$-measurable affine function mapping U to Q and V to R. Conditioning on $\mathcal{F}_{n-1}$ kills the term AX_n, so $W = B$, and W is $\mathcal{E}_{n-1}$-measurable. This proves $\mathcal{F}_{n-1} = \mathcal{E}_{n-1}$.

PROOF OF LEMMA 3. — a) If X is any $\mathcal{E}$-novation, then, owing to the filtration $\mathcal{E}$ being dyadic, $X_n = A\varepsilon_n + B$, where A and B are $\mathcal{E}_{n-1}$-measurable. Condition (N1) implies $X_n = A\left(\varepsilon_n - \mathbb{E}[\varepsilon_n|\mathcal{E}_{n-1}]\right)$, and (N2) then yields $1 = A^2 \operatorname{Var}[\varepsilon_n|\mathcal{E}_{n-1}]$. If furthermore $\operatorname{sgn} X = \varepsilon$, A cannot be negative, and one gets

$$X_n = \frac{\varepsilon_n - \mathbb{E}[\varepsilon_n|\mathcal{E}_{n-1}]}{\sqrt{\operatorname{Var}[\varepsilon_n|\mathcal{E}_{n-1}]}} .$$

Conversely, X defined by this formula is an $\mathcal{E}$-novation; and as $\left|\mathbb{E}[\varepsilon_n|\mathcal{E}_{n-1}]\right| < 1$ and $|\varepsilon_n| = 1$, one has $\operatorname{sgn} X_n = \varepsilon_n$. This proves existence and uniqueness.

Since $\operatorname{sgn} X = \varepsilon$, the natural filtration $\mathcal{E}$ of ε is included in that of X; but the explicit formula for X_n shows that X is adapted to $\mathcal{E}$; so X generates $\mathcal{E}$.

b) (ii) $\Rightarrow$ (i) is an immediate consequence of a). Conversely, if an $\mathcal{F}$-novation X and its sign ε have the same natural filtration $\mathcal{E}$, $\mathcal{E}$ is included in $\mathcal{F}$, so X is also an $\mathcal{E}$-novation, and $X = X^\varepsilon$ by uniqueness in a). ∎

PROPOSITION 6. — *Let $\varepsilon = (\varepsilon_n)_{n\in\mathbb{Z}}$ be a process with values in $\{-1,1\}$ and call $\mathcal{F}$ its natural filtration. Suppose*

(i) *the process ε is Markov* (but not necessarily homogeneous)*;*

(ii) *the σ-field $\mathcal{F}_{-\infty}$ is degenerate;*

(iii) $\operatorname{Var}[\varepsilon_n|\mathcal{E}_{n-1}] > 0$ *a.s. for each $n \in \mathbb{Z}$.*

Under these assumptions, the $\mathcal{F}$-novation X^ε (defined in the previous lemma) has the chaotic representation property.

PROOF. — We shall simply write X instead of X^ε. Notice that all ten conditions of Proposition 3 hold. If $(Z_n)_{n\in\mathbb{Z}}$ is any process, we shall set $Z_A = \prod_{n\in A} Z_n$ for $A \in \mathcal{P}$.

As ε is a Markov process, the conditional expectation $\mathbb{E}[\varepsilon_n|\mathcal{F}_{n-1}]$ is a function of ε_{n-1}; it takes values in $[-1,1]$, and more precisely in the open interval $(-1,1)$ since $\operatorname{Var}[\varepsilon_n|\mathcal{F}_{n-1}] > 0$. Hence we may put $\mathbb{E}[\varepsilon_n|\mathcal{F}_{n-1}] = \sin\Theta_n$, for some random variable $\Theta_n = \theta_n(\varepsilon_{n-1})$, depending on ε_{n-1} only, and with values in $(-\frac{\pi}{2}, \frac{\pi}{2})$. And as ε_{n-1} takes only the values -1 and 1, $\Theta_n = \alpha_n \varepsilon_{n-1} + \beta_n$ for two real numbers

$$\alpha_n = \frac{\theta_n(1) - \theta_n(-1)}{2} \qquad \text{and} \qquad \beta_n = \frac{\theta_n(1) + \theta_n(-1)}{2}$$

that are both in the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. With these notations, the formula giving X_n in the proof of Lemma 3 becomes $X_n = (\varepsilon_n - \sin\Theta_n)/\cos\Theta_n$. This implies $\varepsilon_n = X_n \cos\Theta_n + \sin\Theta_n$; squaring both sides gives $X_n^2 = 1 - 2X_n \tan\Theta_n$ and shows that the predictable process H appearing in the structure equation satisfied by the novation X is $H = -2\tan\Theta$.

For $m \leqslant n$ in $\mathbb{Z}$, call χ^n (respectively χ^n_m) the closed subspace of $\chi(X)$ with orthonormal basis $\{X_A,\ A \in \mathcal{P},\ A \subset]-\infty, n]\}$ (respectively $\{X_A,\ A \subset]m,n]\}$); notice that χ^n_m has finite dimension 2^{n-m} and that for $U \in \chi^m$ and $V \in \chi^n_m$ the product UV belongs to χ^n.

As $\mathcal{F}_\infty = \sigma(\varepsilon_n, n \in \mathbb{Z})$, the chaotic representation property will be established if we show that the chaotic space $\chi(X)$ contains every random variable of the form $f(\varepsilon_{m+1}, \ldots, \varepsilon_n)$. For fixed m and n, those random variables form a finite-dimensional vector space, with basis $\{\varepsilon_A,\ A \subset]m,n]\}$; so it suffices to show that each ε_A belongs to $\chi(X)$.

The first step of the proof will consist in establishing that for every $A \subset]m,n]$, there exist Q and R in χ^n_m such that $\varepsilon_A = Q + R\varepsilon_m$. For fixed n, this will be shown by induction on $m \leqslant n$. If $m = n$, the only possible A is $A = \varnothing$, and the property holds trivially with $Q = 1 = X_\varnothing$ and $R = 0$. Suppose now it holds for some $m \leqslant n$. Replacing Θ_m by $\alpha_m \varepsilon_{m-1} + \beta_m$ in the formula $\varepsilon_m = X_m \cos\Theta_m + \sin\Theta_m$ gives an expression of the form $\varepsilon_m = (aX_m + b) + (cX_m + d)\varepsilon_{m-1}$. Now every subset A of $]m-1, n]$ is either of the form B, or of the form $\{m\} \cup B$, for some $B \subset]m,n]$. By induction hypothesis, $\varepsilon_B = Q + R\varepsilon_m$ with Q and R in χ^n_m; so ε_A is either $Q + R\varepsilon_m$ or $Q\varepsilon_m + R$, and replacing ε_m by $(aX_m + b) + (cX_m + d)\varepsilon_{m-1}$ establishes the claim.

Owing to this property, to show that ε_A is in $\chi(X)$ for $A \subset]m,n]$, it suffices to show that ε_m is in χ^m. Without loss of generality, we shall do it for $m = 0$ only: the rest of the proof will consist in establishing that ε_0 *belongs to the chaotic space* χ^0 *generated by* $(X_n)_{n \leqslant 0}$.

Set

$$\begin{pmatrix} Q_n \\ R_n \end{pmatrix} = \begin{pmatrix} \cos\beta_n & \sin\beta_n \\ -\sin\beta_n & \cos\beta_n \end{pmatrix} \begin{pmatrix} X_n \\ 1 \end{pmatrix}.$$

Rewriting (N1) and (N2) as

$$\mathbb{E}^{\mathcal{F}_{n-1}} \left[\begin{pmatrix} X_n \\ 1 \end{pmatrix} (X_n \ \ 1) \right] = \mathrm{Id},$$

one has

$$\mathbb{E}^{\mathcal{F}_{n-1}} \left[\begin{pmatrix} Q_n \\ R_n \end{pmatrix} (Q_n \ \ R_n) \right] = \begin{pmatrix} \cos\beta_n & \sin\beta_n \\ -\sin\beta_n & \cos\beta_n \end{pmatrix} \mathrm{Id} \begin{pmatrix} \cos\beta_n & -\sin\beta_n \\ \sin\beta_n & \cos\beta_n \end{pmatrix} = \mathrm{Id},$$

whence $\mathbb{E}[Q_n^2 | \mathcal{F}_{n-1}] = \mathbb{E}[R_n^2 | \mathcal{F}_{n-1}] = 1$ and $\mathbb{E}[Q_n R_n | \mathcal{F}_{n-1}] = 0$. Consequently, by induction on $n \leqslant 0$, if A and B (respectively A' and B') are two disjoint sets with union $A \cup B = A' \cup B' =]n, 0]$,

$$\mathbb{E}[(Q_A R_B)(Q_{A'} R_{B'}) | \mathcal{F}_n] = \begin{cases} 1 & \text{if } A = A' \text{ and } B = B' \\ 0 & \text{else;} \end{cases}$$

and when (A,B) ranges over all pairs of complementary subsets of $]n,0]$, the r.v.'s $Q_A R_B$ form an orthonormal basis of the subspace χ_n^0. The orthogonal projection of ε_0 on this subspace is

$$\mathrm{Proj}_{\chi_n^0}\,\varepsilon_0 = \sum_{\substack{A\cup B=]n,0]\\ A\cap B=\varnothing}} \mathbb{E}[\varepsilon_0 Q_A R_B]\, Q_A R_B\ .$$

To show that ε_0 is in the chaotic space, it suffices to show that it is the L^2-limit of $\mathrm{Proj}_{\chi_n^0}\,\varepsilon_0$ when $n\to-\infty$; as $\|\varepsilon_0\|^2=1$, this reduces to proving that $\|\mathrm{Proj}_{\chi_n^0}\,\varepsilon_0\|^2$ tends to 1, or equivalently that

$$\sum_{\substack{A\cup B=]n,0]\\ A\cap B=\varnothing}} \big(\mathbb{E}[\varepsilon_0 Q_A R_B]\big)^2 \longrightarrow 1 \qquad \text{when} \quad n\to-\infty.$$

Set $U_n = Q_n\,\cos\alpha_n$ and $V_n = R_n\,\sin\alpha_n$. One has

$$\begin{aligned}
\varepsilon_n &= X_n\,\cos\Theta_n + \sin\Theta_n\\
&= \big(\cos(\alpha_n\varepsilon_{n-1}+\beta_n)\quad \sin(\alpha_n\varepsilon_{n-1}+\beta_n)\big)\begin{pmatrix} X_n\\ 1\end{pmatrix}\\
&= \big(\cos(\alpha_n\varepsilon_{n-1})\quad \sin(\alpha_n\varepsilon_{n-1})\big)\begin{pmatrix} \cos\beta_n & \sin\beta_n\\ -\sin\beta_n & \cos\beta_n\end{pmatrix}\begin{pmatrix} X_n\\ 1\end{pmatrix}\\
&= \big(\cos\alpha_n \quad \varepsilon_{n-1}\sin\alpha_n\big)\begin{pmatrix} Q_n\\ R_n\end{pmatrix} \;=\; U_n+\varepsilon_{n-1}V_n
\end{aligned}$$

Iterating this formula, one obtains

$$\begin{aligned}
\varepsilon_0 &= U_0 + U_{-1}V_0 + U_{-2}V_{-1}V_0 + \ldots + U_{n+1}V_{n+2}\ldots V_0 + \varepsilon_n V_{n+1}\ldots V_0\\
&= \cos\alpha_0\,Q_0 + \cos\alpha_{-1}\,\sin\alpha_0\,Q_{-1}R_0 + \ldots\\
&\qquad\qquad + \cos\alpha_{n+1}\sin\alpha_{n+2}\ldots\sin\alpha_0\,Q_{n+1}R_{n+2}\ldots R_0\\
&\qquad\qquad\qquad + \varepsilon_n\sin\alpha_{n+1}\ldots\sin\alpha_0 R_{n+1}\ldots R_0\ .
\end{aligned}$$

Multiplying by $Q_A R_B$ (where $A\cup B =]n,0]$ and $A\cap B=\varnothing$) and conditioning by $\mathcal{F}_n$, all terms cancel but one, and there only remains

- if $A=\varnothing$ and $B=]n,0]$,

$$\begin{aligned}
\mathbb{E}\big[\varepsilon_0 Q_A R_B\,\big|\,\mathcal{F}_n\big] &= \mathbb{E}[\varepsilon_n\sin\alpha_{n+1}\ldots\sin\alpha_0 R_{n+1}^2\ldots R_0^2|\mathcal{F}_n]\\
&= \varepsilon_n\sin\alpha_{n+1}\ldots\sin\alpha_0\ ;
\end{aligned}$$

- if $A\neq\varnothing$ and $\sup A = m\in\,]n,0]$,

$$\begin{aligned}
&\mathbb{E}[\varepsilon_0 Q_A R_B|\mathcal{F}_n]\\
&\quad= \mathbb{E}\big[\cos\alpha_m\,\sin\alpha_{m+1}\cdots\sin\alpha_0\,Q_{A\cap]n,m-1]}R_{B\cap]n,m-1]}Q_m^2R_{m+1}^2\cdots R_0^2\,\big|\,\mathcal{F}_n\big]\\
&\quad= \cos\alpha_m\,\sin\alpha_{m+1}\cdots\sin\alpha_0\,\mathbb{E}\big[Q_{A\cap]n,m-1]}R_{B\cap]n,m-1]}\,\big|\,\mathcal{F}_n\big]\\
&\quad= \cos\alpha_m\,\sin\alpha_{m+1}\cdots\sin\alpha_0 \prod_{\substack{a\in A\\ a<m}}\sin\beta_a \prod_{\substack{b\in B\\ b<m}}\cos\beta_b
\end{aligned}$$

(the latter follows from $\mathbb{E}[Q_\ell|\mathcal{F}_{\ell-1}] = \sin\beta_\ell$ and $\mathbb{E}[R_\ell|\mathcal{F}_{\ell-1}] = \cos\beta_\ell$). Taking expectations, squaring and summing gives $\|\mathrm{Proj}_{\chi_n^0}\,\varepsilon_0\|^2$ as the sum of two terms. The first term, corresponding to $A=\varnothing$, is

$$\big(\mathbb{E}[\varepsilon_n]\big)^2\,\sin^2\alpha_{n+1}\ldots\sin^2\alpha_0\ ;$$

the second term is the sum

$$\sum_{m\in]n,0]} \cos^2\alpha_m \sin^2\alpha_{m+1}\dots\sin^2\alpha_0 \sum_{\substack{A\cup B=]n,m-1]\\ A\cap B=\varnothing}} \prod_{a\in A}\sin^2\beta_a \prod_{b\in B}\cos^2\beta_b$$

$$= \sum_{m\in]n,0]} \cos^2\alpha_m \sin^2\alpha_{m+1}\dots\sin^2\alpha_0 \prod_{a\in]n,m-1]} (\sin^2\beta_a+\cos^2\beta_a)$$

$$= \sum_{m\in]n,0]} \cos^2\alpha_m \sin^2\alpha_{m+1}\dots\sin^2\alpha_0$$

$$= \sum_{m\in]n,0]} \big(\sin^2\alpha_{m+1}\dots\sin^2\alpha_0 - \sin^2\alpha_m\sin^2\alpha_{m+1}\dots\sin^2\alpha_0\big)$$

$$= 1-\sin^2\alpha_{n+1}\dots\sin^2\alpha_0 \,.$$

Putting both terms together gives

$$\|\mathrm{Proj}_{\mathcal{X}_n^0}\,\varepsilon_0\|^2 = 1-\big(1-\mathbb{E}[\varepsilon_n]^2\big)\sin^2\alpha_{n+1}\dots\sin^2\alpha_0$$
$$= 1-\sin^2\alpha_{n+1}\dots\sin^2\alpha_0 \operatorname{Var}\varepsilon_n \,,$$

and to establish the chaotic representation property it suffices to verify that

$$\sin^2\alpha_{n+1}\dots\sin^2\alpha_0\operatorname{Var}\varepsilon_n \longrightarrow 0 \qquad \text{when} \quad n \text{ tends to } -\infty.$$

Clearly, this holds if the product $\sin^2\alpha_{n+1}\dots\sin^2\alpha_0$ tends to 0 (for $\operatorname{Var}\varepsilon_n \leqslant 1$); hence we only have to consider the case when the infinite product $\prod_{n\leqslant 0}\sin^2\alpha_n$ is convergent.

We shall show that in that case, condition (iii) of Proposition 5 is fulfilled; as the chaotic representation property always holds in the degenerate situation considered in that proposition, our Proposition 6 will thus be established in full generality.

That condition says that the series $\sum_{n\leqslant 0}(1+H_n^2)^{-1}$ converges a.s., where H is the predictable process featuring in the structure equation satisfied by the novation X. At the beginning of the proof, we saw that $H_n=-2\tan\Theta_n$; consequently

$$\frac{1}{1+H_n^2}=\frac{1}{1+4\tan^2\Theta_n}\leqslant\frac{1}{1+\tan^2\Theta_n}=\cos^2\Theta_n\,,$$

and it only remains to establish that the sum $\sum_{n\leqslant 0}\cos^2\Theta_n$ is a.s. finite.

Put $\bar\alpha_n=\frac{\pi}{2}-|\alpha_n|$. Since the values assumed by Θ_n are $\beta_n-\alpha_n$ and $\beta_n+\alpha_n$, one has $-\frac{\pi}{2}<\beta_n-|\alpha_n|<\beta_n+|\alpha_n|<\frac{\pi}{2}$, whence

$$\begin{cases}\frac{\pi}{2}-2\bar\alpha_n=-\frac{\pi}{2}+2|\alpha_n|<\beta_n+|\alpha_n|<\frac{\pi}{2}\\ -\frac{\pi}{2}+2\bar\alpha_n=\frac{\pi}{2}-2|\alpha_n|>\beta_n-|\alpha_n|>-\frac{\pi}{2}\end{cases}$$

and $\Theta_n=\beta_n\pm\alpha_n\in\big(-\frac{\pi}{2},-\frac{\pi}{2}+2\bar\alpha_n\big)\cup\big(\frac{\pi}{2}-2\bar\alpha_n,\frac{\pi}{2}\big)$. As the infinite product is convergent, $\sin^2\alpha_n\to 1$, so $\bar\alpha_n\to 0$, and $2\bar\alpha_n<\frac{\pi}{2}$ for all n small enough. For these n one has $|\cos\Theta_n|<|\cos(\frac{\pi}{2}-2\bar\alpha_n)|=\sin(2|\alpha_n|)=2\sin|\alpha_n|\cos\alpha_n\leqslant 2\cos\alpha_n$, hence also $\cos^2\Theta_n<4\cos^2\alpha_n=4(1-\sin^2\alpha_n)$; so convergence of the infinite product $\prod_{n\leqslant 0}\sin^2\alpha_n$ implies convergence of the series $\sum_{n\leqslant 0}\cos^2\Theta_n$. ∎

The Markov hypothesis (i) in Proposition 6 has been used to perform explicit computations on the process ε; it is not clear whether the result remains true or not when this hypothesis is dropped.

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