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Koichiro Takaoka

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Some Remarks on the Uniform Integrability of Continuous Martingales

Koichiro TAKAOKA

In this article we show a property on the tails of the supremum and the quadratic variation of real-valued continuous (local) martingales, and furthermore use the property to give a characterization of uniform integrable martingales. Our result refines or generalizes the main theorems of the following three papers: Azéma-Gundy-Yor [1], Elworthy-Li-Yor [2], and the continuous martingale version of Galtchouk-Novikov [4]. The present article is also closely related to a recent paper of Elworthy-Li-Yor [3].

We should mention two more works on related topics. H. Sato [7] gave a result on the uniform integrability of stochastically continuous additive martingales. Concerning exponential local martingales, see Kazamaki's book [5] and its references to earlier papers.

The author obtained the idea of using the technique in Step 4 of the proof of our main theorem when he attended a course on mathematical finance given by Professor Freddy Delbaen in Tokyo, February–March 1998. The author also would like to thank Professor Marc Yor and Professor Kohei Uchiyama for their helpful comments.

Our result is the following

Theorem. *Let $M = (M_t)_{t \in \boldsymbol{R}_+}$ be a real-valued continuous local martingale, with $M_0 = 0$, on a certain filtered probability space satisfying the usual conditions. Assume $M_\infty \stackrel{\text{def}}{=} \lim_{t\to\infty} M_t$ exists a.s. and $E\big[\,|M_\infty|\,\big] < \infty$. Then both*

$$\ell \stackrel{\text{def}}{=} \lim_{\lambda\to\infty} \lambda\, P\big[\, \sup\nolimits_t |M_t| > \lambda\big] \quad \textit{and} \quad \sigma \stackrel{\text{def}}{=} \lim_{\lambda\to\infty} \lambda\, P\big[\,\langle M\rangle_\infty^{1/2} > \lambda\big]$$

exist in $\boldsymbol{R}_+ \cup \{\infty\}$, and satisfy

$$\ell = \sqrt{\frac{\pi}{2}}\,\sigma = \sup_{U\in\mathcal{T}(M)} E\big[\,|M_U|\,\big] - E\big[\,|M_\infty|\,\big],$$

where $\mathcal{T}(M)$ is the set of all reducing stopping times for M. Furthermore, M is a uniformly integrable martingale if and only if $\ell = \sigma = 0$.

Remarks.

1. The expression $\sup_{U\in\mathcal{T}(M)} E\big[\,|M_U|\,\big]$ is less complicated than it looks since, as we will see later in this paper,

$$\sup_{U \in \mathcal{T}(M)} E\big[\,|M_U|\,\big] = \sup_{U \in \mathcal{T}} E\big[\,|M_U|\,\big] = E[L_\infty],$$

where $\mathcal{T}$ is the set of all stopping times and L_t is the local time of M at the origin up to time t.

2. Azéma-Gundy-Yor [1] proved the equivalence

"M is a u.i. martingale" $\Leftrightarrow$ "ℓ exists and $\ell = 0$" $\Leftrightarrow$ "σ exists and $\sigma = 0$"

under the assumptions that M is a martingale and that $\sup_t E\big[\,|M_t|\,\big] < \infty$.

3. The continuous martingale version of the main theorem of Galtchouk-Novikov [4] shows that, under the same assumptions as ours, $E[M_\infty] = 0$ is implied by a condition weaker than "σ exists and $\sigma = 0$" and stronger than "$\liminf_{\lambda \to \infty} \lambda P\big[\langle M \rangle_\infty^{1/2} > \lambda\big] = 0.$"

4. Theorem 1 of Elworthy-Li-Yor [2] and Lemma 1 of Galtchouk-Novikov [4] use a variant of the Tauberian theorem to prove that if M is bounded below (actually in a somewhat more general setting), then the two limits ℓ and σ exist and satisfy $\ell = \sqrt{\frac{\pi}{2}}\,\sigma = -E[M_\infty]$. Note that in this case our result agrees with theirs. Indeed, if $(M_t^-)_t$ is of class (D), then $M_{T_n}^- \to M_\infty^-$ in L^1 for every sequence T_n in $\mathcal{T}(M)$ increasing to ∞ a.s., so we find that

$$\begin{aligned}\sup_{U \in \mathcal{T}(M)} E\big[\,|M_U|\,\big] - E\big[\,|M_\infty|\,\big] &= \lim_{n \to \infty} E\big[\,|M_{T_n}|\,\big] - E\big[\,|M_\infty|\,\big] \\ &= 2 \lim_{n \to \infty} E\big[M_{T_n}^-\big] - E\big[\,|M_\infty|\,\big] \\ &= 2\,E\big[M_\infty^-\big] - E\big[\,|M_\infty|\,\big] \\ &= -E\big[M_\infty\big]\end{aligned}$$

(for the validity of the first equality see Lemma below).

5. The assumption $E\big[\,|M_\infty|\,\big] < \infty$ is essential for our theorem. See Azéma-Gundy-Yor [1] for an example of how things would go wrong without this hypothesis.

The rest of this paper is devoted to the proof of our theorem. We need the following easy lemma.

Lemma. *For every sequence $(T_n)_{n=1}^\infty$ in $\mathcal{T}(M)$ increasing to ∞ a.s., we have*

$$\lim_{n \to \infty} E\big[\,|M_{T_n}|\,\big] = \sup_n E\big[\,|M_{T_n}|\,\big] = \sup_{U \in \mathcal{T}(M)} E\big[\,|M_U|\,\big] = \sup_{U \in \mathcal{T}} E\big[\,|M_U|\,\big],$$

where $\mathcal{T}$ is the set of all stopping times.

Proof. For every stopping time U, observe that $U \wedge T_n \in \mathcal{T}(M)$ and that $M_U = \lim_{n \to \infty} M_{U \wedge T_n}$ a.s. Hence, by Fatou's lemma,

$$E\big[\,|M_U|\,\big] \le \lim_{n \to \infty} E\big[\,|M_{U \wedge T_n}|\,\big] \le \lim_{n \to \infty} E\big[\,|M_{T_n}|\,\big]. \qquad \square$$

Proof of the main theorem. We divide the proof into six steps.

Step 1. We first show the existence of the limit ℓ and the equality $\ell = \sup_{U\in\mathcal{T}(M)} E[\,|M_U|\,] - E[\,|M_\infty|\,]$. For $\lambda > 0$, define the stopping time

$$T_\lambda \stackrel{\text{def}}{=} \inf\{\,t\,:\,|M_t| > \lambda\,\}; \qquad (\inf\emptyset \stackrel{\text{def}}{=} \infty)$$

then $T_\lambda \in \mathcal{T}(M)$ and

$$E[\,|M_{T_\lambda}|\,] \;=\; \lambda P[\,\sup_t |M_t| > \lambda\,] \;+\; E[\,|M_\infty|\,;\,\sup_t |M_t| \le \lambda\,].$$

Here the left-hand side increases with λ, and the second term on the right-hand side converges to $E[\,|M_\infty|\,]$ $(<\infty)$ as $\lambda \to \infty$. Therefore the limit of the first term on the right-hand side also exists. The desired equality also follows from this together with the above Lemma.

Step 2. For the proof of the equivalence "M is a u.i. martingale" $\Leftrightarrow$ "$\ell = 0$", we make the following observation:

$$\begin{aligned}
\ell = 0 \;\;&\Leftrightarrow\;\; \sup_{U\in\mathcal{T}(M)} E[\,|M_U|\,] = E[\,|M_\infty|\,] \quad \text{(by Step 1)}\\
&\Leftrightarrow\;\; \lim_{n\to\infty} E[\,|M_{T_n}|\,] = E[\,|M_\infty|\,] \;\text{ for every sequence } T_n \text{ in } \mathcal{T}(M)\\
&\qquad \text{increasing to } \infty \text{ a.s.} \quad \text{(by Lemma)}\\
&\Leftrightarrow\;\; \lim_{n\to\infty} M_{T_n} = M_\infty \;\text{ in } L^1 \;\text{ for every sequence } T_n \text{ in } \mathcal{T}(M)\\
&\qquad \text{increasing to } \infty \text{ a.s.} \quad (\text{since } E[\,|M_\infty|\,] < \infty)\\
&\Leftrightarrow\;\; M \text{ is a u.i. martingale.}
\end{aligned}$$

Step 3. Next we show the inequality

$$\ell \le \sqrt{2\pi}\,\liminf_{\lambda\to\infty} \lambda P[\,\langle M\rangle_\infty^{1/2} > \lambda\,],$$

which gives the implication "$\ell = \infty$" $\Rightarrow$ "σ exists and $\sigma = \infty$." We apply an argument similar to the proof of the main theorem of Galtchouk-Novikov [4]. For $x > 0$, define the stopping time

$$S_x \stackrel{\text{def}}{=} \inf\{\,t\,:\,M_t > x\,\}. \qquad (\inf\emptyset \stackrel{\text{def}}{=} \infty)$$

Since $(M_{t\wedge S_x})_t$ is a continuous local martingale bounded above, it is proved in Elworthy-Li-Yor [2] and Galtchouk-Novikov [4] that

$$\sqrt{\tfrac{\pi}{2}}\,\lim_{\lambda\to\infty} \lambda P[\,\langle M\rangle_{S_x}^{1/2} > \lambda\,] \;=\; E[\,M_{S_x}\,]$$

(see Remark 4 after the statement of our Theorem). Also, observe that

$$E[\,M_{S_x}\,] \;=\; x P[\,\sup_t M_t > x\,] \;+\; E[\,M_\infty\,;\,\sup_t M_t \le x\,],$$

and thus

$$\sqrt{\tfrac{\pi}{2}}\ \liminf_{\lambda\to\infty}\ \lambda P\big[\langle M\rangle_\infty^{1/2} > \lambda\big]$$
$$\geq\ \sqrt{\tfrac{\pi}{2}}\ \lim_{\lambda\to\infty}\ \lambda P\big[\langle M\rangle_{S_x}^{1/2} > \lambda\big]$$
$$=\ x P\big[\ \sup_t M_t\ > x\big]\ +\ E\big[M_\infty\,;\ \sup_t M_t\ \leq x\big].$$

Likewise, replacing $(M_t)_t$ with $(-M_t)_t$ we have

$$\sqrt{\tfrac{\pi}{2}}\ \liminf_{\lambda\to\infty}\ \lambda P\big[\langle M\rangle_\infty^{1/2} > \lambda\big]$$
$$\geq\ x P\big[\ \inf_t M_t\ < -x\big]\ -\ E\big[M_\infty\,;\ \inf_t M_t\ \geq -x\big].$$

Therefore

$$\sqrt{2\pi}\ \liminf_{\lambda\to\infty}\ \lambda P\big[\langle M\rangle_\infty^{1/2} > \lambda\big]$$
$$\geq\ x\Big\{P\big[\ \sup_t M_t\ > x\big]\ +\ P\big[\ \inf_t M_t\ < -x\big]\Big\}$$
$$+E\big[M_\infty\,;\ \sup_t M_t\ \leq x\big]\ -\ E\big[M_\infty\,;\ \inf_t M_t\ \geq -x\big]$$
$$\geq\ x P\big[\ \sup_t |M_t|\ > x\big]$$
$$+E\big[M_\infty\,;\ \sup_t M_t\ \leq x\big]\ -\ E\big[M_\infty\,;\ \inf_t M_t\ \geq -x\big],$$

and by letting $x\to\infty$ we get the desired inequality.

Step 4. In this step we make some preparations for the proof of the existence of σ (in Step 5) and the equality $\sqrt{\tfrac{\pi}{2}}\,\sigma\ =\ \sup_{U\in\mathcal{T}(M)} E\big[\,|M_U|\,\big] - E\big[\,|M_\infty|\,\big]$ (in Step 6). By virtue of Step 3, we need to consider only the case $\ell<\infty$ (or equivalently $\sup_{U\in\mathcal{T}(M)} E\big[\,|M_U|\,\big]\ <\ \infty$), which we will assume for the rest of the proof. Define the local martingale $N=(N_t)_{t\in\boldsymbol{R}_+}$ by

$$N_t \stackrel{\text{def}}{=} -\int_0^t \operatorname{sgn}(M_s)\,dM_s.$$

Note that $\langle N\rangle_\cdot = \langle M\rangle_\cdot$. It also follows from Tanaka's formula that

$$|M_t|\ =\ -N_t + L_t,\qquad t\geq 0,\quad \text{a.s.},$$

where L_t is the local time of M at the origin up to time t. Since this is the Doob-Meyer decomposition of the local submartingale $(|M_t|)_t$, we see that

$$\forall U\in\mathcal{T}(M),\qquad E[\,|M_U|\,]\ =\ E[\,L_U\,],$$

and hence

$$E[\,L_\infty\,]\ =\ \sup_{U\in\mathcal{T}(M)}\ E\big[\,|M_U|\,\big]\ <\ \infty.$$

Furthermore, it follows from the Skorohod equation argument that

$$L_t\ =\ \sup_{u\in[0,t]}\ N_u,\qquad t\geq 0,\quad \text{a.s.}$$

Therefore

$$E\left[\,\sup_t N_t\,\right] \;<\; \infty,$$

which is crucial for the rest of the proof.

Step 5. With the assumption in Step 4 and the notations there assumed, we will here prove

$$\limsup_{\lambda\to\infty}\;\lambda\,P\left[\,\langle N\rangle_\infty^{1/2} > \lambda\,\right] \;\le\; \lim_{x\to\infty}\;\lim_{\lambda\to\infty}\;\lambda\,P\left[\,\langle N\rangle_{S_x{}'}^{1/2} > \lambda\,\right],$$

where

$$S_x{}' \stackrel{\text{def}}{=} \inf\{\,t\,:\,N_t > x\,\}. \qquad (\inf\emptyset \stackrel{\text{def}}{=} \infty)$$

Note that this implies the existence of the limit σ since it is trivial that

$$\liminf_{\lambda\to\infty}\;\lambda\,P\left[\,\langle N\rangle_\infty^{1/2} > \lambda\,\right] \;\ge\; \lim_{x\to\infty}\;\lim_{\lambda\to\infty}\;\lambda\,P\left[\,\langle N\rangle_{S_x{}'}^{1/2} > \lambda\,\right].$$

It suffices to show that, for each fixed $0 < a < 1$,

$$\limsup_{\lambda\to\infty}\;\lambda\,P\left[\,\langle N\rangle_\infty^{1/2} > \lambda\,\right] \;\le\; \frac{1}{a}\;\lim_{x\to\infty}\;\lim_{\lambda\to\infty}\;\lambda\,P\left[\,\langle N\rangle_{S_x{}'}^{1/2} > \lambda\,\right].$$

For $x > 0$, we have

$$P\left[\,\langle N\rangle_\infty^{1/2} > \lambda\,\right] \;\le\; P\left[\,\langle N\rangle_{S_x{}'}^{1/2} \le a\,\lambda,\;\; \langle N\rangle_\infty^{1/2} > \lambda\,\right] + P\left[\,\langle N\rangle_{S_x{}'}^{1/2} > a\,\lambda\,\right]$$

and hence

$$\begin{aligned}\limsup_{\lambda\to\infty}\;\lambda\,P\left[\,\langle N\rangle_\infty^{1/2} > \lambda\,\right] \;&\le\; \frac{1}{a}\;\lim_{\lambda\to\infty}\;\lambda\,P\left[\,\langle N\rangle_{S_x{}'}^{1/2} > \lambda\,\right] \\ &\quad + \sup_\lambda\;\lambda\,P\left[\,\langle N\rangle_{S_x{}'}^{1/2} \le a\,\lambda,\;\; \langle N\rangle_\infty^{1/2} > \lambda\,\right].\end{aligned}$$

Thus it suffices to show

$$\lim_{x\to\infty}\;\sup_\lambda\;\lambda\,P\left[\,\langle N\rangle_{S_x{}'}^{1/2} \le a\,\lambda,\;\; \langle N\rangle_\infty^{1/2} > \lambda\,\right] \;=\; 0.$$

Fix $x > 0$ for the moment. For $t \ge 0$, define

$$\tilde N_t^{(x)} \stackrel{\text{def}}{=} N_{S_x{}'+t} - N_{S_x{}'} \qquad\text{and}\qquad \tilde{\mathcal F}_t^{(x)} \stackrel{\text{def}}{=} \mathcal F_{S_x{}'+t}.$$

Note that $\left(\tilde N_t^{(x)}\right)_t$ is a continuous local martingale w.r.t. the filtration $\left(\tilde{\mathcal F}_t^{(x)}\right)_t$. Also, observe that

$$\begin{aligned}&\sup_\lambda\;\lambda\,P\left[\,\langle N\rangle_{S_x{}'}^{1/2} \le a\,\lambda,\;\; \langle N\rangle_\infty^{1/2} > \lambda\,\right] \\ &\quad\le\; \sup_\lambda\;\lambda\,P\left[\,\langle \tilde N^{(x)}\rangle_\infty^{1/2} > \sqrt{1-a^2}\,\lambda\,\right] \\ &\quad=\; \tfrac{1}{\sqrt{1-a^2}}\;\sup_\lambda\;\lambda\,P\left[\,\langle \tilde N^{(x)}\rangle_\infty^{1/2} > \lambda\,\right] \\ &\quad\le\; \tfrac{C}{\sqrt{1-a^2}}\;\sup_\lambda\;\lambda\,P\left[\,\sup_t |\tilde N_t^{(x)}| > \lambda\,\right], \qquad (*)\end{aligned}$$

where the last inequality follows from the well-known good λ inequality (see e.g. §IV.4 of Revuz-Yor [6]), with the constant C universal; in particular, C does not depend on x. Since

$$\forall\lambda > 0, \qquad \lambda P[\,\sup_t |\tilde{N}_t^{(x)}| > \lambda\,] \;\le\; E[\,|\tilde{N}_{\tilde{T}_\lambda}^{(x)}|\,],$$
$$(\,\tilde{T}_\lambda \stackrel{\text{def}}{=} \inf\{\,t\,:\,|\tilde{N}_t^{(x)}| > \lambda\,\}\,)$$

it follows that

$$\begin{aligned}(*) \;&\le\; \tfrac{C}{\sqrt{1-a^2}} \sup_{U\in\mathcal{T}(\tilde{N}^{(x)})} E[\,|\tilde{N}_U^{(x)}|\,] \\ &\qquad (\text{ where } \mathcal{T}(\tilde{N}^{(x)}) \text{ is defined the same way as } \mathcal{T}(M)\,) \\ &=\; \tfrac{C}{\sqrt{1-a^2}}\, 2 \sup_{U\in\mathcal{T}(\tilde{N}^{(x)})} E[\,\tilde{N}_U^{(x)\,+}\,] \\ &\le\; \tfrac{C}{\sqrt{1-a^2}}\, 2\, E[\,(\sup_t N_t - x)^+\,].\end{aligned}$$

The last expression converges to 0 as $x \to \infty$, since $E[\,\sup_t N_t\,] < \infty$.

Step 6. It remains to prove the equality $\sqrt{\frac{\pi}{2}}\,\sigma = \sup_{U\in\mathcal{T}(M)} E[\,|M_U|\,] - E[\,|M_\infty|\,]$. We assume the notations in the previous two steps. For $x > 0$, the same argument as in Step 3 gives

$$\begin{aligned}&\sqrt{\tfrac{\pi}{2}} \lim_{\lambda\to\infty} \lambda P[\,\langle N\rangle_{S_x'}^{1/2} > \lambda\,] \\ &\quad = x P[\,\sup_t N_t > x\,] + E[\,N_\infty\,;\,\sup_t N_t \le x\,].\end{aligned}$$

Here the first term on the right-hand side converges to 0 as $x \to \infty$, since $E[\,\sup_t N_t\,] < \infty$. The second term converges to

$$\begin{aligned}E[\,N_\infty\,] \;&=\; E[\,L_\infty\,] - E[\,|M_\infty|\,] \\ &=\; \sup_{U\in\mathcal{T}(M)} E[\,|M_U|\,] - E[\,|M_\infty|\,].\end{aligned}$$

Therefore

$$\lim_{x\to\infty} \sqrt{\tfrac{\pi}{2}} \lim_{\lambda\to\infty} \lambda P[\,\langle N\rangle_{S_x'}^{1/2} > \lambda\,] \;=\; \sup_{U\in\mathcal{T}} E[\,|M_U|\,] - E[\,|M_\infty|\,],$$

which together with Step 5 completes the proof. □

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