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A Unified Approach to Several Inequalities for Gaussian and Diffusion Measures

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Abstract

This paper presents a simple unified approach to several inequalities for Gaussian and diffusion measures. They include hypercontractive inequalities, logarithmic Sobolev inequalities, FKG inequalities, and correlation inequalities.

1 Introduction

This paper is concerned with several inequalities associated with Gaussian measures and measures generated by diffusions. We are mainly interested in hypercontractive inequalities, Poincaré inequalities (spectral gap inequalities), logarithmic Sobolev inequalities, FKG inequalities, and correlation inequalities. A unified approach is developed to produce all these inequalities at the same time. This approach is the so-called semigroup approach [9], [10], [6]. However, one usually obtains hypercontractive inequalities from logarithmic Sobolev inequalities. An interesting point of this paper is to illustrate that the hypercontractive inequalities can be obtained in a somewhat simpler way. We prove hypercontractivity by differentiating an auxiliary function once and (while if we differentiate it twice we obtain the logarithmic Sobolev inequalities). This approach is natural since it is well-known that the logarithmic Sobolev inequality is the infinitesimal form of hypercontractivity. If we are only interested in Gaussian measures, our approach for hypercontractivity is as simple as the famous Neveu approach ([11], see also [7]).

2 Hypercontractivity for Diffusions

Let $(E, \mathcal{E}, \mu)$ be a measure space and let $L^2(\mu)$ be the set of all square integrable functions from E to $\mathbb{R}$. Let L be the Markov generator associated with a continuous Markov semigroup P_t in $L^2(\mu)$. Denote $\mu(f) = \int_E f(x)\mu(dx)$.

Assumptions: *μ is invariant with respect to L and $(P_t)_{t\geq 0}$. There is a nice algebra $\mathcal{A}$ of bounded functions on E which is dense in the L^2 domain of L, stable by L, by P_t and by the action of composition with C^∞ real functions which are 0 at 0. Let also $\mathcal{A}$ be dense in $L^p(\mu)$ for any $1 < p < \infty$. P_t is ergodic in the sense that for any f, $P_t f \to \mu(f)$ μ-almost surely.*

Following P.-A. Meyer, we introduce the so-called "carré du champ" operator Γ as the symmetric bilinear form on $\mathcal{A}$ defined by

$$\Gamma(f,g) = \frac{1}{2}\{L(fg) - fLg - gLf\}, \quad \forall\, f,g \in \mathcal{A}. \tag{2.1}$$

Denote $\Gamma(f) = \Gamma(f,f)$. It is well-known that Γ is positive-definite.

We shall consider a Markov semigroup whose generator is a diffusion in the following sense:

For every C^∞ function ψ on $\mathbb{R}^n$, and for every finite family $F = (f_1, \cdots, f_n)$, where $f_1, \cdots, f_n \in \mathcal{A}$,

$$L\psi(F) = \sum_{i=1}^{n} \frac{\partial \psi}{\partial x_i}(f_1, \cdots, f_n)L(f_i) + \sum_{i,j=1}^{n} \frac{\partial^2 \psi}{\partial x_i \partial x_j}(f_1, \cdots, f_n)\Gamma(f_i, f_j). \tag{2.2}$$

The iterated "carré du champ" operator is defined as

$$\Gamma_2(f,g) := \frac{1}{2}\{L\Gamma(f,g) - \Gamma(f,Lg) - \Gamma(g,Lf)\}, \ \forall f,g \in \mathcal{A}. \tag{2.3}$$

Let $\mathrm{Rg}(f)$ denote the closure of the range of f.

The main resut of this work is the following statement.

Theorem 2.1 *Let R, τ be fixed real numbers. Let ϕ and ψ be C^∞ functions on an open interval $J \subset \mathbb{R}$. If $\Gamma_2(f) \geq R\Gamma(f)$, $\forall f \in \mathcal{A}$, and if $\phi(x) \geq 0, \psi(x) \geq 0, \phi''(x) \leq 0, \psi''(x) \leq 0$, and*

$$\left(e^{-R\tau}\phi'(x)\psi'(y)\right)^2 \leq \phi(x)\phi''(x)\psi(y)\psi''(y), \quad \forall\, x,y \in J, \tag{2.4}$$

then for all f, $g \in \mathcal{A}$ such that $\mathrm{Rg}(P_t(f)) \subset J$, $\mathrm{Rg}(P_t(g)) \subset J$,

$$\mu\{\psi(g)\ P_\tau(\phi(f))\} \leq \phi(\mu(f))\psi(\mu(g)). \tag{2.5}$$

Proof It is easy to see that for every such ϕ, and $f \in \mathcal{A}$ with $\mathrm{Rg}(f) \subset J$, $\phi(f)$ is well-defined and (2.2) holds. Denote $F_t = P_t f$ and $G_t = P_t g$. For any $T > 0$, introduce an auxiliary function of t,

$$h_t := P_{T-t}\{[P_\tau(\phi(F_t))]\psi(G_t)\}, \quad 0 \leq t \leq T.$$

Let us compute its derivative. Let $L_t = L - \frac{\partial}{\partial t}$. By (2.2) it is easy to check that for any f_t, g_t defined on $\mathbb{R}_+ \times E$,

$$L_t\phi(f_t) = \phi''(f_t)\Gamma(f_t), \quad L_t(f_t g_t) = f_t L_t g_t + g_t L_t f + 2\Gamma(f_t, g_t), \tag{2.6}$$

Set $\Phi = \phi(F_t)$, $\Psi = \psi(G_t)$, $\Phi' = \phi'(F_t)$, $\Psi' = \psi'(G_t)$, $\Phi'' = \phi''(F_t)$, and $\Psi'' = \psi''(G_t)$. By (2.6),

$$\begin{aligned}\frac{dh_t}{dt} &= -P_{T-t}\{L_t[(P_\tau(\Phi))\Psi]\}\\ &= -P_{T-t}\{L_t(P_\tau(\Phi))\Psi + P_\tau(\Phi)L_t(\Psi) + 2\Gamma(P_\tau(\Phi),\Psi)\}\\ &= -P_{T-t}\{P_\tau[\Phi''\Gamma(F_t)]\Psi + P_\tau[\Phi]\Psi''\Gamma(G_t) + 2\Gamma(P_\tau(\Phi),\Psi)\}\,. \quad (2.7)\end{aligned}$$

Since Γ is positive definite, we have that

$$|\Gamma(P_\tau(\Phi),\Psi)|^2 \leq \Gamma(P_\tau(\Phi))\Gamma(\Psi)\,.$$

It is proved in [1] (page 149) that if $\Gamma_2(f) \geq R\Gamma(f)$ for all $f \in \mathcal{A}$, then

$$\sqrt{\Gamma(P_T f)} \leq e^{-RT}P_T\sqrt{\Gamma(f)}\,, \quad \forall\, f \in \mathcal{A}\,. \tag{2.8}$$

Thus we have

$$\begin{aligned}|\Gamma(P_\tau(\Phi),\Psi)| &\leq \sqrt{P_\tau[\Gamma(\Phi)]}\sqrt{\Gamma(\Psi)} \leq e^{-R\tau}P_\tau\sqrt{\Gamma(\Phi)}\sqrt{\Gamma(\Psi)}\\ &= e^{-R\tau}P_\tau\left[|\Phi'|\sqrt{\Gamma(F_t)}\right]|\Psi'|\sqrt{\Gamma(G_t)}\,.\end{aligned}$$

The last inequality follows from $\Gamma(\phi(f)) = \phi'(f)^2\Gamma(f)$, which is an easy consequence of (2.1) and (2.2). Let us then introduce the operator $\tilde{P}_\tau$ acting on $\mathcal{A}\times\mathcal{A}$ by $\tilde{P}_\tau(f\times g) = P_\tau(f)(x)g(x)$ (where $(f\times g)(x,y) = f(x)g(y)\,,\ x,y \in E$). Since P_τ is positivity preserving, the same is true for $\tilde{P}_\tau$. Then we have

$$\begin{aligned}\frac{dh_t}{dt} \geq\ & -P_{T-t}\{\tilde{P}_\tau[\Phi''\Gamma(F_t)\times\Psi + \Phi\times\Psi''\Gamma(G_t)\\ &+2e^{-R\tau}|\Phi'||\Psi'|\sqrt{\Gamma(F_t)}\times\sqrt{\Gamma(G_t)}]\}\,. \quad (2.9)\end{aligned}$$

If (2.4) holds, then $\frac{dh_t}{dt} \geq 0$. Thus $h_0 \leq h_T$. Computing h_T and h_0, we obtain the inequality

$$P_T\{(P_\tau(\phi(f)))\psi(g)\} \leq P_\tau(\phi(P_T(f)))\psi(P_T g)\,. \tag{2.10}$$

Letting $T \to \infty$, the theorem is proven. □

This theorem is a generalization of the hypercontractivity theorem. Set $||f||_p^p = \mu(|f|^p)$.

Corollary 2.2 *Let $p > 1$ and $q > 1$. If $\Gamma_2(f) \geq R\Gamma(f)$ and if $e^{-2R\tau} \leq \dfrac{p-1}{q-1}$, then $||P_\tau f||_q \leq ||f||_p$, $\forall f \in \mathcal{A}$.*

Proof Let us take $J_\varepsilon = (-\varepsilon/2, +\infty)$, $\phi_\varepsilon(x) = |x+\varepsilon|^{1/p}$, and $\psi_\varepsilon(y) = |y+\varepsilon|^{1/q'}$ with $1/q + 1/q' = 1$. Therefore (2.4) is true for ϕ_ε and ψ_ε if and only if $e^{-2R\tau} \leq (p-1)/(q-1)$. Thus (2.5) holds for ϕ_ε and ψ_ε. Letting $\varepsilon \to 0$, we obtain that for all $f, g \in \mathcal{A}$ and $f, g \geq 0$,

$$\mu\left\{P_\tau(f^{1/p})g^{1/q'}\right\} \leq (\mu(f))^{1/p}(\mu(g))^{1/q'}\,.$$

In the above inequality replacing f by $(f^2+\varepsilon)^{p/2}-\varepsilon^{p/2}$ and g by $(g^2+\varepsilon)^{q'/2}-\varepsilon^{q'/2}$ and then letting $\varepsilon \to 0$, we get

$$\mu((P_\tau|f|)|g|) \le (\mu(|f|^p))^{1/p}(\mu(|g|^{q'}))^{1/q'}, \quad \forall\ f,g\in\mathcal{A}.$$

The corollary is thus established. □

3 Logarithmic Sobolev Inequalities

It is not so easy to compute the second derivative of h_t introduced in the previous section. We shall compute it in the next section for a Gaussian measure. Here we take $\psi(x) = x$ and $\tau = 0$ and compute h_t''. In this case we have $\frac{dh_t}{dt} = P_{T-t}\{\phi''\Gamma(F)\}$. Differentiating this identity once more with respect to t and using (2.2) for $\Psi(x,y) = \phi''(x)y$ and $F_1 = F$ and $F_2 = \Gamma(F)$, we obtain

$$\begin{aligned} \frac{d^2h_t}{dt^2} &= P_{T-t}\{L_t[\phi''\Gamma(F)]\} \\ &= P_{T-t}\left\{\phi^{(4)}\Gamma(F)^2 + \phi''L_t\Gamma(F) + 2\phi^{(3)}\Gamma(F,\Gamma(F))\right\} \\ &= P_{T-t}\left\{\phi^{(4)}\Gamma(F)^2 + 2\phi''\Gamma_2(F) + 2\phi^{(3)}\Gamma(F,\Gamma(F))\right\} \\ &= P_{T-t}\{\mathrm{Tr}(AB) + 2R\phi''\Gamma(F)\}, \end{aligned}$$

where $A = \begin{pmatrix} \phi''/2 & \phi^{(3)} \\ \phi^{(3)} & \phi^{(4)} \end{pmatrix}$ and $B = \begin{pmatrix} 4[\Gamma_2(F) - R\Gamma(F)] & \Gamma(F,\Gamma(F)) \\ \Gamma(F,\Gamma(F)) & \Gamma(F)^2 \end{pmatrix}$. B is positive definite by [1], p.149. If (3.2) below holds, A is also positive definite. Thus we have

$$\frac{d^2h_t}{dt^2} \ge 2RP_{T-t}[\phi''(F)\Gamma(F)] = 2R\frac{dh_t}{dt}. \tag{3.1}$$

This implies that the derivative of $h_t - h_t'\frac{1-e^{-2Rt}}{2R}$ is non-negative. Hence

$$h_T - h_0 \le h_T'(1-e^{-2RT})/(2R).$$

Thus we have

Theorem 3.1 *Let ϕ be a C^∞ function on some open interval $J \subset \mathbb{R}$ satisfying*

$$\phi''(x) \ge 0, \quad \phi^{(4)}(x) \ge 0, \quad 2(\phi^{(3)}(x))^2 \le \phi''(x)\phi^{(4)}(x), \quad \forall x \in J. \tag{3.2}$$

Assume that $\Gamma_2(f) \ge R\Gamma(f)$, $\forall f \in \mathcal{A}$. Then for every $f \in \mathcal{A}$ such that $\mathrm{Rg}(P_t f) \subset J$,

$$P_T(\phi(f)) \le \phi(P_T(f)) + \frac{1-e^{-2RT}}{2R}P_T(\phi''(f)\Gamma(f)), \quad \forall\, T > 0. \tag{3.3}$$

Letting $T \to \infty$, we have

$$\mu(\phi(f)) \le \phi(\mu(f)) + \frac{1}{2R}\mu(\phi''(f)\Gamma(f)). \tag{3.4}$$

Remark 1 *In (3.4) letting $\phi(x) = x^2$, one gets the Poincaré inequality and letting $\phi(x) = x \log x$ (using the argument of Corollary 2.2), one gets the logarithmic Sobolev inequality, see [6]. For the Ornstein-Uhlenbeck semigroup, (2.8) is obvious so that the preceding proof of hypercontractivity for Gaussian measures is indeed rather simple.*

4 Gaussian Measures

For a Gaussian measure, we may also use the heat semigroup [8] instead of the Ornstein-Uhlenbeck semigroup. Let μ be the standard Gaussian measure on $\mathbb{R}^d$. Let P_t be heat kernel semigroup associated with the standard Laplacian: $\frac{\partial P_t}{\partial t} = \Delta P_t$ (We omit the factor 1/2 for simplicity). Let $\Delta_t = \Delta - \frac{\partial}{\partial t}$. Denote by $\langle \cdot, \cdot \rangle$ the inner product of two vectors or the Hilbert-Schmidt product of two matrices. The same computation rule (2.2) applies to Δ_t and moreover, for any functions f and g on $\mathbb{R}^d$ (in this section, f and g always denote C^∞ functions with compact supports except otherwise stated), if $F_t = P_t f$ and $G_t = P_t g$ as before,

$$\begin{aligned} \Delta_t \langle \nabla F_t \,, \nabla G_t \rangle &= \langle \nabla(\Delta_t F_t) \,, \nabla G_t \rangle + \langle \nabla F_t \,, \nabla(\Delta_t G_t) \rangle + \langle \nabla^2 F_t \,, \nabla^2 G_t \rangle \\ &= 2 \langle \nabla^2 F_t \,, \nabla^2 G_t \rangle \,. \end{aligned} \tag{4.1}$$

Fix $T \geq 0$. Consider $h_t = P_{T-t} \{\phi(F_t)\psi(G_t)\}$, $0 \leq t \leq T$. Then it is easy to see that

$$\begin{aligned} \frac{dh_t}{dt} &= -P_{T-t}\{\phi''(F_t)\psi|\nabla F_t|^2 + \phi(F_t)\psi''(G_t)|\nabla G_t|^2 \\ &\quad + \phi'(F_t)\psi'(G_t)\langle \nabla F_t \,, \nabla G_t \rangle\} \,. \end{aligned}$$

If we take $\phi(x) = \psi(x) = x$, and if $\nabla f \geq 0$ and $\nabla g \geq 0$ component-wise, then we see $h_t' \geq 0$. Thus $h_T \geq h_0$. Thus we obtain in this way the classical FKG inequality for Gaussian measures.

Proposition 4.1 *If $\nabla f \geq 0$, $\nabla g \geq 0$ component-wise, then*

$$\mu(f)\mu(g) \leq \mu(fg) \,.$$

Now, compute the second derivative of h_t for general ϕ and ψ. We have that

$$\begin{aligned} \frac{d^2 h_t}{dt^2} &= P_{T-t}\Big\{\Delta_t \left(\phi''(F_t)\psi(G_t)|\nabla F_t|^2\right) + \Delta_t \left(\phi(F_t)\psi''(G_t)|\nabla G_t|^2\right) \\ &\quad + 2\Delta_t \left(\phi'(F_t)\psi'(G_t)\langle \nabla F_t \,, \nabla G_t \rangle\right) \Big\} \,. \end{aligned} \tag{4.2}$$

Apply then (2.2) to $\Psi = \phi''(x_1)\psi(x_2)x_3$ and use (4.1) to compute $\Delta_t |\nabla F_t|^2$. It follows that

$$\begin{aligned} \Delta_t \left(\phi''(F_t)\psi(G_t)|\nabla F_t|^2\right) &= 2\phi''\psi\langle \nabla^2 F_t, \nabla^2 F_t \rangle + \phi^{(4)}\psi|\nabla F_t|^4 \\ &\quad + \phi''\psi''|\nabla F_t|^2|\nabla G_t|^2 + 2\phi^{(3)}\psi'\langle \nabla F_t \,, \nabla G_t \rangle|\nabla F_t|^2 \\ &\quad + 4\phi^{(3)}\psi\langle \nabla F_t \otimes \nabla F_t \,, \nabla^2 F_t \rangle \\ &\quad + 4\phi''\psi'\langle \nabla F_t \otimes \nabla G_t \,, \nabla^2 F_t \rangle \,. \end{aligned}$$

In a similar way we can compute the other terms in (4.2). If we introduce $X^T = [X_1, X_2, X_3, X_4, X_5]$, where

$$X_1 = \nabla F_t \otimes \nabla F_t\,, \quad X_2 = \nabla G_t \otimes \nabla G_t\,, \quad X_3 = \nabla F_t \otimes \nabla G_t\,,$$

$$X_4 = \nabla^2 F_t\,, \quad X_5 = \nabla^2 G_t\,,$$

then we can write

$$\frac{d^2 h_t}{dt^2} = \langle X\,, AX\rangle\,, \tag{4.3}$$

where $A = B + C$, B is a 5×5 matrix with $B_{45} = B_{54} = 2\phi'\psi'$ and the other elements 0 and C is defined as

$$C = \begin{pmatrix} \phi^{(4)}\psi & 2\phi''\psi'' & 2\phi^{(3)}\psi' & 2\phi^{(3)}\psi & 2\phi''\psi' \\ 2\phi''\psi'' & \phi\psi^{(4)} & 2\phi'\psi^{(3)} & 2\phi'\psi'' & 2\phi\psi^{(3)} \\ 2\phi^{(3)}\psi' & 2\phi'\psi^{(3)} & 2\phi''\psi'' & 4\phi''\psi' & 4\phi'\psi'' \\ 2\phi^{(3)}\psi & 2\phi'\psi'' & 4\phi''\psi' & 2\phi''\psi & 0 \\ 2\phi''\psi' & 2\phi\psi^{(3)} & 4\phi'\psi'' & 0 & 2\phi\psi'' \end{pmatrix}. \tag{4.4}$$

If A is positive definite, then $h_t'' \geq 0$. Namely, $h_0 \leq h_T - h_0'$. Thus we proved

Theorem 4.2 *If ϕ and ψ are C^∞ functions on some open interval $J \subset \mathbb{R}$ such that A is positive definite on J, then for all f, g such that $\mathrm{Rg}(P_t(f)) \subset J$, $\mathrm{Rg}(P_t(g)) \subset J$,*

$$\begin{aligned} \mu(\phi(f)\psi(g)) \quad &\leq \quad \phi(\mu(f))\psi(\mu(g)) + \mu\Big\{\phi''(f)\psi(g)|\nabla f|^2 + \phi(f)\psi''(g)|\nabla g|^2 \\ &\quad +2\phi'(f)\psi'(g)\langle \nabla f\,, \nabla g\rangle\Big\}\,. \end{aligned} \tag{4.5}$$

Remark 2 *1) Take $\phi(x) = \psi(x) = x$. If f and g are convex functions and if $\mu(\nabla f) = 0$ and $\mu(\nabla g) = 0$, then (4.5) implies the following correlation inequality [8]:*

$$\mu(fg) \geq \mu(f)\mu(g)\,.$$

2) Let $\psi(x) \equiv 1$ and let ϕ satisfy (3.2). Then it is easy to check that A is positive definite. Thus (4.5) implies

$$\mu(\phi(f)) \leq \phi(\mu(f)) + \mu(\phi''(f)|\nabla f|^2)\,. \tag{4.6}$$

As we mentioned, this implies the Poincaré and the logarithmic Sobolev inequalities. In fact, if $\phi(x) = x^2$, then ϕ satisfies (3.2). (4.6) becomes the Poincaré inequality

$$\mu(|f|^2) \leq \mu(f)^2 + \mu(|\nabla f|^2)\,.$$

If $\phi(x) = x\log^+ x$, then ϕ satisfies (3.2) (using the argument of Corollary 2.2). (4.6) implies the logarithmic Sobolev inequality

$$\mu(f\log f) \leq \mu(f)\log\mu(f) + \mu(\frac{1}{f}|\nabla f|^2)\,, \quad \forall f \geq 0\,.$$

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Note: More complete references can be found in [2], [5].