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A short proof of decomposition of strongly reduced martingales

Michał Morayne and Krzysztof Tabisz

Abstract

A short proof of the following theorem is given: If M is a martingale, $T > 0$ is a stopping time, $M = M^T$ and $E(|M_T||\mathcal{F}_t)1_{[t < T]}$ is bounded, then M is a sum of a BMO (and, thus, square-integrable) martingale and a martingale of integrable variation.

The purpose of this note is to give a short proof of P.A. Meyer's theorem ([Me]) stated in the abstract. Although the proof given here is very much in the spirit of that of [Me] it seems to be simpler and shorter (in particular, we do not use potentials and Riesz decomposition in the proof). The proof presented here reduces to a sequence of easy inequalities. A shortcut has been possible because of the fact that the dual predictable projection of a reduced process is reduced by the same stopping time.

Let $\mathbf{R}_+ = [0, \infty)$. Let $(\mathcal{F}_t, t \in \mathbf{R}_+)$ be a fixed right-hand side continuous complete filtration. We shall consider martingales with respect to this filtration, assuming always that they are CADLAG (i.e. that almost all their trajectories are right-hand side continuous and have left-hand side finite limits). For a process X by X_∞ we denote $\lim_{t \rightarrow \infty} X_t$, when such a limit exists a.s.

BMO denotes the class of those uniformly integrable martingales M for which $EM_\infty^2 < \infty$ and there exists a constant c such that for each stopping time S the following inequality is satisfied: $E((M_\infty - M_{S-})^2|\mathcal{F}_S) < c$. \mathcal{A} denotes the class of the processes of integrable variation i.e. $A \in \mathcal{A}$ if $E \text{Var}_0^\infty A_t < \infty$.

For a stopping time T we shall put $[[0, T]] = \{(\omega, t) : t \leq T(\omega)\}$, $[[T, \infty)) = \{(\omega, t) : t \geq T(\omega)\}$, $((T, \infty)) = \{(\omega, t) : t > T(\omega)\}$ and $[[T]] = \{(\omega, t) : T(\omega) = t\}$.

We say that a process X is *reduced* by a stopping time T if $X^T = X$, where $X_t^T = X_{\min(t, T)}$. If, in addition, $T > 0$ and for some constant C we have $\sup_{t < T} E(|X_T||\mathcal{F}_t) < C$ a.s. we say that X is *strongly reduced* by T .

We are going to prove the following theorem ([Me], Chap. IV, 8, p. 294 and Chap. V, 5c, p. 335).

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Theorem. If $T > 0$ is a stopping time and M is a martingale strongly reduced by T , then M can be expressed as a sum $M = N + A$, where $N \in BMO$ and $A \in \mathcal{A}$.

The following lemma is crucial for our proof.

Lemma. Let $T > 0$ be a stopping time, $\Phi \in L_1$, Φ be \mathcal{F}_T -measurable, $E(|\Phi||\mathcal{F}_t)$ be strongly reduced by T , U be the dual predictable projection of the process $\Phi \mathbf{1}_{[[T, \infty))}$. Then there exists a constant c such that for each stopping time S

$$E((U_\infty - U_{S-})^2 | \mathcal{F}_S) < c.$$

Proof. First we shall show that there exists such a constant c that for each stopping time S

$$E((U_\infty - U_S)^2 | \mathcal{F}_S) < c. \quad (1)$$

Let W be the dual predictable projection of the process $|\Phi| \mathbf{1}_{[[T, \infty))}$. Let $P_t = E(|\Phi||\mathcal{F}_t)$. We have for a set $B \in \mathcal{F}_S$:

$$\begin{aligned} E((U_\infty - U_S)^2 \mathbf{1}_B) &\leq E((W_\infty - W_S)^2 \mathbf{1}_B) \leq 2E \int_0^\infty \mathbf{1}_{((S, \infty))} \mathbf{1}_B (W_t - W_S) dW_t \\ &= 2E \int_0^\infty \mathbf{1}_{((S, \infty))} \mathbf{1}_B (W_t - W_S) d(|\Phi| \mathbf{1}_{[[T, \infty))})_t = 2E(\mathbf{1}_B |\Phi| (W_T - W_S)) = \\ &\quad 2E \int_0^\infty \mathbf{1}_{((S, \infty))} \mathbf{1}_B |\Phi| dW_t = 2E \int_0^\infty \mathbf{1}_{((S, \infty))} \mathbf{1}_B P_{t-} dW_t \\ & (= 2E \int_0^\infty \mathbf{1}_{((S, \infty))} \mathbf{1}_B P_{t-} d(|\Phi| \mathbf{1}_{[[T, \infty))})_t \leq 2E(P_{T-} |\Phi|) \leq 2CE|\Phi| < \infty) \\ &\leq 2CE(\mathbf{1}_B \mathbf{1}_B (W_\infty - W_S)) \leq 2C(P(B))^{1/2} (E(\mathbf{1}_B (W_\infty - W_S)^2))^{1/2}. \end{aligned}$$

To get the second and the third equality and the next inequality we used the fact that the process W is reduced by T . The last step was possible by virtue of Schwarz inequality.

Comparing the second and the last term of the sequence of (in)equalities above and repeating the first inequality we get

$$E((U_\infty - U_S)^2 \mathbf{1}_B) \leq 4C^2 P(B)$$

and this gives (1).

We still need to show that all the jumps of the process U are uniformly bounded. First let us notice that the process U , being predictable, possibly jumps only at countable number of graphs of predictable stopping times. So let us assume that S is a predictable stopping time which means that there exists a nondecreasing sequence of stopping times S_n such that $S_n < S$ on the set $[S > 0]$ and $\lim_n S_n = S$. Let $B \in \mathcal{F}_S$. Let $R_t = E(\mathbf{1}_B | \mathcal{F}_t)$. We have:

$$E(\mathbf{1}_B | U_S - U_{S-}) \leq E(\mathbf{1}_B (W_S - W_{S-})) = E \int_0^\infty \mathbf{1}_{[[S]]} \mathbf{1}_B dW_t =$$

$$\begin{aligned}
E \int_0^\infty \mathbf{1}_{[[S]]} R_t - d(|\Phi| \mathbf{1}_{[[T, \infty]])}_t &= E(\mathbf{1}_{[S=T]} R_{S-} |\Phi|) = EE(\mathbf{1}_{[S=T]} R_{S-} |\Phi| | \mathcal{F}_{S-}) = \\
&E(R_{S-} E(\mathbf{1}_{[S=T]} |\Phi| | \mathcal{F}_{S-})) = E(R_{S-} \lim_n E(\mathbf{1}_{[S=T]} |\Phi| | \mathcal{F}_{S_n})) \leq \\
&E(R_{S-} \sup_n (\mathbf{1}_{[S_n < T]} P_{S_n})) \leq CER_{S-} = CP(B).
\end{aligned}$$

This finishes the proof.

Now we can give a proof of Theorem.

Proof of Theorem. Let U now denote the dual predictable projection of $M_T \mathbf{1}_{[[T, \infty))}$. The decomposition of M is defined as in [Me] as $M = N + A$, where $N = M - M_T \mathbf{1}_{[[T, \infty))} + U$ and $A = M_T \mathbf{1}_{[[T, \infty))} - U$ and we apply Lemma to get $N \in BMO$. It is obvious that $A \in \mathcal{A}$. This finishes the proof.

References

- [Me] P.A. Meyer, *Un cours sur les integrales stochastiques*, Séminaire de Probabilités X, Lecture Notes in Mathematics 511, Berlin, Heidelberg, New York 1976.