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Asymptotic estimates for the first hitting time of fluctuating additive functionals of Brownian motion

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1 Introduction

In [3], we obtained the following estimates for the first hitting time of the integrated Brownian motion: Let $B(t)$ be the linear Brownian motion started at 0. It holds with some explicit constant $k > 0$

$$(1.1) \quad P \left[\int_0^u B(s) ds < r \text{ for all } 0 \leq u \leq t \right] \sim kr^{1/6} t^{-1/4} \text{ as } r^{1/6} t^{-1/4} \rightarrow 0,$$

which is a refinement of Sinai's estimates[12].

The above formula as well as the other ones follow systematically from the theorem in [3]: Let $(X(t), Y(t))$ be the Kolmogorov diffusion ([5]).

$$(1.2) \quad Y(t) = y + B(t), \quad X(t) = x + \int_0^t Y(s) ds.$$

Let T be the first hitting time to the positive y -axis:

$$(1.3) \quad T = \inf\{t \geq 0; X(t) = 0, Y(t) \geq 0\}.$$

Hence $Y(T)$ is the hitting place on the positive y -axis. We denote by $E_{(x,y)}$ and $P_{(x,y)}$ the expectation and the probability measure for this diffusion respectively.

Theorem ([3]) *For $\mu, \kappa \geq 0$ and $x \leq 0, y \in \mathbb{R}$ it holds*

$$(1.4) \quad 1 - E_{(\bar{\sigma}x, \bar{\sigma}^{1/3}y)} \left[\exp \left\{ -\sigma\mu T - \sqrt{\sigma}\sqrt{2\kappa}Y(T) \right\} \right] \sim \tilde{K}(x, y) \bar{\sigma}^{1/6} K(\kappa, \mu) \sigma^{1/4}$$

as $\bar{\sigma}^{1/6} \sigma^{1/4}$ tends to 0, where

$$K(\kappa, \mu) = \frac{3(\sqrt{2\kappa} + \sqrt{2\mu})\Gamma(\frac{1}{3})3^{1/3}}{\sqrt{\pi}\sqrt{\sqrt{2\kappa} + 2\sqrt{2\mu}}\Gamma(\frac{1}{6})2^{1/6}}$$

and

$$\tilde{K}(x, y) = \frac{|x|^{5/6} e^{-2(y^+)^3/9|x|}}{\Gamma(\frac{1}{3})} \int_0^\infty dt e^{-t} (|x|t + 2|y^-|^3/9)^{1/6} (|x|t + 2(y^+)^3/9)^{-5/6}.$$

The proof depends heavily on a formula obtained by McKean[8].

We considered in [4] a generalization for this problem. We redefine $(X(t), Y(t))$, the odd additive functional, as

$$(1.5) \quad Y(t) = y + B(t), \quad X(t) = x + \int_0^t |Y(s)|^\alpha \operatorname{sgn}(Y(s)) ds.$$

and we retain the notations T , $E_{(x,y)}$ and $P_{(x,y)}$. In [4], we were able to prove some weaker estimates:

Theorem ([4]) *For $\alpha \geq 0$, $\nu := 1/(\alpha + 2)$, $x \leq 0$ and $y = 0$, there exist positive constants $k'(\alpha)$, $k''(\alpha)$ such that*

$$(1.6) \quad k'(\alpha)|x|^{\nu/2}t^{-1/4} < P_{(x,0)}[T > t] < k''(\alpha)|x|^{\nu/2}t^{-1/4}$$

for all small $|x|^{\nu/2}t^{-1/4}$.

The present paper proves the existence of the limit value for $|x|^{-\nu/2}t^{1/4}P_{(x,0)}[T > t]$, and more generally, we obtain similar results for some additive functionals that are not odd, or symmetric. We shall observe that the exponent $-1/4$ of time parameter in the above theorems varies between 0 and $-1/2$ in accordance with the skewness of additive functionals.

There are at least two approaches for our problem: the analytical one using Krein's spectral theory of strings(cf. Kotani-Watanabe[6]) and the probabilistic one based on the excursion theory, among which we mainly take the latter course.

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2 The main theorem

In the remainder of this paper, almost all quantities depend on the parameter $\alpha > -1$ and $\bar{c} > 0$ without any mentioning. Let V be a function on the real line which is positive on $(0, \infty)$ and negative on $(-\infty, 0)$.

$$(2.7) \quad V(x) = x^\alpha \text{ for } x > 0; V(0) = 0; V(x) = -|x|^\alpha/\bar{c} \text{ for } x < 0.$$

We define a diffusion $(X(t), Y(t))$ on \mathbb{R}^2 in a similar way and denote it by the same symbol:

$$(2.8) \quad Y(t) = y + B(t), \quad X(t) = x + \int_0^t V(Y(s)) ds.$$

We denote by $E_{(x,y)}$ and $P_{(x,y)}$ the expectation and the probability measure for the diffusion started at $(x, y) \in \mathbb{R}^2$. Let T be the first hitting time to the positive y -axis as usual. Let T_0^Y be the first hitting time to x -axis:

$$(2.9) \quad T_0^Y = \inf\{t \geq 0; Y(t) = 0\}$$

and for $\kappa, \lambda, \mu \geq 0$, $x \leq 0$, $y \in \mathbb{R}$ define $u_0(x, y) \equiv u_0(x, y; \mu)$ by

$$(2.10) \quad u_0(x, y) = E_{(x,y)}[\exp(-\mu T)]$$

and more generally $u(x, y) \equiv u(x, y; \kappa, \lambda, \mu)$ by

$$(2.11) \quad u(x, y) = E_{(x,y)} \left[\exp \left\{ -\mu T - \lambda X(T_0^Y \circ \theta_T) - \kappa(T_0^Y \circ \theta_T - T) \right\} \right]$$

$$(2.12) \quad \equiv E_{(x,y)} [\exp \{-\mu T\} F(\lambda V + \kappa; Y(T))]$$

here θ_t is the usual shift operator on the path space and the function $F(\lambda V + \kappa; z)$ is the unique bounded solution of $\frac{1}{2}F''(z) = (\lambda V(z) + \kappa)F(z)$ on $(0, \infty)$ with $F(0) = 1$. It is clear that $0 \leq u(x, y) \leq 1$, $u(0, 0) = 1$ and $u_0(0, y) = 1$ for $y > 0$.

Theorem 1 Define positive numbers $0 < \nu < 1$, $0 < \rho < 1$ by $\nu = 1/(\alpha + 2)$ and $\bar{c}^\nu \sin \pi\nu(1 - \rho) = \sin \pi\nu\rho$. Then for $\kappa, \lambda, \mu \geq 0$ there exists a positive constant $C(\kappa, \lambda, \mu)$ such that it holds

$$(2.13) \quad 1 - u(x, 0; \sigma\kappa, \sigma^{1/2\nu}\lambda, \sigma\mu) \sim |x|^{\nu\rho} C(\kappa, \lambda, \mu) \sigma^{\rho/2}$$

as $|x|^{\nu\rho} \sigma^{\rho/2}$ tends to 0.

Corollary 1 It holds that

$$(2.14) \quad 1 - u_0(x, 0; \mu) \sim C(0, 0, 1) |x|^{\nu\rho} \mu^{\rho/2}$$

as $|x|^{\nu\rho} \mu^{\rho/2}$ tends to 0, in other words,

$$(2.15) \quad P \left[\int_0^s V(B(u)) du < |x| \text{ for all } 0 \leq s \leq t \right] \sim \frac{C(0, 0, 1)}{\Gamma(1 - \rho/2)} |x|^{\nu\rho} t^{-\rho/2}$$

as $|x|^{\nu\rho} t^{-\rho/2}$ tends to 0.

We have, more generally, the following theorem.

Theorem 2 There exist a positive constant $\tilde{C}(x, y)$ such that, for $\kappa, \lambda, \mu \geq 0$, $x \leq 0$ and $y \in \mathbb{R}$, it holds that

$$(2.16) \quad 1 - u(\bar{\sigma}x, \bar{\sigma}^{1/\nu}y; \sigma\kappa, \sigma^{1/2\nu}\lambda, \sigma\mu) \sim \tilde{C}(x, y) \bar{\sigma}^{\nu\rho} C(\kappa, \lambda, \mu) \sigma^{\rho/2}$$

for positive $\sigma, \bar{\sigma}$ such that $\bar{\sigma}^{\nu\rho} \sigma^{\rho/2}$ tends to 0, where $C(\kappa, \lambda, \mu)$ is the same as in Theorem 1 and $\tilde{C}(x, y)$ is given by

$$\tilde{C}(x, y) = \frac{|x|^{1-\nu+\nu\rho} \exp \left\{ -2\nu^2(y^+)^{1/\nu} / |x| \right\}}{\Gamma(\nu)} \int_0^\infty dt e^{-t} \left(|x|t + \frac{2\nu^2}{\bar{c}} |y^-|^{1/\nu} \right)^{\nu\rho} (|x|t + 2\nu^2(y^+)^{1/\nu})^{-1+\nu-\nu\rho}.$$

Remark 1. The function u has the following scaling property: for any $c > 0$

$$\begin{aligned} u(x, y; \kappa, \lambda, \mu) &\equiv u(c^{1/\nu}x, cy; c^{-2}\kappa, c^{-1/\nu}\lambda, c^{-2}\mu) \\ &\equiv E_{(c^{1/\nu}x, cy)} \left[\exp \left\{ -c^{-2}\mu T \right\} F(\lambda V + \kappa; c^{-1}Y(T)) \right] \end{aligned}$$

and the theorems are stated accordingly.

Remark 2. The distribution of $Y(T)$ under $P_{(0,y)}$ is known explicitly by Rogers-Williams[10], see also McGill[7]: For $y < 0$,

$$(2.17) \quad P_{(0,y)}[Y(T) \in d\eta] = \frac{\sin \pi\nu\rho}{\pi\nu\bar{c}^{\nu\rho}} |y|^\rho \eta^{1/\nu-1-\rho} \frac{d\eta}{\bar{c}^{-1}|y|^{1/\nu} + \eta^{1/\nu}}, \text{ on } \{\eta > 0\}.$$

Their methods do not seem to cover, however, the cases involving the stopping time T .

Remark 3. We denote by $\tau(t)$ the inverse of the local time of Y at 0. It is well known that $\int_0^{\tau(t)} V(B_u)du$ is a stable process with index ν and it holds

$$(2.18) \quad P \left[\int_0^{\tau(s)} V(B_u)du < |x| \text{ for all } s \leq t \right] \sim \text{const } |x|^{\nu\rho} t^{-\rho}$$

as $|x|^{\nu\rho} t^{-\rho}$ tends to 0. See e.g. Bertoin[2]. This result has the same order as our Corollary 1 in the space variable $|x|$, but differs in the time variable t .

Remark 4. Note also that ρ is equal to the probability $P[\int_0^{\tau(t)} V(B_u)du > 0]$ independent of t , which can be proved using the result by Zolotarev[13].

3 Proof of Theorem 1

We denote by $L(t)$ the local time at 0 of $Y(T)$: $L_t = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{(-\varepsilon,\varepsilon)}(Y(u))du$ and by τ_t or $\tau(t)$ the right continuous inverse of L_t : $\tau_t \equiv \tau(t) = \inf\{u > 0; L_u > t\}$. Let n^+ and n^- be the Itô measure for positive and negative excursions respectively, and set $n = n^+ + n^-$.

We denote a general excursion by $\varepsilon = (\varepsilon_t; t \geq 0)$, its lifetime by $\zeta = \zeta(\varepsilon)$ and define a random time for $x \leq 0$,

$$(3.19) \quad T(\varepsilon, x) = \inf \left\{ 0 \leq t \leq \zeta; x + \int_0^t V(\varepsilon_s)ds \geq 0 \right\}.$$

We set $T(\varepsilon, x) = \zeta$ if there is no such t . It follows, through calculations of the Lévy measure of $X(\tau_t)$, that

$$\begin{aligned} n^+ \left[1 - \exp \left\{ -\lambda \int_0^\zeta V(\varepsilon_s)ds \right\} \right] &= \frac{\nu^{2\nu-1} 2^\nu \Gamma(1-\nu)}{\Gamma(\nu)} \lambda^\nu, \\ n^- \left[1 - \exp \left\{ \lambda \int_0^\zeta V(\varepsilon_s)ds \right\} \right] &= \frac{\nu^{2\nu-1} 2^\nu \Gamma(1-\nu)}{\Gamma(\nu)} (\lambda/\bar{c})^\nu \end{aligned}$$

for positive λ and that

$$(3.20) \quad n^+ \left[\int_0^\zeta V(\varepsilon_s)ds > \xi \right] = \frac{\nu^{2\nu-1} 2^\nu}{\Gamma(\nu)} \xi^{-\nu},$$

$$(3.21) \quad n^- \left[\int_0^\zeta V(\varepsilon_s)ds < -\xi \right] = \frac{\nu^{2\nu-1} 2^\nu}{\Gamma(\nu)} (\bar{c}\xi)^{-\nu}$$

for positive ξ .

We have an integral equation for $u(x, 0)$.

Lemma 1 We extend u for positive x by $u(x, 0) = 1$. Then it holds for $x < 0$

$$(3.22) \quad n \left[u \left(x + \int_0^\zeta V(\varepsilon_s) ds, 0 \right) - u(x, 0) \right] \\ = n \left[u \left(x + \int_0^\zeta V(\varepsilon_s) ds, 0 \right) \{ 1 - e^{-\mu T(\varepsilon, x)} F(\lambda V + \kappa; \varepsilon(T(\varepsilon, x))) \} \right].$$

Proof. Let $F(z) = F(\lambda V + \kappa; z)$. Define $a \vee b = \max(a, b)$, $a \wedge b = \min(a, b)$ and

$$M(t) = u(X(\tau_t \wedge T), Y(\tau_t \wedge T)) e^{-\mu(\tau(t) \wedge T)}$$

then $u(x, 0) = E_{(x,0)}[M(t)]$ holds for any $t \geq 0$ and $x \leq 0$.

If $T \leq \tau_{t-}$ then $M(t) - M(t-) = 0$.

If $\tau_{t-} < T \leq \tau_t$ then τ_t is the first hitting time of 0 by Y after T , i.e., $\tau_t = T_0^Y \circ \theta_T$. In this case,

$$M(t) - M(t-) = e^{-\mu T} F(Y(T)) - e^{-\mu\tau(t-)} u(X(\tau_{t-}), 0)$$

and $T - \tau_{t-} = T(\varepsilon, X(\tau_{t-}))$, here ε denotes the excursion started at τ_{t-} and ended at τ_t : $\varepsilon_s = Y(s + \tau_{t-})$, $s < \tau_t - \tau_{t-}$.

Finally if $\tau_t < T$ then

$$M(t) - M(t-) = e^{-\mu\tau(t)} u((X(\tau_t), 0) - e^{-\mu\tau(t-)} u((X(\tau_{t-}), 0)$$

and $\tau_t - \tau_{t-} = \zeta(\varepsilon)$. The master formula of excursion theory (cf. Revuz-Yor[11] page 439) tells us

$$E_{(x,0)} [M(s) - M(0)] = \int_0^s dt E_{(x,0)} \left[\right. \\ e^{-\mu\tau(t-)} n^+ \left[e^{-\mu T(\varepsilon, X(\tau_{t-}))} F(\varepsilon(T(\varepsilon, X(\tau_{t-}))) - u((X(\tau_{t-}), 0); T(\varepsilon, X(\tau_{t-})) < \zeta) \right] \\ \left. + e^{-\mu\tau(t-)} n \left[e^{-\mu\zeta} u \left(X(\tau_{t-}) + \int_0^\zeta V(\varepsilon_s) ds, 0 \right) - u(X(\tau_{t-}), 0); T(\varepsilon, X(\tau_{t-})) = \zeta \right] \right].$$

Recalling $u(x, 0) = E_{(x,0)} [M(s)]$, we know that the integrand of the right hand side is identically null.

Since $X(\tau_t)$ is a ν -stable Lévy process, the paths are right continuous and the transition density decays as t goes to 0 uniformly outside any neighborhood of $X(0)$. The proof is hence complete if we show

$$n^+ \left[e^{-\mu T(\varepsilon, x)} F(\varepsilon(T(\varepsilon, x))) - u(x, 0); T(\varepsilon, x) < \zeta \right] \\ + n \left[e^{-\mu\zeta} u \left(x + \int_0^\zeta V(\varepsilon_s) ds, 0 \right) - u(x, 0); T(\varepsilon, x) = \zeta \right],$$

which coincides with

$$n \left[u \left(x + \int_0^\zeta V(\varepsilon_s) ds, 0 \right) e^{-\mu T(\varepsilon, x)} F(\varepsilon(T(\varepsilon, x))) - u(x, 0) \right],$$

is continuous on $\{x < 0\}$ and its absolute value is dominated by an integrable function plus a constant. We need the following lemma.

Lemma 2 The function $u(x, 0)$ is infinitely differentiable on $\{x < 0\}$ and $\frac{\partial u}{\partial x}$ is positive.

Moreover, if $\alpha \geq 0$, $\nu \leq 1/2$ then $\frac{\partial^2 u}{\partial x^2}$ is positive, in particular $\frac{\partial u}{\partial x} = o(1/|x|)$ as $x \rightarrow -\infty$. If $-1 < \alpha < 0$, $1/2 < \nu < 1$ then $\frac{\partial u}{\partial x} = O(1/|x|)$ as $x \rightarrow -\infty$.

Remark. It can be proved for any $m > 0$ and $n > 0$, $\frac{\partial^m u}{\partial x^n} = O(|x|^{-m})$. However the statement above is sufficient for our purpose.

Proof. Let $F(z) = F(\lambda V + \kappa; z)$. By the scaling property it holds that

$$u(x, 0; \kappa, \lambda, \mu) = E_{(-1,0)}[e^{-|x|^{2\nu}\mu T} F(|x|^\nu Y(T))].$$

Since $F(z)$ decays exponentially as $z \rightarrow \infty$, the differentiation inside the expectation can be justified. Hence

$$\begin{aligned} \frac{\partial u}{\partial x}(x, 0; \kappa, \lambda, \mu) &= E_{(-1,0)} \left[2\nu|x|^{2\nu-1}\mu T e^{-|x|^{2\nu}\mu T} F(|x|^\nu Y(T)) \right. \\ &\quad \left. + e^{-|x|^{2\nu}\mu T} \nu|x|^{\nu-1} Y(T) (-F'(|x|^\nu Y(T))) \right]. \end{aligned}$$

Here $-F'(z)$ is a positive decreasing function. The integrand is obviously positive and if $2\nu - 1 \leq 0$ it is strictly decreasing in $|x|$. If $\nu > 1/2$, we use again the scaling property:

$$\frac{\partial u}{\partial x}(x, 0; \kappa, \lambda, \mu) = \frac{1}{|x|} E_{(x,0)} \left[2\nu\mu T e^{-\mu T} F(Y(T)) + e^{-\mu T} \nu Y(T) (-F'(Y(T))) \right].$$

The integrand is a bounded function of two variables T and $Y(T)$. \square

End of the proof of Lemma 1. The difference between

$$n \left[u \left\{ x + \int_0^\zeta V(\varepsilon_s) ds, 0 \right\} e^{-\mu T(\varepsilon, x)} F(\varepsilon(T(\varepsilon, x))) - u(x, 0) \right]$$

and $n \left[u(x + \int_0^\zeta V(\varepsilon_s) ds, 0) - u(x, 0) \right]$ is bounded since it is dominated by

$$\begin{aligned} &n \left[1 - e^{-\mu T(\varepsilon, x)} F(\varepsilon(T(\varepsilon, x))) \right] \\ &\equiv n \left[1 - \exp \left\{ -\mu T(\varepsilon, x) - \kappa(\zeta - T(\varepsilon, x)) - \lambda \int_{T(\varepsilon, x)}^\zeta V(\varepsilon_s) ds \right\} \right], \end{aligned}$$

which is also bounded by

$$\begin{aligned} &n \left[1 - \exp \left\{ -(\mu \vee \kappa)\zeta - \lambda \int_0^\zeta V(\varepsilon_s) \vee 0 ds \right\} \right] \\ &< n \left[1 - \exp \{ -(\mu \vee \kappa)\zeta \} \right] + n \left[1 - \exp \left\{ -\lambda \int_0^\zeta V(\varepsilon_s) \vee 0 ds \right\} \right] < \infty. \end{aligned}$$

We divide $n \left[u(x + \int_0^\zeta V(\varepsilon_s) ds, 0) - u(x, 0) \right]$ into two parts.

$n \left[|u(x + \int_0^\zeta V(\varepsilon_s) ds, 0) - u(x, 0)|; |\int_0^\zeta V(\varepsilon_s) ds| > 1 \right]$ is bounded because $0 \leq u \leq 1$ and $n \left[|\int_0^\zeta V(\varepsilon_s) ds| > 1 \right] < \infty$ by (3.20) and (3.21). Integrating by parts,

$$\begin{aligned} & n \left[\left| u \left\{ x + \int_0^\zeta V(\varepsilon_s) ds, 0 \right\} - u(x, 0) \right|; \left| \int_0^\zeta V(\varepsilon_s) ds \right| < 1 \right] \\ &= \int_0^1 d\xi \frac{\partial u}{\partial x}(x + \xi, 0) n^+ \left[\xi < \int_0^\zeta V(\varepsilon_s) ds < 1 \right] \\ &\quad - \int_{-1}^0 d\xi \frac{\partial u}{\partial x}(x + \xi, 0) n^- \left[-1 < \int_0^\zeta V(\varepsilon_s) ds < \xi \right], \end{aligned}$$

which is integrable, since it is a convolution of two integrable functions $\frac{\partial u}{\partial x}$ and $n^\pm[\xi < |\int_0^\zeta V(\varepsilon_s) ds|]$.

The continuity also follows using the above arguments since $T(\varepsilon, x)$ and $\varepsilon(T(\varepsilon, x))$ are continuous in x . \square

Putting the explicit value of $n^\pm[\xi < |\int_0^\zeta V(\varepsilon_s) ds|]$ into the left side of Lemma 1, we have

$$\begin{aligned} & n \left[u(x + \int_0^\zeta V(\varepsilon_s) ds, 0) - u(x, 0) \right] \\ &= \frac{\nu^{2\nu-1} 2^\nu}{\Gamma(\nu)} |x|^{-\nu} \left(\{1 - u(x, 0)\} - \nu \int_0^1 |1 - t|^{-\nu-1} (\{1 - u(xt, 0)\} - \{1 - u(x, 0)\}) dt \right. \\ &\quad \left. - \nu \int_1^\infty \frac{|1 - t|^{-\nu-1}}{\bar{c}^\nu} (\{1 - u(xt, 0)\} - \{1 - u(x, 0)\}) dt \right) \end{aligned}$$

The integral transform on this right side can be inverted.

Lemma 3 For $v \in C^1((-\infty, 0))$ such that $\frac{dv}{dx}$ is integrable, define $Lv(x) \in C((-\infty, 0))$ by

$$Lv(x) = \nu \int_0^1 |1 - t|^{-\nu-1} (v(xt) - v(x)) dt + \nu \int_1^\infty \frac{|1 - t|^{-\nu-1}}{\bar{c}^\nu} (v(xt) - v(x)) dt.$$

If $v(x) - Lv(x) = f(x)$ then it holds

$$(3.23) \quad v(x) = \int_{-\infty}^0 \frac{dt}{|t|} f(t) G\left(-\frac{|t|}{|x|}\right)$$

with a function $G(b)$ defined by

$$\begin{aligned} G(b) &= \tilde{G}(-\log(-b)), \quad b < 0, \\ \tilde{G}(\xi) &= \lim_{A \rightarrow +\infty} \int_{-A}^A \frac{e^{-i\xi x}}{2\pi r(ix)} dx, \quad \xi \in \mathbb{R} \\ r(z) &= \frac{1}{\Gamma(\nu) \sin \pi \nu \rho} \Gamma(1 - z) \Gamma(\nu + z) \sin \pi(\nu \rho + z), \quad z \in \mathbb{C} \end{aligned}$$

and with $\rho \in (0, 1)$ defined by $\bar{c}^\nu = \frac{\sin \pi \nu \rho}{\sin \pi \nu (1 - \rho)}$.

Moreover, if $\int_{-\infty}^0 |x|^{-1-\nu\rho} |f(x)| dx < \infty$ then

$$(3.24) \quad \lim_{x \rightarrow -0} \frac{v(x)}{|x|^{\nu\rho}} = \frac{\Gamma(\nu) \sin \pi\nu\rho}{\pi\nu\rho\Gamma(\nu\rho)\Gamma(\nu - \nu\rho)} \int_{-\infty}^0 |x|^{-1-\nu\rho} f(x) dx.$$

Remark. The Markov process associated to L turns into a Lévy process by taking the logarithm. This property enables us to calculate $\tilde{G}(\xi)$ and $r(z)$ explicitly. We prove this lemma at the end of this section.

Proof of Theorem 1. We set

$$f(x) = \frac{\Gamma(\nu)|x|^\nu}{\nu^{2\nu-1}2^\nu} n \left[u \left(x + \int_0^\zeta V(\varepsilon_s) ds, 0 \right) (1 - e^{-\mu T(\varepsilon, x)}) F(\varepsilon(T(\varepsilon, x))) \right]$$

for $x < 0$. It is obvious that $f(x)$ is positive everywhere and continuous. As we saw in the proof of Lemma 2, $n[1 - e^{-\mu T(\varepsilon, x)} F(\varepsilon(T(\varepsilon, x)))]$ is bounded, hence $f(x) = O(|x|^\nu)$ as x tends to 0.

We have also

$$f(x) = \frac{\Gamma(\nu)|x|^\nu}{\nu^{2\nu-1}2^\nu} n \left[u \left(x + \int_0^\zeta V(\varepsilon_s) ds, 0 \right) - u(x, 0) \right].$$

By integration by parts, $n \left[|u(x + \int_0^\zeta V(\varepsilon_s) ds, 0) - u(x, 0)|; \frac{|x|}{2} > \left| \int_0^\zeta V(\varepsilon_s) ds \right| \right]$ is dominated by $\text{const} \int_{-|x|/2}^{|x|/2} d\xi |\xi|^{-\nu} \frac{\partial u}{\partial x}(x + \xi, 0)$. It is shown in Lemma 2 that $\frac{\partial u}{\partial x} = O(1/|x|)$ as $x \rightarrow -\infty$, which implies

$$n \left[\left| u \left(x + \int_0^\zeta V(\varepsilon_s) ds, 0 \right) - u(x, 0) \right|; \frac{|x|}{2} > \left| \int_0^\zeta V(\varepsilon_s) ds \right| \right] = O(|x|^{-\nu}).$$

Finally, $n \left[|u(x + \int_0^\zeta V(\varepsilon_s) ds, 0) - u(x, 0)|; \frac{|x|}{2} < \left| \int_0^\zeta V(\varepsilon_s) ds \right| \right]$ is easily dominated by $n \left[\frac{|x|}{2} < \left| \int_0^\zeta V(\varepsilon_s) ds \right| \right] = O(|x|^{-\nu})$ since $0 \leq u(x, 0) \leq 1$ for every x .

Therefore we have shown that $f(x) = O(1)$ as $x \rightarrow -\infty$, hence the integrability of $f(x)$ with respect to $|x|^{-1-\nu\rho} dx$ and the existence of the limit value for $(1 - u(x, 0))/|x|^{\nu\rho}$ as $x \rightarrow -0$.

The statement of the theorem follows from the scaling property of u : for any $c > 0$,

$$u(x, y; c^2\kappa, c^{1/\nu}\lambda, c^2\mu) = u(c^{1/\nu}x, cy; \kappa, \lambda, \mu).$$

Setting $y = 0$, $c = \sqrt{\sigma}$, $u(x, 0; \sigma\kappa, \sigma^{1/2\nu}\lambda, \sigma\mu)$ is equal to $u(\sigma^{1/2\nu}x, 0; \kappa, \lambda, \mu)$, which satisfies $1 - u(\sigma^{1/2\nu}x, 0; \kappa, \lambda, \mu) \sim \text{const} |\sigma^{1/2\nu}x|^{\nu\rho}$ as $\sigma^{1/2\nu}|x|$ tends to 0. \square

Proof of Lemma 3. Define the functions \tilde{v} and \tilde{f} on \mathbb{R} by $\tilde{v}(x) = v(-\exp(-x))$ and $\tilde{f}(x) = f(-\exp(-x))$. Define the integral operators \tilde{L} and \tilde{L}' by

$$(3.25) \quad \begin{aligned} \tilde{L}g(x) &:= \nu \int_0^\infty |1 - e^{-X}|^{-\nu-1} e^{-X} \{g(x+X) - g(x)\} dX \\ &\quad + \frac{\nu}{\tilde{c}^\nu} \int_{-\infty}^0 |1 - e^{-X}|^{-\nu-1} e^{-X} \{g(x+X) - g(x)\} dX \end{aligned}$$

$$(3.26) \quad \begin{aligned} \tilde{L}'g(x) &:= \frac{\nu}{\tilde{c}^\nu} \int_0^\infty |1 - e^X|^{-\nu-1} e^X \{g(x+X) - g(x)\} dX \\ &\quad + \nu \int_{-\infty}^0 |1 - e^X|^{-\nu-1} e^X \{g(x+X) - g(x)\} dX \end{aligned}$$

for $g \in C^1(\mathbb{R} \rightarrow \mathbb{C})$ with integrable $\frac{dg}{dx}$. It is obvious that $\tilde{L}\tilde{v}(x) \equiv Lv(-e^{-x})$.

To prove the lemma, it is sufficient to show that

$$(3.27) \quad \tilde{v}(x) = \int_{-\infty}^{\infty} \tilde{f}(y)\tilde{G}(y-x)dy.$$

We can show, by an standard argument, that \tilde{f}, \tilde{v} and $\tilde{L}\tilde{v}$ belong to $\mathcal{S}'(\mathbb{R})$ and for any $\phi \in \mathcal{S}(\mathbb{R})$ it holds

$$(3.28) \quad (\tilde{v} - \tilde{L}\tilde{v})(\mathcal{F}\phi) = \tilde{v}((1 - \tilde{L}')\mathcal{F}\phi)$$

It is elementary but tedious to verify that

$$(3.29) \quad 1 - r(i\xi) = \frac{\nu}{c^\nu} \int_0^\infty |1 - e^X|^{-\nu-1} e^X \{e^{-i\xi X} - 1\} dX + \nu \int_{-\infty}^0 |1 - e^X|^{-\nu-1} e^X \{e^{-i\xi X} - 1\} dX,$$

$$(3.30) \quad (1 - \tilde{L}')\mathcal{F}\phi(x) = \mathcal{F}[\phi(\xi)r(i\xi)](x), \quad \phi \in \mathcal{S}$$

and that the function $\frac{1}{r(i\xi)}$ on \mathbb{R} is infinitely differentiable and

$$(3.31) \quad r(-i\xi) = \overline{r(i\xi)}, \quad \frac{1}{r(i\xi)} = \frac{\text{const}}{x^\nu} + O(x^{-1-\nu}) \text{ as } x \rightarrow \infty.$$

We next show for any $\chi \in \mathcal{S}(\mathbb{R})$

$$(3.32) \quad \mathcal{F} \left[\frac{\chi(x)}{r(ix)} \right] = \mathcal{F}\chi * \tilde{G}.$$

We start with

$$(3.33) \quad \mathcal{F} \left[\chi(x) \frac{1_{[-A,A]}(x)}{r(ix)} \right] = \frac{1}{\sqrt{2\pi}} \mathcal{F}\chi * \mathcal{F} \left[\frac{1_{[-A,A]}(x)}{r(ix)} \right], \quad A > 0.$$

It is clear that the left side of (3.33) converges to the left side of (3.32) as A tends to ∞ .

The difference between $\tilde{G}(y)$ and $\frac{1}{\sqrt{2\pi}} \mathcal{F} \left[\frac{1_{[-A,A]}(x)}{r(ix)} \right](y)$ is dominated by $c_0 + c_1|y|^{\nu-1}$. To see this, it is sufficient to estimate $\int_A^\infty \frac{\exp(-iyx)}{r(ix)} dx$ for positive A .

By (3.31), $|\int_A^\infty \frac{\exp(-iyx)}{r(ix)} dx|$ is less than $c_2 \int_A^\infty \frac{|\exp(-iyx)|}{x^{1+\nu}} dx + c_3 |\int_A^\infty \frac{\exp(-iyx)}{x^\nu} dx|$ with some positive constants c_2, c_3 . $\int_A^\infty \frac{\exp(-iyx)}{x^\nu} dx = |y|^{\nu-1} \int_{|y|A}^\infty \frac{\exp(-i(\text{sgn } y)x)}{x^\nu} dx$ is dominated by $M|y|^{\nu-1}$ with

$$M = \sup_{\xi > 0} \left| \int_\xi^\infty \frac{\exp(-ix)}{x^\nu} dx \right|.$$

Since $\frac{1}{\sqrt{2\pi}} \mathcal{F} \left[\frac{1_{[-A,A]}(x)}{r(ix)} \right](y)$ converges to $\tilde{G}(y)$ for fixed $y > 0$, the right side of (3.33) also converges to the right side of (3.32) as A tends to ∞ . Hence we have established the equation (3.32).

We now show (3.27). Set $\chi(x) = \phi(x)r(ix) \in \mathcal{S}(\mathbb{R})$. $\tilde{v}(\mathcal{F}\chi)$ is equal to $\tilde{v}((1 - \tilde{L}')\mathcal{F}\phi)$ by (3.30), which is further equal to the left side of (3.28). Since $\tilde{v} - \tilde{L}\tilde{v} = \tilde{f}$, we have

$$\tilde{v}(\mathcal{F}\chi) = \tilde{f}(\mathcal{F}\phi).$$

Here $\mathcal{F}\phi$ is equal to the left side of (3.32). Hence we have

$$\begin{aligned} \tilde{v}(\mathcal{F}\chi) &= \tilde{f}(\mathcal{F}\chi * \tilde{G}) \\ &= \int_{-\infty}^{\infty} dy \tilde{f}(y) \int_{-\infty}^{\infty} d\xi \mathcal{F}\chi(\xi) \tilde{G}(y - \xi) \\ &= \int_{-\infty}^{\infty} d\xi \mathcal{F}\chi(\xi) \int_{-\infty}^{\infty} dy \tilde{f}(y) \tilde{G}(y - \xi). \end{aligned}$$

The both sides of (3.27) are continuous and bounded, and coincide in $S'(\mathbb{R})$, hence they also coincide in $C_b(\mathbb{R})$.

If, moreover, $f(x)$ is integrable with respect to $|x|^{-1-\nu\rho}dx$ on the negative half line, then $\int_{-\infty}^0 \frac{dt}{|t|} f(t) \frac{G(-|t|/|x|)}{|x|^{\nu\rho}}$ converges to the right side of (3.24) as x tends to -0 because of the following asymptotics:

$$\begin{aligned} G(b) &\sim \frac{\Gamma(\nu) \sin \pi\nu\rho}{\pi\nu\rho\Gamma(\nu\rho)\Gamma(\nu - \nu\rho)} |b|^{-\nu\rho} \text{ as } b \rightarrow -\infty, \\ G(b) &\sim \frac{\Gamma(\nu) \sin \pi\nu\rho}{\pi\nu(1 - \rho)\Gamma(\nu\rho)\Gamma(\nu - \nu\rho)} |b|^{1-\nu\rho} \text{ as } b \rightarrow -0, \\ G(b) &\sim O(|b + 1|^{\nu-1}) \text{ as } b \rightarrow -1. \end{aligned}$$

□

4 Proof of Theorem 2.

By the scaling property of Brownian motion, we have for positive c

$$u^c(x, y) := u(c^{1/\nu}x, cy; \kappa, \lambda, \mu) = u(x, y; c^2\kappa, c^{1/\nu}\lambda, c^2\mu).$$

In the previous section it is established with some constant $C > 0$

$$(4.34) \quad 1 - u^c(x, 0) \sim Cc^\rho|x|^{\nu\rho} \text{ as } c \rightarrow +0$$

while in this section we prove

$$(4.35) \quad 1 - u^c(x, y) \sim Cc^\rho\tilde{C}(x, y) \text{ as } c \rightarrow +0$$

for fixed $x \leq 0, y \in \mathbb{R}$.

4.1 The case of the starting point (x, y) in the third quadrant.

Let $Y_0 = y < 0$. In this case Y_t is negative until the hitting time T_0^Y . Applying the optional sampling theorem to the martingale $F(\lambda V_-; |Y_t|) \exp \left\{ \lambda \int_0^t V(Y(s)) ds \right\}$, $\lambda > 0$, we obtain

$$E_{(0,y)}[\exp \{ \lambda X(T_0^Y) \}] = F(\lambda V_-; |y|),$$

where $F(\lambda V_-; z)$ is the unique bounded solution of $\frac{1}{2}F''(z) = \frac{\lambda}{\varepsilon}z^\alpha F(z)$ on $\{z > 0\}$ with $F(0) = 1$.

The function $F(\lambda V_-; z)$ is expressed in terms of modified Bessel functions:

$$F(\lambda V_-; z) = \frac{2\nu^\nu}{\Gamma(\nu)} (2\lambda/\bar{c})^{\nu/2} \sqrt{z} K_\nu(2\nu z^{1/2\nu} (2\lambda/\bar{c})^{1/2}).$$

Here $\nu = 1/(2 + \alpha)$ as usual. Using the formula (2.13.42) in Oberhettinger-Badii [9], we can invert the Laplace transform to obtain

$$(4.36) \quad E_{(0,y)}[X(T_0^Y) \in d\xi] = \frac{\nu^{2\nu} 2^\nu |y|}{\Gamma(\nu) \bar{c}^\nu |\xi|^{1+\nu}} \exp\left\{-\frac{2\nu^2 |y|^{1/\nu}}{\bar{c} |\xi|}\right\} d\xi \text{ on } \{\xi < 0\}.$$

It is obvious that the law of $X(T_0^Y)$ under $P_{(x,y)}$ is identical to that of $x + X(T_0^Y)$ under $P_{(0,y)}$.

By the strong Markov property of $(X(t), Y(t))$,

$$\begin{aligned} 1 - u^c(x, y) &= 1 - E_{(x,y)}[u^c(X(T_0^Y), 0) \exp\{-c^2 \mu T_0^Y\}] \\ &= E_{(x,y)}[1 - u^c(X(T_0^Y), 0)] + O(E[1 - \exp\{-c^2 \mu T_0^Y\}]). \end{aligned}$$

We see from (4.35) that $\frac{1 - u^c(x, 0)}{C c^\rho}$ is dominated by $C'|x|^{\nu\rho}$ with some constant C' , and it is well known that $E[1 - \exp\{-c^2 \mu T_0^Y\}] = 1 - \exp\{-\sqrt{2\mu}c|y|\} = O(c)$.

Combining this with the integrability of $|x + X(T_0^Y)|^{\nu\rho}$ we know

$$\lim_{c \rightarrow +0} \frac{1 - u^c(x, y)}{C c^\rho} = E_{(0,y)} [|x + X(T_0^Y)|^{\nu\rho}].$$

Putting (4.36) into the right hand side,

$$\begin{aligned} \tilde{C}(x, y) &= E_{(0,y)} [|x + X(T_0^Y)|^{\nu\rho}] \\ &= \int_0^\infty d\xi (|x| + \xi)^{\nu\rho} \frac{\nu^{2\nu} 2^\nu |y|}{\Gamma(\nu) \bar{c}^\nu |\xi|^{1+\nu}} \exp\left\{-\frac{2\nu^2 |y|^{1/\nu}}{\bar{c} |\xi|}\right\}. \end{aligned}$$

Replacing $\frac{2\nu^2 |y|^{1/\nu}}{\bar{c} |\xi|}$ by t , we obtain

$$\tilde{C}(x, y) = \Gamma(\nu)^{-1} \int_0^\infty dt e^{-t} \left(|x|t + \frac{2\nu^2 |y|^{1/\nu}}{\bar{c}}\right)^{\nu\rho} t^{-1+\nu-\nu\rho}. \quad \square$$

4.2 The case of the starting point (x, y) in the second quadrant.

The function u^c satisfies in the left half plain $\{x < 0\}$ the differential equation

$$\frac{1}{2} \frac{\partial^2 u^c}{\partial y^2} + V(y) \frac{\partial u^c}{\partial y} = c^2 \mu u^c$$

with the boundary condition on the positive y -axis:

$$u^c(0, y) = F(c^{1/\nu} \lambda V + c^2 \kappa; y) \equiv F(\lambda V + \kappa; cy), \quad y > 0, c > 0.$$

Let $U_c(y) = \int_{-\infty}^0 dx e^{zx} u^c(x, y)$, $z \geq 0$. It follows from Theorem 1

$$(4.37) \quad 1/z - U_c(0) = Cc^\rho \Gamma(1 + \nu\rho) z^{-1-\nu\rho} \text{ as } c \rightarrow 0.$$

An integration by parts shows

$$\frac{1}{2} U_c''(y) = (zy^\alpha + c^2\mu)U_c(y) - y^\alpha F(\lambda V + \kappa; cy), \quad y > 0.$$

Let $\phi_c(y)$, $\psi_c(y)$, $F_c(y)$ be the solutions of the equation $\frac{1}{2}f''(y) = (zy^\alpha + c^2\mu)f(y)$ on $(0, \infty)$ determined by the following conditions:

$$\begin{aligned} \phi_c(0) &= 1, & \phi'_c(0) &= 0 \\ \psi_c(0) &= 0, & \psi'_c(0) &= 1 \\ F_c(0) &= 1, & F_c(y) &\text{ is bounded, i.e., } F_c(y) = F(zV + c^2\mu; y). \end{aligned}$$

Let $\phi_0(y)$, $\psi_0(y)$, $F_0(y)$ be the solutions of $\frac{1}{2}f''(y) = zy^\alpha f(y)$ normalized similarly.

We have by the method of variation of constants that

$$\begin{aligned} U_c(y) &= U_c(0)F_c(y) + 2F_c(y) \int_0^y \psi_c(\xi)\xi^\alpha F(\lambda V + \kappa; c\xi)d\xi \\ &\quad + 2\psi_c(y) \int_y^\infty F_c(\xi)\xi^\alpha F(\lambda V + \kappa; c\xi)d\xi. \end{aligned}$$

Since $F(\lambda V + \kappa; c\xi)$ is a convex decreasing function it holds the inequality $0 < 1 - F(\lambda V + \kappa; c\xi) < \left| \frac{dF}{d\xi}(\lambda V + \kappa; 0) \right| c\xi$. Hence we have, for each fixed $y > 0$,

$$\begin{aligned} \int_0^y \psi_c(\xi)\xi^\alpha F(\lambda V + \kappa; c\xi)d\xi &= \int_0^y \psi_c(\xi)\xi^\alpha d\xi + O(c) \\ \int_y^\infty F_c(\xi)\xi^\alpha F(\lambda V + \kappa; c\xi)d\xi &= \int_y^\infty F_c(\xi)\xi^\alpha d\xi + O(c) \end{aligned}$$

as c tends to 0. Noting the differential equation of ψ_c and F_c we obtain

$$\begin{aligned} &2F_c(y) \int_0^y \psi_c(\xi)\xi^\alpha F(\lambda V + \kappa; c\xi)d\xi + 2\psi_c(y) \int_y^\infty F_c(\xi)\xi^\alpha F(\lambda V + \kappa; c\xi)d\xi \\ &= \frac{F_c(y)(\psi'_c(y) - 1) - \psi_c(y)F'_c(y)}{z} + O(c) = \frac{1 - F_c(y)}{z} + O(c). \end{aligned}$$

We need to prove that, for each fixed $y > 0$, $F_c(y) - F_0(y) = O(c)$ as $c \rightarrow 0$. By the Feynmann-Kac formula, $F_c(y) \equiv F(zV + c^2\mu; y)$ is the same as $E_y[\exp(-\int_0^{T_0} (zV(B_s) + c^2\mu)ds)]$. Here T_0 is the first hitting time to 0 by a standard Brownian motion B_s . Now it is clear that $0 < F_0(y) - F_c(y) = E_y[\exp(-\int_0^{T_0} (zV(B_s)ds)(1 - \exp(-c^2\mu T_0))] < E_y[1 - \exp(-c^2\mu T_0)] = 1 - \exp(-cy\sqrt{2\mu}) = O(c)$.

Combining these with (4.37) we have

$$(4.38) \quad 1/z - U_c(y) = Cc^\rho \Gamma(1 + \nu\rho) F(zV; y) z^{-1-\nu\rho} + O(c) \text{ as } c \rightarrow +0.$$

We can conclude by a standard argument that

$$1 - u^c(x, y) \sim Cc^\rho \tilde{C}(x, y) \text{ as } c \rightarrow +0$$

with

$$\int_0^\infty e^{-zx} dx \tilde{C}(x, y) = \Gamma(1 + \nu\rho) F(zV; y) z^{-1-\nu\rho}.$$

Since $F(zV; y) = \frac{2\nu^\nu}{\Gamma(\nu)} (2z)^{\nu/2} \sqrt{y} K_\nu(2\nu y^{1/2\nu} (2z)^{1/2})$, we can invert the Laplace transform (see Oberhettinger-Badii [9] (13.45)) to obtain

$$\tilde{C}(x, y) = \frac{\Gamma(1 + \nu\rho) |x|^{1/2 + \nu\rho - \nu/2}}{\Gamma(\nu) 2^{(1-\nu)/2} \nu^{1-\nu} y^{(1-\nu)/2\nu}} \exp\left\{-\frac{\nu^2 y^{1/\nu}}{|x|}\right\} W_{\frac{\nu}{2} - \frac{1}{2} - \nu\rho, \frac{\nu}{2}}(2\nu^2 y^{1/\nu} / |x|)$$

where $W_{\kappa, \mu}(z)$ is a Whittaker function defined by (see Abramowitz-Stegun [1] (13.1.33) and (13.2.5))

$$W_{\kappa, \mu}(z) = \frac{z^{1/2 + \mu} e^{-z/2}}{\Gamma(1/2 + \mu - \kappa)} \int_0^\infty dt e^{-zt} t^{-1/2 + \mu - \kappa} (1+t)^{\mu + \kappa - 1/2}.$$

Replacing $2\nu^2 y^{1/\nu} t / |x|$ by t , we obtain

$$\tilde{C}(x, y) = \frac{|x|^{1-\nu+2\nu\rho} \exp\{-2\nu^2 y^{1/\nu} / |x|\}}{\Gamma(\nu)} \int_0^\infty dt e^{-t} t^{\nu\rho} (|x|t + 2\nu^2 y^{1/\nu})^{-1+\nu-\nu\rho}. \quad \square$$

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