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EXPONENTIAL INEQUALITIES FOR BESSEL PROCESSES

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Abstract : Let $R_d^*(t)$ be the supremum at time t of a Bessel process with dimension d . For T a stopping time, Burkholder has compared the expectation of $\left(\frac{R_d^*(T)}{\sqrt{d}}\right)^p$ and $(\sqrt{T})^p$ for $p > 0$. Replacing the function x^p by exponential functions, we obtain some variant of his results.

I - Introduction and notations

Let $(B_s)_{s \geq 0}$ be a linear Brownian motion starting from zero. Let R_d be a positive process such that R_d^2 is a solution of the equation :

$$X_t = 2 \int_0^t \sqrt{|X_s|} dB_s + dt, \quad d > 0;$$

i.e. : R_d is a Bessel process of dimension d starting from 0.

We set : $R_d^*(t) = \sup_{0 \leq s \leq t} R_d(s)$.

Let T be a stopping time with respect to the natural filtration of B .

Burkholder [B] established for any $p > 0$, that $\left(E\left(\frac{R_d^*(T)}{\sqrt{d}}\right)^p / E(\sqrt{T})^p\right)_{d \in \mathbb{N}^*}$ converges to 1, uniformly in T , as d tends to ∞ .

Refinements of this convergence have since been proved by Davis [D].

Making use of Poincaré's Lemma, Yor ([Y], p.55) could prove adequate modifications of these results for other times than stopping times.

A consequence of Burkholder's result is that $\left(E\left(\frac{R_d^*(T)}{d^{1/2}}\right)^p / E(T^{p/2})\right)_{d \in \mathbb{N}^*}$ is uniformly bounded in d and T . In this paper we consider the following question : What happens if the moderate function x^p is replaced by an exponential function ? For example, is there a function F such that the

sequence $(E(\exp(\frac{R_d^*(T)}{d^{1/2}})) / E(F(\sqrt{T})))_{d \geq 0}$ is uniformly bounded in d and T , or even converging as d tends to ∞ .

We will see that the answer is affirmative for the uniform boundedness question and negative for the convergence question.

II - Exponential inequalities for Bessel processes

Theorem 1 :

(i) *There exist two strictly positive constants c and β such that for any $\lambda > 0$ and any d in \mathbb{N}^* , we have for any stopping time T :*

$$E(\exp\{\lambda(\frac{R_d^*(T)}{d^{1/2}})\}) \leq c E(\exp\{\frac{\beta^2}{2} \lambda^2 T\})$$

Moreover β can be taken equal to $2\sqrt{e}$.

(ii) *For any p in $(0,2)$ there exist two strictly positive constants b_p and β_p such that for any $\lambda > 0$ and any d in \mathbb{N}^* , we have for any stopping time T :*

$$E(\exp\{\lambda(\frac{R_d^*(T)}{d^{1/2}})^p\}) \leq b_p E(\exp\{\beta_p \lambda^{\frac{2}{2-p}} T^{\frac{p}{2-p}}\})$$

Moreover β_p can be taken equal to $(\frac{1}{p} - \frac{1}{2}) p^{\frac{2}{2-p}} (4e)^{\frac{p}{2-p}}$.

The proof of Theorem 1 is based on the following result.

Theorem 2. *There exists a strictly positive constant c such that for every stopping time T , every d in \mathbb{N}^* and every $p > 0$, we have :*

$$(1)_p \quad E(R_d^*(T))^p \leq c (2\sqrt{e})^p d^{p/2} E((B_1^*)^p) E(T^{p/2})$$

Proof of Theorem 2 : Jacka and Yor have proved (see [J-Y] section 4) that there exists a constant a_p such that for all stopping times T with respect to the natural filtration of B and every d in \mathbb{N}^* :

$$E(R_d^*(T))^p \leq a_p d^{p/2} E(T^{p/2})$$

with $a_p \leq 2e 2^p (p + \frac{1}{2})^{p/2}$ when $p \geq 2$.

Thus, we are looking for $\beta > 0$ such that there exists $c > 0$ with :

$$c \cdot \beta^p E(B_1^*)^p \geq a_p$$

Since for any $p > 0$: $E(|B_1|^{2p}) = \frac{2^p}{\pi^{1/2}} \Gamma(p + \frac{1}{2})$,

Stirling's asymptotic formula gives the following equivalency :

$$E(|B_1|^{2p}) \underset{p \rightarrow \infty}{\sim} 2^{p+\frac{1}{2}} (p + \frac{1}{2})^p \exp\{-(p + \frac{1}{2})\}$$

Hence, for any $\beta \geq 2\sqrt{e}$, there exists a constant $c > 0$ such that for every $p > 0$

$$c \cdot \beta^p E(B_1^*)^p \geq a_p. \quad \square$$

We note that (1)_p can be rewritten as follows :

$$(2)_p \quad E(R_d^*(T))^p \leq c (2\sqrt{e})^p d^{p/2} E(\tilde{B}_1^*)^p T^{p/2}$$

where \tilde{B} is an independent copy of B .

Summing the inequalities (2)_{np} , n running through \mathbb{N} and p being a fixed value in $(0, 2)$, we obtain the following result :

$$E(\exp\{\lambda (R_d^*(T))^p\}) \leq c E(\exp\{\lambda (2\sqrt{e})^p d^{p/2} (\tilde{B}_1^*)^p T^{p/2}\})$$

We then use the following majorizations already established in [D-E] :

$$E(\exp\{\lambda B_1^*\}) \leq 4 \exp\left(\frac{\lambda^2}{2}\right) \quad \text{and}$$

$$E(\exp\{\lambda (B_1^*)^p\}) \leq b_p \exp\left\{\left(\frac{1}{p} - \frac{1}{2}\right)(\lambda p)^{\frac{2}{2-p}}\right\}$$

where b_p is a strictly positive constant.

to obtain Theorem 1.

We can write similar relations for exponential functions vanishing at zero. As an example, we have the following theorem .

Theorem 3 : *There exist two strictly positive constants c and β such that*

for any $\lambda > 0$ and any d in \mathbb{N}^* , we have for any stopping time T :

$$E\left(\cosh\left\{\lambda\left(\frac{R_d^*(T)}{d^{1/2}}\right)\right\} - 1\right) \leq c E\left(\exp\left\{\frac{\beta^2}{2} \lambda^2 T\right\} - 1\right)$$

Moreover: $\beta \leq 2\sqrt{e}$.

Proof : By summing the inequalities $(2)_{2n}$, n running through \mathbb{N}^* , we obtain :

$$E\left(\cosh\left\{\lambda\left(\frac{R_d^*(T)}{d^{1/2}}\right)\right\} - 1\right) \leq c E\left(\cosh\{\lambda\beta \tilde{B}_1^* \cdot T^{1/2}\} - 1\right)$$

We then note that : $E\left(\cosh\{\lambda B_1^*\} - 1\right) \leq 2\left(\exp\left(\frac{\lambda^2}{2}\right) - 1\right)$.

Remark : In view of the results of Burkholder [B] and Davis [D] it is natural to look for a function F such that, for example,

$E\left(\exp\left(\frac{R_d^*(T)}{d^{1/2}}\right)\right) / E(F(T^{1/2}))$ would converge to 1, uniformly in T , when d tends to infinity.

Assuming such a function F exists, we would obtain that for a given $\epsilon > 0$, if d is big enough, for every stopping time T :

$$(*) \quad (1-\epsilon) E(F(T^{1/2})) \leq E\left(\exp\left(\frac{R_d^*(T)}{d^{1/2}}\right)\right).$$

We would also have : $\lim_{d \rightarrow \infty} E\left(\exp\left(\frac{R_d^*(t)}{d^{1/2}}\right)\right) = F(t^{1/2})$, for $t > 0$.

But we know that for every p in \mathbb{N}^* , $E\left(\frac{R_d^*(t)}{d^{1/2}}\right)^p$ converges to $t^{p/2}$.

Consequently for every n in \mathbb{N}^* :

$$\lim_{d \rightarrow \infty} E\left(\exp\left(\frac{R_d^*(t)}{d^{1/2}}\right)\right) \geq \lim_{d \rightarrow \infty} \sum_{p=0}^n \frac{1}{p!} E\left(\frac{R_d^*(t)}{d^{1/2}}\right)^p = \sum_{p=0}^n \frac{t^{p/2}}{p!}.$$

We finally obtain : $F(t^{1/2}) \geq \exp(t^{1/2})$, for every $t > 0$.

Hence the inequality (*) implies :

$$(1-\epsilon) E(\exp(T^{1/2})) \leq E\left(\exp\left(\frac{R_d^*(T)}{d^{1/2}}\right)\right).$$

In [J-Y] Jacka and Yor have proved that such an inequality can not hold

when $d=1$. Since their argument is exclusively based on the scaling property of the Brownian motion, we can easily extend their result to any $d>1$.

In conclusion, there is no function F verifying such hypothesis.

Moreover, we see thanks to the same kind of argument, that there is not

even a function F such that $E(\exp(-\frac{R_d^*(T)}{d^{1/2}}))/E(F(T^{1/2}))$ would be uniformly minorized by a strictly positive constant.

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