Lester E. Dubins

Paths of finitely additive brownian motion need not be bizarre

Séminaire de probabilités (Strasbourg), tome 33 (1999), p. 395-396
<http://www.numdam.org/item?id=SPS_1999__33__395_0>

PATHS OF FINITELY ADDITIVE BROWNIAN MOTION NEED NOT BE BIZARRE

by
Lester E. Dubins

Abstract. Each stochastic process, in particular the Wiener process, has a finitely additive cousin whose paths are polynomials, and another cousin whose paths are step functions.

Notation. R is the real line; T is the half-ray of nonnegative moments of time; a path, w, is a mapping of T into R; W is the set of paths; I is the identity map of W onto itself.

Plainly, I is essentially the same as the one-parameter family of evaluation maps, $I(t)$ or $I(t, .)$, defined for t in T, by $I(t, w) = w(t)$.

Of course, once W, the space of paths, is endowed with a sufficiently rich probability measure, I becomes a stochastic process. Probabilities in this note are not required to be countably additive; those on W are assumed to be defined (at least) on F, the set of finite-dimensional (Borel) subsets of W. As always, to a stochastic process, X, is associated its family $J = J(X)$ of finite-dimensional joint distributions, one such distribution $J(t)$ for each n-tuple t of distinct moments of time. Of course, $J(X)$ is a consistent family, which has the usual meaning that, if t is a subsequence of t', then $J(t)$ is the t-marginal of $J(t')$.

Definition. Two stochastic processes are *cousins* if the J of one of the processes is the same as the J of the other process.

Of interest herein are those subsets H of W that satisfy:

Condition *. Each stochastic process X has a cousin almost all of whose paths are in H.

Throughout this note, J designates a consistent family of finite-dimensional joint distributions, and a stochastic process X is a J-process if $J(X) = J$.

Record here the following alternative formulation of Condition *.

Condition **. For each J, there is a J-process almost all of whose paths are in H.

That ** suffices for * is a triviality. That * suffices for ** becomes a triviality once one recalls that, for each J, there is a J-process. So the conditions are equivalent.

As a preliminary to characterizing the H that satisfy Condition *, introduce for each n-tuple t of distinct time-points, $t = (t_1, \dots, t_n)$, and each n-tuple x of possible positions, $x = (x_1, \dots, x_n)$, the set $S[t, x]$ of all paths w such that, for each i from 1 to n, $w(t_i)$ is x_i.

Condition ***. H has a nonempty intersection with each $S[t, x]$.

Proposition 1. *A set H of paths satisfies Condition * if and only if it satisfies Condition ***.*

Proof. Suppose H satisfies *. Then, for each probability P on F, these three equivalent conditions hold: [i] There is a probability Q that agrees with P on F for which $QH = 1$; [ii] H has outer P-probability 1; [iii] the inner P-probability of the complement of H is zero. As [iii] implies, for no finite-dimensional set S disjoint from H is $P(S)$ strictly positive. A fortiori, for no such S does $P(S) = 1$. In particular, no $S[t, x]$ disjoint from H has P-probability 1. This implies that there is no $S[t, x]$ disjoint from H. For, as is easily verified, for each $S[t, x]$ there is a P under which $S[t, x]$ has probability 1. Consequently, each $S[t, x]$ has nonempty intersection with H, or, what is the same thing, H satisfies ***.

For the converse, suppose that H satisfies ***, or equivalently, that no $S[t, x]$ is included in the complement, H' of H. Surely then, no nonempty union of the $S[t, x]$ is included in H'. Since, as is easily verified, each finite-dimensional set is such a union, no nonempty, finite-dimensional set is included in H'. Since the empty set is the only finite-dimensional set included in H', the only finite-dimensional set that includes H is the complement of the empty set, namely, W. Now fix a consistent family J, and let P be the corresponding probability on F. For this P, as for all P on F, the outer P-probability of H is necessarily 1. Therefore, P has an extension that assigns probability 1 to H. Equivalently, there is a J-process, almost all of whose paths are in H. So H satisfies *. ∎

A step function is one that, on each bounded time-interval, has only a finite number of values, each assumed on a finite union of intervals.

Theorem 1. *Each stochastic process, in particular the Wiener process, has a cousin almost all of whose paths are polynomials, another cousin almost all of whose paths are step functions that are continuous on the right (on the left), and a fourth cousin almost all of whose paths are continuous, piecewise-linear functions.*

Proof of Theorem 1. Plainly, each of the four sets of paths satisfies Condition ***. Therefore, Proposition 1 applies. ∎

A remark (informal). The (strong) Markov property need not be inherited by a cousin of a process, or, as is closely related, the existence of proper disintegrations (proper conditional distributions) of the future given the past need not transfer to the cousin. An example is provided by a cousin of Brownian Motion whose paths are polynomials. On the other hand, those properties are inheritable by those cousins of Brownian Motion whose paths are step functions, or piecewise-linear functions. Definition of proper, and of disintegration, may, amongst other places, be seen in the two references.

References

[1] David Blackwell and Lester Dubins. On existence and non-existence of proper, regular, conditional distributions. *Ann. Prob. 3* (1975), 741–752.

[2] Lester Dubins and Karel Prickry. On the existence of disintegrations. *Séminaire de Probabilités XXIX* (1995), 248–259.