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# ON SUMS OF IID RANDOM VARIABLES INDEXED BY $N$ PARAMETERS\*

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**Summary.** Motivated by the works of J.L. DOOB and R. CAIROLI, we discuss reverse  $N$ -parameter inequalities for sums of i.i.d. random variables indexed by  $N$  parameters. As a corollary, we derive SMYTHE's law of large numbers.

## 1. INTRODUCTION

For any integer  $N \geq 1$ , let us consider  $\mathbb{Z}_+^N \triangleq \{1, 2, \dots\}^N$  and endow it with the following partial order: for all  $\mathbf{n}, \mathbf{m} \in \mathbb{Z}_+^N$ ,

$$\mathbf{n} \preceq \mathbf{m} \iff n_i \leq m_i, \quad \text{for all } 1 \leq i \leq N.$$

Suppose  $\{X, X(\mathbf{k}); \mathbf{k} \in \mathbb{Z}_+^N\}$  is a sequence of independent, identically distributed random variables, indexed by  $\mathbb{Z}_+^N$ . The corresponding random walk  $S$  is given by:

$$S(\mathbf{n}) \triangleq \sum_{\mathbf{k} \preceq \mathbf{n}} X(\mathbf{k}), \quad \mathbf{n} \in \mathbb{Z}_+^N.$$

According to CAIROLI AND DALANG [CD], for all  $p > 1$ ,

$$\begin{aligned} \mathbb{E} \sup_{\mathbf{n}} \left| \frac{S(\mathbf{n})}{\langle \mathbf{n} \rangle} \right| < \infty &\iff \mathbb{E}[|X|(\log_+ |X|)^N] < \infty, \\ \mathbb{E} \sup_{\mathbf{n}} \left| \frac{S(\mathbf{n})}{\langle \mathbf{n} \rangle} \right|^p < \infty &\iff \mathbb{E}|X|^p < \infty. \end{aligned} \tag{1.1}$$

Here and throughout, for all  $x > 0$ ,

$$\log_+ x \triangleq \begin{cases} \ln(x), & \text{if } x > e \\ 1, & \text{if } 0 < x \leq e \end{cases},$$

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and for all  $\mathbf{n} \in \mathbb{Z}_+^N$ ,  $\langle \mathbf{n} \rangle \triangleq \prod_{j=1}^N n_j$ . When  $N = 1$ , this is classical. In this case, J.L. DOOB has given a more probabilistic interpretation of this fact by observing that  $S(n)/n$  is a reverse martingale; cf. CHUNG [Ch] for this and more. The goal of this note is to show how a quantitative version of the method of DOOB can be carried out, even when  $N > 1$ . Our approach involves projection arguments which are reminiscent of some old ideas of R. CAIROLI; see CAIROLI [Ca], CAIROLI AND DALANG [CD] and WALSH [W].

Perhaps the best way to explain the proposed approach is by demonstrating the following result which may be of independent interest. For related results and a wealth of further references, see [CD], SHORACK AND SMYTHE [S1] and SMYTHE [S2].

**Theorem 1.** *For all  $p > 1$ ,*

$$\mathbb{E} \sup_{\mathbf{n} \in \mathbb{Z}_+^N} \left| \frac{S(\mathbf{n})}{\langle \mathbf{n} \rangle} \right|^p \leq \left( \frac{p}{p-1} \right)^{Np} \mathbb{E} |X|^p. \quad (1.2)$$

Moreover, the corresponding  $L^1$  norm has the following bound:

$$\mathbb{E} \sup_{\mathbf{n} \in \mathbb{Z}_+^N} \left| \frac{S(\mathbf{n})}{\langle \mathbf{n} \rangle} \right| \leq \left( \frac{e}{e-1} \right)^N \left\{ N + \mathbb{E} [ |X| (\log_+ |X|)^N ] \right\}. \quad (1.3)$$

Theorem 1 implies the “hard” half of both displays in eq. (1.1). The easy half is obtained upon observing that for all  $p \geq 1$ ,

$$\mathbb{E} \sup_{\mathbf{n}} \left| \frac{S(\mathbf{n})}{\langle \mathbf{n} \rangle} \right|^p \geq 2^{-p} \mathbb{E} \sup_{\mathbf{n}} \left| \frac{X(\mathbf{n})}{\langle \mathbf{n} \rangle} \right|^p,$$

and directly calculating the above.

An enhanced version of Theorem 1 is stated and proved in Section 2. There, we also demonstrate how to use Theorem 1 together with Banach space arguments to obtain the law of large numbers for  $S(\mathbf{n})$  due to SMYTHE [S2].

## 2. PROOF OF THEOREM 1

I will prove (1.3) of Theorem 1. Eq. (1.2) follows along similar lines. In fact, it turns out to be a lot simpler to prove more. Define for all  $p \geq 0$ ,

$$\Psi_p(x) \triangleq x (\log_+ x)^p, \quad x > 0.$$

I propose to prove the following extension of Theorem 1:

**Theorem 1-bis.** *For all  $p \geq 0$ ,*

$$\mathbb{E} \sup_{\mathbf{n} \in \mathbb{Z}_+^N} \Psi_p \left( \frac{S(\mathbf{n})}{\langle \mathbf{n} \rangle} \right) \leq (p+1)^N \left( \frac{e}{e-1} \right)^N \left\{ N + \mathbb{E} \Psi_{p+N}(|X|) \right\}.$$

Setting  $p \equiv 0$  in Theorem 1-bis, we arrive at Theorem 1.

Let us recall the following elementary fact:

**Lemma 2.1.** *Suppose  $\{M_n; n \geq 1\}$  is a reverse martingale. Then for all  $p > 1$ ,*

$$\mathbb{E} \sup_{n \geq 1} |M_n|^p \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}|M_1|^p. \quad (2.1)$$

For any  $p \geq 0$ ,

$$\mathbb{E} \sup_{n \geq 1} \Psi_p(|M_n|) \leq (p+1) \left(\frac{e}{e-1}\right) \left\{1 + \mathbb{E}\Psi_{p+1}(|X|)\right\}. \quad (2.2)$$

**Proof.** Eq. (2.1) follows from integration by parts and the maximal inequality of DOOB. Likewise, one shows that

$$\mathbb{E} \sup_{n \geq 1} \Psi_p(|M_n|) \leq \left(\frac{e}{e-1}\right) \left\{1 + \mathbb{E}\left[\Psi_p(|M_1|) \ln_+ \Psi_p(|M_1|)\right]\right\}.$$

For all  $x > 0$ ,  $\ln_+ \Psi_p(x) \leq \ln_+ x + p \ln_+ \ln_+ x$ . Eq. (2.2) follows easily.  $\diamond$

Now, each  $\mathbf{n} \in \mathbb{Z}_+^N$  can be thought of as  $\mathbf{n} = (\hat{\mathbf{n}}, n_N)$ , where  $\hat{\mathbf{n}}$  is defined by  $\hat{\mathbf{n}} \triangleq (n_1, \dots, n_{N-1}) \in \mathbb{Z}_+^{N-1}$ . For all  $\mathbf{n} \in \mathbb{Z}_+^N$  and all  $1 \leq j \leq n_N$ , define

$$Y(\hat{\mathbf{n}}, j) \triangleq \frac{1}{\prod_{j=1}^{N-1} n_j} \sum_{i_1=1}^{n_1} \cdots \sum_{i_{N-1}=1}^{n_{N-1}} X(\hat{\mathbf{i}}, j).$$

Clearly,

$$\frac{S(\mathbf{n})}{\langle \mathbf{n} \rangle} = \frac{1}{n_N} \sum_{j=1}^{n_N} Y(\hat{\mathbf{n}}, j), \quad \mathbf{n} \in \mathbb{Z}_+^N. \quad (2.3)$$

Let

$$\mathcal{R}(k) \triangleq \sigma\{X(\mathbf{m}); m_N > k\} \vee \sigma\{S(\mathbf{m}); m_N = k\}, \quad k \geq 1,$$

where  $\sigma\{\dots\}$  represents the ( $\mathbb{P}$ -completed)  $\sigma$ -field generated by  $\{\dots\}$ .

**Lemma 2.2.**  $\{\mathcal{R}(k); k \geq 1\}$  is a reverse filtration indexed by  $\mathbb{Z}_+^1$ .

**Proof.** This means that  $\mathcal{R}(k) \supset \mathcal{R}(k+1)$  — a simple fact.  $\diamond$

**Lemma 2.3.** For all  $\mathbf{n} \in \mathbb{Z}_+^N$ ,

$$\frac{S(\mathbf{n})}{\langle \mathbf{n} \rangle} = \mathbb{E}[Y(\hat{\mathbf{n}}, 1) \mid \mathcal{R}(n_N)].$$

Assuming Lemma 2.3 for the moment, let us prove Theorem 1.

**Proof of Theorem 1-bis.** Without loss of generality, we can and will assume that

$$\mathbb{E}\Psi_{p+N}(|X|) < \infty. \quad (2.4)$$

Otherwise, there is nothing to prove. When  $N = 1$ , the result follows immediately from Lemma 2.1. Our proof proceeds by induction over  $N$ . Suppose Theorem 1-bis holds for all sums of iid random variables indexed by  $\mathbb{Z}_+^{N-1}$  whose incremental distribution is the same as that of  $X$ . We will prove it holds for  $N$ . By Lemma 2.3,

$$\mathbb{E} \sup_{\mathbf{n} \in \mathbb{Z}_+^N} \Psi_p \left( \frac{S(\mathbf{n})}{\langle \mathbf{n} \rangle} \right) \leq \mathbb{E} \sup_{k \geq 1} \Psi_p \left( \mathbb{E}[W \mid \mathcal{R}(k)] \right),$$

where

$$W \triangleq \sup_{n_1, \dots, n_{N-1} \geq 1} |Y(\hat{\mathbf{n}}, 1)|.$$

However,  $\{Y(\hat{\mathbf{n}}, 1); \hat{\mathbf{n}} \in \mathbb{Z}_+^{N-1}\}$  is the average of a random walk indexed by  $\mathbb{Z}_+^{N-1}$  with the same increments as  $S$ . Therefore, by the induction assumption,

$$\mathbb{E} \Psi_p(W) \leq (p+1)^{N-1} \left( \frac{e}{e-1} \right)^{N-1} \left\{ N-1 + \mathbb{E} \Psi_{p+N}(|X|) \right\}. \quad (2.5)$$

In particular,  $\mathbb{E}W < \infty$ . Together with with Lemma 2.1's eq. (2.2), this implies that  $M_k \triangleq \mathbb{E}[W \mid \mathcal{R}(k)]$  is a reverse martingale, By eq. (2.2) of Lemma 2.1,

$$\mathbb{E} \left[ \sup_{\mathbf{n} \in \mathbb{Z}_+^N} \Psi_p \left( \frac{S(\mathbf{n})}{\langle \mathbf{n} \rangle} \right) \right] \leq (p+1) \left( \frac{e}{e-1} \right) \left\{ 1 + \mathbb{E}[\Psi_p(W)] \right\}.$$

Note that  $(p+1)e(e-1)^{-1} \geq 1$ . Therefore, applying (2.5) to this inequality, we obtain Theorem 1-bis.  $\diamond$

**Proof of Lemma 2.3.** Recall (2.3). It remains to show that for  $1 \leq j \leq n_N$ ,

$$\mathbb{E}[Y(\hat{\mathbf{n}}, j) \mid \mathcal{R}(n_N)] = \mathbb{E}[Y(\hat{\mathbf{n}}, 1) \mid \mathcal{R}(n_N)]. \quad (2.6)$$

To this end, we observe that  $\{Y(\hat{\mathbf{n}}, j); 1 \leq j \leq n_N\}$  is a sequence of iid random variables. By exchangeability,

$$\mathbb{E}[Y(\hat{\mathbf{n}}, j) \mid \mathcal{B}(\mathbf{n})] = \mathbb{E}[Y(\hat{\mathbf{n}}, 1) \mid \mathcal{B}(\mathbf{n})], \quad (2.7)$$

where for all  $\mathbf{n} \in \mathbb{Z}^N$ ,

$$\mathcal{B}(\mathbf{n}) \triangleq \sigma\{S(\mathbf{k}); \mathbf{k} \in \mathbb{Z}_+^N \text{ with } k_N = n_N \text{ and } k_j \leq n_j, \text{ for all } 1 \leq j \leq N-1\}.$$

Let  $\mathcal{C}_0(n_N)$  denote the sigma-field generated by  $\{X(\mathbf{k}); k_N > n_N\}$  and define

$$\mathcal{C}(n_N) \triangleq \mathcal{C}_0(n_N) \vee \sigma\{X(\mathbf{k}); k_N = n_N \text{ and for some } 1 \leq j \leq N-1, k_j > n_j\}.$$

It is easy to see that  $\mathcal{B}(\mathbf{n})$  is independent of  $\mathcal{C}(n_N)$  and

$$\mathcal{R}(n_N) = \mathcal{C}(n_N) \vee \mathcal{B}(\mathbf{n}). \quad (2.8)$$

Eq. (2.6) follows from (2.7), (2.8) and the elementary fact that the collection  $\{Y(\hat{\mathbf{n}}, j); 1 \leq j \leq n_N\}$  is independent of  $\mathcal{C}(n_N)$ .  $\diamond$

**Open Problem.\*** Motivated by the proof of Theorem 1-bis — and in the notation of that proof — consider:

$$T(n_N)(\hat{\mathbf{n}}) \triangleq \frac{1}{n_N} \sum_{j=1}^{n_N} Y(\hat{\mathbf{n}}, j).$$

It is easy to see that  $T(n_N)$  is a reverse martingale which takes its values in the space of all sequences indexed by  $\mathbb{Z}_+^{N-1}$ . For all  $\mathbf{n} \in \mathbb{Z}_+^N$  and any two reals  $a < b$ , define  $U_{a,b}(n_N)(\hat{\mathbf{n}})$  to be the total number of upcrossings of the interval  $[a, b]$  before time  $n_N$  of the (real valued) reverse martingale  $k \mapsto T(k)(\hat{\mathbf{n}})$ . Is it true that there exist constants  $C_1$  and  $C_2$  (which depend **only** on  $N$ ) such that

$$\mathbb{E}\left[\sup_{\hat{\mathbf{n}} \in \mathbb{Z}_+^{N-1}} U_{a,b}(n_N)(\hat{\mathbf{n}})\right] \leq C_1 \frac{\mathbb{E}\left[\sup_{\hat{\mathbf{n}} \in \mathbb{Z}_+^{N-1}} |T(1)(\hat{\mathbf{n}}) - a|\right]}{(b-a)^{C_2}}? \tag{2.9}$$

Note that when  $N = 1$ , the supremum is vacuous. In this case, the above holds with  $C_1 = C_2 = 1$  and is DOOB's upcrossing inequality for the reversed martingale  $T$ . If it holds, (2.9) and Theorem 1 together imply SMYTHE's strong law of large numbers; cf. [S2]. The main part of the aforementioned result is the following:

**Theorem 2.** ([S2]) *Suppose*

$$\mathbb{E}[|X|(\log_+ |X|)^{N-1}] < \infty \quad \text{and} \quad \mathbb{E}X = 0. \tag{2.10}$$

*Then almost surely,*

$$\lim_{\langle \mathbf{n} \rangle \rightarrow \infty} \frac{S(\mathbf{n})}{\langle \mathbf{n} \rangle} = 0.$$

**Remark.** Classical arguments show that condition (2.10) is necessary as well.

**Proof.** I will first prove Theorem 2 for  $N = 2$ . Let  $c_0$  denote the collection of all bounded functions  $a : \mathbb{Z}_+^1 \mapsto \mathbb{R}$  such that  $\lim_{k \rightarrow \infty} |a(k)| = 0$ . Topologize  $c_0$  with the supremum norm:  $\|a\| \triangleq \sup_k |a(k)|$ . Then,  $c_0$  is a separable Banach space. Let

$$\xi_j(k) \triangleq \frac{1}{k} \sum_{i=1}^k X(i, j).$$

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\* **Added Note.** Since this article was accepted for publication, we have found the answer to the open problem above to be affirmative.

Note that  $\xi_j$  are i.i.d. random functions from  $\mathbb{Z}_+^1$  to  $\mathbb{R}$ . By Theorem 1, for all  $j \geq 1$ ,  $\mathbb{E}\|\xi_j\| \leq e^2(e-1)^{-2}\{2 + \mathbb{E}[|X| \log_+ |X|]\} < \infty$ . By the classical strong law of large numbers,  $\xi_1, \xi_2, \dots$  are i.i.d. elements of  $c_0$ . The most elementary law of large numbers on Banach spaces will show that as elements of  $c_0$ , almost surely,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \xi_j = 0.$$

See LEDOUX AND TALAGRAND [LT; Corollary 7.10] for this and much more. In other words, almost surely

$$\lim_{n_1 \rightarrow \infty} \frac{1}{n_1} \sum_{i_1=1}^{n_1} X(i_1, i_2) = 0,$$

uniformly over all  $i_2 \geq 1$ . Plainly, this implies the desired result and much more when  $N = 2$ . The general case follows by inductive reasoning; the details are omitted.  $\diamond$

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