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PATHS OF FINITELY ADDITIVE BROWNIAN MOTION
NEED NOT BE BIZARRERE

by
Lester E. Dubins

Abstract. Each stochastic process, in particular the Wiener process, has a finitely additive cousin whose paths are polynomials, and another cousin whose paths are step functions.

Notation. $\mathbb{R}$ is the real line; $T$ is the half-ray of nonnegative moments of time; a path, $w$, is a mapping of $T$ into $\mathbb{R}$; $W$ is the set of paths; $I$ is the identity map of $W$ onto itself.

Plainly, $I$ is essentially the same as the one-parameter family of evaluation maps, $I(t)$ or $I(t, \cdot)$, defined for $t$ in $T$, by $I(t, w) = w(t)$.

Of course, once $W$, the space of paths, is endowed with a sufficiently rich probability measure, $I$ becomes a stochastic process. Probabilities in this note are not required to be countably additive; those on $W$ are assumed to be defined (at least) on $\mathcal{F}$, the set of finite-dimensional (Borel) subsets of $W$. As always, to a stochastic process, $X$, is associated its family $J = J(X)$ of finite-dimensional joint distributions, one such distribution $J(t)$ for each $n$-tuple $t$ of distinct moments of time. Of course, $J(X)$ is a consistent family, which has the usual meaning that, if $t$ is a subsequence of $t'$, then $J(t)$ is the $t$-marginal of $J(t')$.

Definition. Two stochastic processes are cousins if the $J$ of one of the processes is the same as the $J$ of the other process.

Of interest herein are those subsets $H$ of $W$ that satisfy:
Condition *. Each stochastic process $X$ has a cousin almost all of whose paths are in $H$.

Throughout this note, $J$ designates a consistent family of finite-dimensional joint distributions, and a stochastic process $X$ is a $J$-process if $J(X) = J$.

Record here the following alternative formulation of Condition *.
Condition **. For each $J$, there is a $J$-process almost all of whose paths are in $H$.

That ** suffices for * is a triviality. That * suffices for ** becomes a triviality once one recalls that, for each $J$, there is a $J$-process. So the conditions are equivalent.

As a preliminary to characterizing the $H$ that satisfy Condition *, introduce for each $n$-tuple $t$ of distinct time-points, $t = (t_1, \ldots, t_n)$, and each $n$-tuple $x$ of possible positions, $x = (x_1, \ldots, x_n)$, the set $S[t, x]$ of all paths $w$ such that, for each $i$ from 1 to $n$, $w(t_i)$ is $x_i$.

Condition ***. $H$ has a nonempty intersection with each $S[t, x]$.

**Proposition 1. A set $H$ of paths satisfies Condition * if and only if it satisfies Condition ***.
Proof. Suppose $H$ satisfies *. Then, for each probability $P$ on $F$, these three equivalent conditions hold: [i] There is a probability $Q$ that agrees with $P$ on $F$ for which $QH = 1$; [ii] $H$ has outer $P$-probability 1; [iii] the inner $P$-probability of the complement of $H$ is zero. As [iii] implies, for no finite-dimensional set $S$ disjoint from $H$ is $P(S)$ strictly positive. A fortiori, for no such $S$ does $P(S) = 1$. In particular, no $S[t, x]$ disjoint from $H$ has $P$-probability 1. This implies that there is no $S[t, x]$ disjoint from $H$.

For the converse, suppose that $H$ satisfies ***, or equivalently, that no $S[t, x]$ is included in the complement, $H'$ of $H$. Surely then, no nonempty union of the $S[t, x]$ is included in $H'$. Since, as is easily verified, each finite-dimensional set is such a union, no nonempty, finite-dimensional set is included in $H'$. Since the empty set is the only finite-dimensional set included in $H'$, the only finite-dimensional set that includes $H$ is the complement of the empty set, namely, $W$. Now fix a consistent family $J$, and let $P$ be the corresponding probability on $F$. For this $P$, as for all $P$ on $F$, the outer $P$-probability of $H$ is necessarily 1. Therefore, $P$ has an extension that assigns probability 1 to $H$. Equivalently, there is a $J$-process, almost all of whose paths are in $H$. So $H$ satisfies *. 

A step function is one that, on each bounded time-interval, has only a finite number of values, each assumed on a finite union of intervals.

Theorem 1. Each stochastic process, in particular the Wiener process, has a cousin almost all of whose paths are polynomials, another cousin almost all of whose paths are step functions that are continuous on the right (on the left), and a fourth cousin almost all of whose paths are continuous, piecewise-linear functions.

Proof of Theorem 1. Plainly, each of the four sets of paths satisfies Condition ***. Therefore, Proposition 1 applies.

A remark (informal). The (strong) Markov property need not be inherited by a cousin of a process, or, as is closely related, the existence of proper disintegrations (proper conditional distributions) of the future given the past need not transfer to the cousin. An example is provided by a cousin of Brownian Motion whose paths are polynomials. On the other hand, those properties are inheritable by those cousins of Brownian Motion whose paths are step functions, or piecewise-linear functions. Definition of proper, and of disintegration, may, amongst other places, be seen in the two references.

References
