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Abstract

Fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ and a Brownian motion B on that space and consider any solution process X (on Ω) to a stochastic differential equation (SDE) $dX_t = f(t, X)\,dB_t + g(t, X)\,dt$ (1). A well-known theorem states that pathwise uniqueness implies that the solution X to SDE (1) is strong, i.e., it is adapted to the P-completed filtration generated by B. Pathwise uniqueness means that, on *any* filtered probability space carrying a Brownian motion and for any initial value, SDE (1) has at most one (weak) solution. We present an example that if we only assume that, for any initial value, there is at most one solution process on the *given* space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$), we can no longer conclude that the solution X is strong.

1 Introduction

Consider the following stochastic differential equation (SDE)

$$X_t = X_0 + \int_0^t f(s, X)\,dB_s + \int_0^t g(s, X)\,ds, \tag{1.1}$$

where $f, g : \mathbb{R}^+ \times \mathbb{C}(\mathbb{R}^+) \to \mathbb{R}$ are predictable mappings and B denotes Brownian motion ($\mathbb{C}(\mathbb{R}^+) := \{f : \mathbb{R}^+ \to \mathbb{R} : f \text{ continuous}\}$ denotes Wiener space and predictability is defined as in Revuz & Yor (1994), IX,§1). There are at least two fundamentally different concepts of approaching SDE (1.1).

Firstly, one can start with a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ and a Brownian motion B on that space. SDE (1.1) is then interpreted as an equation only for processes defined on Ω and by B one always refers to the same Brownian motion on Ω. *Existence and uniqueness of a solution* means in this context that, for any initial value X_0, there is (up to indistinguishability) exactly one solution process on Ω satisfying Equation (1.1). This concept is applied e.g. by Protter (1992) and it easily extends to arbitrary semimartingales as driving processes.

Alternatively, one may regard SDE (1.1) independently of a fixed underlying probability space and a fixed Brownian motion. In this context, SDE (1.1) has a *(weak) solution* whenever there is a probability space and two processes X and B on that space such that B is a Brownian motion and Equation (1.1) holds for this particular choice. Here, the space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ and the Brownian motion are part of the solution. *Pathwise uniqueness* holds if, for any two solutions $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P, (X, B))$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{P}, (\tilde{X}, \tilde{B}))$ with $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P) = (\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{P})$, $B = \tilde{B}$, and $X_0 = \tilde{X}_0$, the solutions X and $\tilde{X}$ are indistinguishable. The concept of weak solutions is discussed in many books (see e.g. Revuz & Yor (1994), Karatzas & Shreve (1991)). Clearly, a solution on a fixed space is always a weak solution. Also, *pathwise uniqueness* implies *uniqueness on a fixed space*. For a thorough account of both viewpoints see Jacod (1979).

Following Revuz & Yor (1994) we call a (weak) solution $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P, (X, B))$ *strong* if X is adapted to the P-completed filtration generated by the driving Brownian motion B. A well-known theorem due to Yamada & Watanabe (cf. Revuz & Yor (1994), Theorem IX.1.7; for a generalization to SDE's involving random measures see Jacod (1979), Théorème 14.94) states that pathwise uniqueness implies that any (weak) solution to SDE (1.1) is strong.

Now, consider the following situation. Starting from a fixed probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ and a fixed Brownian motion B, we are given a solution X to SDE (1.1) and we know that X is (up to indistinguishability) the only solution on that space starting in X_0. Is it, in general, true that X is a strong solution? (Note that we do not assume pathwise uniqueness, as pathwise uniqueness involves weak solutions on other spaces as well.) We give an example that the answer is *no*. More precisely, we present a SDE having no strong solution, having exactly one solution (for a fixed initial value) on some probabilty space and more than one solution on others. The example will be closely related to Tsirel'son's SDE (cf. Revuz & Yor (1994), p. 373).

We use the following notation: $[\cdot]$ denotes the integer part of a real number, λ is Lebesgue measure. For random variables U, V we write P^U, $P^{U|V}$, $P^{U|V=v}$ for the distribution (under P) of U, the conditional distribution of U given V, the factorisation of the conditional distribution of U given V, respectively. π_1 and $\pi_2 : \mathbb{R}^2 \to \mathbb{R}$ denote the projection on the first and the second coordinate.

2 The example

Consider the SDE

$$X_t = X_0 + B_t + \int_0^t \tau(s, X)\, ds, \tag{2.1}$$

where B stands for standard Brownian motion and $\tau : \mathbb{R}^+ \times \mathbb{C}(\mathbb{R}^+) \to \mathbb{R}$ is defined by

$$\tau(t, \omega) := \begin{cases} \alpha\left(\left\{\frac{\omega(t_k)-\omega(t_{k-1})}{t_k - t_{k-1}}\right\}\right) & \text{for } t_k < t \leq t_{k+1}, \\ 0 & \text{for } t = 0 \text{ or } t > 1, \end{cases}$$

where $\{x\}$ denotes x modulo 1, the function α is defined by $\alpha(x) := x 1_{[0,1/2)}(x) + (x + 1/4)1_{[1/2,3/4)}(x) + (x - 1/4)1_{[3/4,1)}(x)$, and $(t_k)_{k\in -\mathbb{N}}$ is a strictly increasing sequence of numbers such that $t_0 = 1$ and $\lim_{k\to-\infty} t_k = 0$.

As for Tsirel'son's example (where we have the identity instead of α) τ is predictable and bounded and a weak solution $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P, (X, B))$ to SDE (2.1) with $X_0 = 0$ exists (see e.g. Revuz & Yor (1994), Theorem IX.1.11).

By $(\mathcal{F}_t^B)_{t\geq 0}$ and $(\mathcal{F}_t^X)_{t\geq 0}$ we denote the P-completed natural filtrations of B resp. X. Let $(\tilde{X}_t)_{t\geq 0}$ be another weak solution defined on the same filtered probability space, with respect to the same Brownian motion B, and with $\tilde{X}_0 = X_0 = 0$. If we set for $t_k < t \leq t_{k+1}$

$$\eta_t := \frac{X_t - X_{t_k}}{t - t_k}, \quad \tilde{\eta}_t := \frac{\tilde{X}_t - \tilde{X}_{t_k}}{t - t_k}, \quad \varepsilon_t := \frac{B_t - B_{t_k}}{t - t_k},$$

we have that for $t_k < t \leq t_{k+1}$

$$X_t = B_t + \sum_{l \leq k} \alpha(\{\eta_{t_{l-1}}\})(t_l - t_{l-1}) + \alpha(\{\eta_{t_k}\})(t - t_k) \tag{2.2}$$

and hence

$$\eta_t = \varepsilon_t + \alpha(\{\eta_{t_k}\}), \tag{2.3}$$

and accordingly for $\tilde{X}$ and $\tilde{\eta}$.

Now the following statements hold:

1. For any $0 < s < t \leq 1$ we have $\mathcal{F}_t^X = \sigma(\{\eta_s\}) \vee \mathcal{F}_t^B$.
2. For any $k \in -\mathbb{N}$ there is a probability measure ρ on $[0, 1/4)$ and constants $c_1, c_2 \geq 0$ with $c_1 + c_2 = 1$ (independent of k) such that the distribution of $(\{\eta_{t_k}\}, \{\tilde{\eta}_{t_k} - \eta_{t_k}\})$ is

$$\lambda|_{[0,1)} \otimes \Bigg(\rho|_{(0,1/4)} * \frac{1}{4}(\epsilon_0 + \epsilon_{1/4} + \epsilon_{1/2} + \epsilon_{3/4}) \\ + c_1 \rho|_{\{0\}} + c_2 \rho|_{\{0\}} * \frac{1}{3}(\epsilon_{1/4} + \epsilon_{1/2} + \epsilon_{3/4}) \Bigg),$$

   where the asterisk denotes convolution and ϵ_a is the Dirac measure in a.
3. For any $k \in -\mathbb{N}$ the random vector $(\{\eta_{t_k}\}, \{\tilde{\eta}_{t_k}\})$ is independent of $\mathcal{F}_1^B$.
4. X is not strong.

Since $(\Omega, \mathcal{F}, (\mathcal{F}_t^X)_{t \geq 0}, P, (X, B))$ is a weak solution of SDE (2.1), let us assume $(\mathcal{F}_t)_{t \geq 0} = (\mathcal{F}_t^X)_{t \geq 0}$ for the following. Then we have in addition:

5. For any $k \in -\mathbb{N}$ there is a measureable mapping $\beta : \mathbb{R} \to \mathbb{R}$ such that $\{\tilde{\eta}_{t_k}\} = \beta(\{\eta_{t_k}\})$ P-a.s.
6. $(X_t)_{t \geq 0}$ and $(\tilde{X}_t)_{t \geq 0}$ are indistinguishable.
7. On $(\Omega, \mathcal{F}, (\mathcal{F}_t^X)_{t \geq 0}, P)$ and for any $a \in \mathbb{R}$, the process $X^a := X + a$ is (up to indistinguishability) the unique solution to SDE (2.1) starting at a in $t = 0$, but it is not strong.

Remark. Statement 7 can be strengthened in that, for any $T > 0$ (and for any fixed initial value), there is no other process on that space solving SDE (2.1) on $[0, T]$.

3 Proofs

Proof of Statement 1.
The $\supset$-inclusion follows from the definitions and from Equation (2.2). Since (2.3) implies $\{\eta_{t_k}\} = \alpha(\{\eta_t - \varepsilon_t\})$ for $t_k < t \leq t_{k+1}$, the inclusion "$\subset$" follows easily from Equation (2.2).

Proof of Statement 2.
We will proceed in four steps.

Step 1: Definition of several Markov kernels
We start by defining mappings

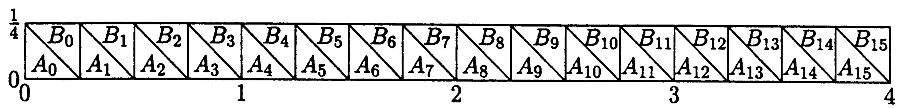
$$S^y : [0,4) \to [0,4),\ x \mapsto \alpha(\{x\}) + [4\{\alpha(\{y + [x]/4 + x\}) - \alpha(\{x\})\}]$$

for any $y \in [0, 1/4)$. For $k = 0, \dots, 15$ we set

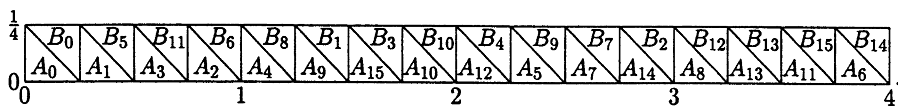
$$A_k := \{(x,y) : y \in [0,1/4), x \in [k/4, (k+1)/4 - y)\},$$

$$B_k := \{(x,y) : y \in [0,1/4), x \in [(k+1)/4 - y, (k+1)/4)\}.$$

With this notation, we have the following simple graphical representation of the mapping $[0,4) \times [0,1/4) \to [0,4,0,1/4), (x,y) \mapsto (S^y(x), y)$. The image of



under this mapping is



.

Consider, for example, $(x, y) \in B_7$, i.e., $x \in [2 - y, 2)$. It follows

$$\begin{aligned} S^y(x) &= \alpha(\{x\}) + [4\{\alpha(\{y + [x]/4 + x\}) - \alpha(\{x\})\}] \\ &= x - 1 - 1/4 + [4\{\alpha(\{y + 1/4 + x\}) - (x - 1 - 1/4)\}] \\ &= x - 5/4 + [4\{y + 1/4 + x - 2 - (x - 5/4)\}] \\ &= x - 5/4 + 2 = x + 3/4. \end{aligned}$$

Hence, B_7 is shifted in x-direction by 3/4 on B_{10}.

Now, we define Markov kernels $K^y, L^{y,b}$ (for any fixed $y \in [0,1/4)$, $b \in \mathbb{R}$) from $[0,4)$ to $[0,4)$ as follows:

$$L^{y,b}(x, A) := \epsilon_{\{b - \alpha(\{x\})\} + [S^y(x)]}(A) \text{ for } x \in [0,4), A \in \mathcal{B}([0,4))$$

and

$$K^y(x, A) := \int_A \kappa^y(x, x')\, dx' \text{ for } x \in [0,4), A \in \mathcal{B}([0,4))$$

with

$$\kappa^y(x, x') := \sum_{n \in \mathbb{Z}} \phi(\{x'\} + n - \alpha(\{x\})) 1_{\{0\}}([S^y(x)] - [x']),$$

where ϕ denotes the density of the standard normal distribution.

Lemma 3.1 *For $x \in [0,1)$, we have $K^0(x, [1,4)) = 0$. For $x \in [1,4)$, we have $K^0(x, [0,1))$* $= 0$.

Proof. Since $[S^0(x)] = 0$ for $x \in [0,1)$, we have $\kappa^0(x,x') = 0$ for $x \in [0,1)$, $x' \in [1,4)$, hence $K^0(x,[1,4)) = \int_{[1,4)} \kappa^0(x,x')\,dx' = 0$ for $x \in [0,1)$. From the graphical representation of S^y one observes $[S^0(x)] \neq 0$ for $x \in [1,4)$, hence $\kappa^0(x,x') = 0$ for $x \in [1,4)$, $x' \in [0,1)$. It follows $K^0(x,[0,1)) = \int_{[0,1)} \kappa^0(x,x')\,dx' = 0$ for $x \in [1,4)$. □

Therefore, we can define Markov kernels K_1 from $[0,1)$ to $[0,1)$ and K_2 from $[1,4)$ to $[1,4)$ by

$$\begin{aligned} K_1(x,A) &:= K^0(x,A) \quad \text{for } x \in [0,1), A \in \mathcal{B}([0,1)); \\ K_2(x,A) &:= K^0(x,A) \quad \text{for } x \in [1,4), A \in \mathcal{B}([1,4)). \end{aligned}$$

Step 2: Fixed points of the Markov kernels defined in Step 1

Notation. Let I be an interval.

1. For any Markov kernel K from I to I, we denote the corresponding Markov operator $\mathcal{M}^1(I) \to \mathcal{M}^1(I)$ again by K (i.e., $KQ : A \mapsto \int_I K(x,A)Q(dx)$ for $Q \in \mathcal{M}^1(I) := \{Q : Q \text{ probability measure on } I\}$, $A \in \mathcal{B}(I)$).

2. We set $\mathcal{D}_I := \{g \in L^1(I) : g \geq 0, \int_I g d\lambda = 1\}$. If a Markov kernel K from I to I has a transition density $\kappa : I \times I \to \mathbb{R}^+$ (i.e., $K(x,A) = \int_A \kappa(x,x')\,dx'$), then we denote the mapping $L^1(I) \to L^1(I)$, $g \mapsto \kappa g$ with

$$(\kappa g)(x') := \int_I \kappa(x,x')g(x)\,dx,$$

   also by κ. Observe that $\kappa|_{\mathcal{D}_I} \subset \mathcal{D}_I$.

3. Powers of a transition density $\kappa : I \times I \to \mathbb{R}^+$ shall be defined recursively by $\kappa^1(x,x') := \kappa(x,x')$ and $\kappa^{n+1}(x,x') := \int_I \kappa(x'',x')\kappa^n(x,x'')\,dx''$.

Lemma 3.2 *1. For any $y \in [0,1/4)$, $b \in \mathbb{R}$, the distribution $\frac{1}{4}\lambda|_{[0,4)} \in \mathcal{M}^1([0,4))$ is a fixed point of the Markov operators K^y and $L^{y,b}$.*

2. *$\lambda|_{[0,1)} \in \mathcal{M}^1([0,1))$ is a fixed point of the Markov operator K_1.*

3. *$\frac{1}{3}\lambda|_{[1,4)} \in \mathcal{M}^1([1,4))$ is a fixed point of the Markov operator K_2.*

4. *For any $b \in \mathbb{R}$ and any $c_1, c_2 > 0$ with $c_1 + c_2 = 1$, the distribution $c_1\lambda|_{[0,1)} + c_2\frac{1}{3}\lambda|_{[1,4)} \in \mathcal{M}^1([0,4))$ is a fixed point of the Markov operator $L^{0,b}$.*

Proof.

1. Fix $y \in [0,1/4)$, $b \in \mathbb{R}$. For any $A \in \mathcal{B}([0,4))$, we have

$$\begin{aligned} K^y\Big(\frac{1}{4}\lambda|_{[0,4)}\Big)(A) &= \frac{1}{4}\int_{[0,4)}\int_A \kappa^y(x,x')\,dx'\,dx \\ &= \frac{1}{4}\int_A\int_{(S^y)^{-1}([[x'],[x']+1))} \sum_{n\in\mathbb{Z}} \phi(\{x'\}+n-\alpha(\{x\}))\,dx\,dx' \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\int_A \int_{[[x'],[x']+1)} \sum_{n\in\mathbb{Z}} \phi(\{x'\}+n-\alpha(\{S^y(x'')\}))\,dx''\,dx' \\
&= \frac{1}{4}\int_A \int_{[[x'],[x']+1)} \sum_{n\in\mathbb{Z}} \phi(\{x'\}+n-\{x''\})\,dx''\,dx' \\
&= \frac{1}{4}\int_A dx' = \frac{1}{4}\lambda(A),
\end{aligned}
$$

where the third equation follows from the fact that λ is invariant under $S^y = (S^y)^{-1}$ (because S^y is a permutation of the intervals A_k^y, B_k^y, $k = 0, \dots, 15$), and the fourth equation follows from $\{S^y(x'')\} = \alpha(\{x''\})$ for any $x'' \in [0,4)$. Similarly, we obtain for any $A \in \mathcal{B}([0,4))$

$$
L^{y,b}\Big(\frac{1}{4}\lambda|_{[0,4)}\Big)(A) \;=\; \frac{1}{4}\lambda(A).
$$

2., 3., and 4. follow along the same lines (Observe that $S^0(x) \in [0,1)$ for $x \in [0,1)$ and $S^0(x) \in [1,4)$ for $x \in [1,4)$). □

Step 3: Convergence of iterates of the Markov kernels defined in Step 1

Lemma 3.3 *Let I be an interval, K a Markov kernel from I to I defined by a transition density $\kappa : I \times I \to \mathbb{R}$, and suppose that there are $j \in \mathbb{N}$, $s > 0$ such that $\kappa^j(x,x') > s$ for any $x, x' \in I$. Further assume that $\tilde{g}$ is a fixed point of $\kappa|_{\mathcal{D}_I}$. Then we have*

$$
\sup_{g\in\mathcal{D}_I} \|\kappa^n g - \tilde{g}\|_{L^1(I)} \to 0 \textit{ for } n \to \infty.
$$

Proof. Since $\kappa^n g - \tilde{g} = \kappa^n(g-\tilde{g})$ and $\|\kappa^{n+1}(g-\tilde{g})\|_{L^1(I)} = \|\kappa(\kappa^n(g-\tilde{g}))\|_{L^1(I)} \le \|\kappa^n(g-\tilde{g})\|_{L^1(I)}$ for $g \in \mathcal{D}_I$, $n \in \mathbb{N}$ (cf. Lasota & Mackey (1985), Prop. 3.1.1), it suffices to show that $\|\kappa^j h\|_{L^1(I)} \le (1-\lambda(I)s)\|h\|_{L^1(I)}$ for any $h \in L^1(I)$ with $\int_I h\,d\lambda = 0$.

Let $h \in L^1(I)\setminus\{0\}$ with $\int_I h\,d\lambda = 0$, and denote $c := \|h^+\|_{L^1(I)} = \|h^-\|_{L^1(I)}$. For any $g \in \mathcal{D}_I$, $x' \in I$, we have $\kappa^j g(x') = \int_I \kappa^j(x,x')g(x)\,dx \ge s$, hence $(\kappa^j g - s)^- = 0$. By $h^+/c \in \mathcal{D}_I$, $h^-/c \in \mathcal{D}_I$, it follows $\kappa^j(h^+/c) \ge s$, $\kappa^j(h^-/c) \ge s$. Therefore,

$$
\|\kappa^j(h^+/c) - s\|_{L^1(I)} = \int_I (\kappa^j(h^+/c)(x) - s)\,dx = 1 - \lambda(I)s,
$$

and accordingly $\|\kappa^j(h^-/c) - s\|_{L^1(I)} = 1 - \lambda(I)s$. Together, we obtain

$$
\begin{aligned}
\|\kappa^j h\|_{L^1(I)} &= \|c(\kappa^j(h^+/c) - s) - c(\kappa^j(h^-/c) - s)\|_{L^1(I)} \\
&\le c(\|\kappa^j(h^+/c) - s\|_{L^1(I)} + \|\kappa^j(h^-/c) - s\|_{L^1(I)}) \\
&= 2c(1-\lambda(I)s) = (1-\lambda(I)s)\|h\|_{L^1(I)}.
\end{aligned}
$$

□

In order to apply the preceding lemma to the kernels K^y, K_1, K_2, we state

Lemma 3.4 *1. Let $y \in (0, 1/4)$. There is a $s > 0$ such that, for any $x, x' \in [0,4)$, we have $(\kappa^y)^3(x,x') > s$.*

2. *There is a* $s > 0$ *such that, for any* $x, x' \in [0,1)$, *we have* $\kappa^0(x,x') > s$ *and, for any* $x, x' \in [1,4)$, *we have* $(\kappa^0)^3(x,x') > s$.

Proof. Since the mapping $[0,1] \times [0,1] \to \mathbb{R}$, $(u,v) \mapsto \sum_{n\in\mathbb{Z}} \phi(u+n-v)$, is positive and continuous, it has a lower bound $m > 0$. Hence, we have $\kappa^y(x,x') \geq m$ for any $y \in [0,1/4)$ and any $x, x' \in [0,4)$ with $[S^y(x)] = [x']$.

For $y \in [0,1/4)$, $k = 0, \ldots, 15$ define the sets

$$A_k^y := \{x : (x,y) \in A_k\}, \ B_k^y := \{x : (x,y) \in B_k\}.$$

In the following cases (among others), we have $\kappa^y(x,x') \geq m$:

$$\begin{array}{ll|ll|ll} x \in (S^y)^{-1}([0,1)) & ,x' \in B_1^y & x \in B_1^y & ,x' \in [1,2) & x \in B_5^y & ,x' \in [0,1) \\ x \in (S^y)^{-1}([1,2)) & ,x' \in A_4^y & x \in A_4^y & ,x' \in [1,2) & x \in A_4^y & ,x' \in [1,2) \\ x \in (S^y)^{-1}([2,3)) & ,x' \in A_9^y & x \in A_9^y & ,x' \in [1,2) & x \in A_5^y & ,x' \in [2,3) \\ x \in (S^y)^{-1}([3,4)) & ,x' \in A_{15}^y & x \in A_{15}^y & ,x' \in [1,2) & x \in A_6^y & ,x' \in [3,4). \end{array}$$

1. Fix $y \in [0,1/4)$. There is a $\delta > 0$ such that $\lambda(A_k^y) > \delta$, $\lambda(B_k^y) > \delta$ for $k = 0, \ldots, 15$. Define $s := m^3\delta^2$ and observe that, for $x' \in [0,1)$,

$$\int_{[1,2)} \kappa^y(v,x')\,dv \geq \int_{B_5^y} m\,dv \geq m\delta,$$

and accordingly for $x' \in [1,2)$, $[2,3)$, $[3,4)$ (with A_4^y, A_5^y, A_6^y instead of B_5^y). It follows for $x \in (S^y)^{-1}([0,1))$, $x' \in [0,4)$:

$$\begin{aligned} (\kappa^y)^3(x,x') &= \int_{[0,4)}\int_{[0,4)} \kappa^y(x,u)\kappa^y(u,v)\kappa^y(v,x')\,du\,dv \\ &\geq \int_{[0,4)}\int_{B_1^y} m\kappa^y(u,v)\kappa^y(v,x')\,du\,dv \\ &\geq \int_{B_1^y} du \int_{[1,2)} m^2\kappa^y(v,x')\,dv \\ &\geq m^3\delta^2, \end{aligned}$$

and accordingly for $x \in (S^y)^{-1}([1,2))$, $x \in (S^y)^{-1}([2,3))$, $x \in (S^y)^{-1}([3,4))$ (with A_4^y, A_9^y, A_{15}^y instead of B_1^y).

2. Obviously, $\lambda(A_k^0) = 1/4$ for $k = 0, \ldots, 15$. Define $s := \min\{m, m^3/16\}$. For $x \in [0,1)$, we have $S^0(x) = \alpha(\{x\})$, hence $[S^y(x)] = 0$. Therefore, $\kappa^0(x,x') = \sum_{n\in\mathbb{Z}} \phi(\{x'\} + n - \alpha(\{x\})) \geq m$ for any $x, x' \in [0,1)$. The second statement follows as in 1. (but this time with $B_1^y = \emptyset = B_5^y$). □

Corollary 3.5 *If we denote the transition densities of* K_1, K_2 *by* κ_1, κ_2 *(i.e.,* $\kappa_1 : [0,1) \times [0,1) \to \mathbb{R}$, $\kappa_1(x,x') = \kappa(x,x')$*;* $\kappa_2 : [1,4) \times [1,4) \to \mathbb{R}$, $\kappa_2(x,x') = \kappa(x,x')$*), we obtain*

1. $\sup_{g\in\mathcal{D}_{[0,4)}} \|(\kappa^y)^n g - 1/4\|_{L^1([0,4))} \to 0$ *for* $n \to \infty$, *for any* $y \in (0,1/4)$,

2. $\sup_{g\in\mathcal{D}_{[0,1)}} \|(\kappa_1)^n g - 1\|_{L^1([0,1))} \to 0$ *for* $n \to \infty$,

3. $\sup_{g\in\mathcal{D}_{[1,4)}} \|(\kappa_2)^n g - 1/3\|_{L^1([1,4))} \to 0$ *for* $n \to \infty$.

Proof. Lemma 3.2, Lemma 3.3, Lemma 3.4. □

Step 4: The joint distribution of $(\{\eta_{t_k}\}, \{\tilde{\eta}_{t_k} - \eta_{t_k}\})$
Define the mapping $\psi : [0,1)\times[0,1) \to [0,4)\times[0,1/4)$ by $(x', y') \mapsto (x' + [4y'], \{4y'\}/4)$. ψ is a bijection with converse $\psi^{-1} : [0,4) \times [0,1/4) \to [0,1) \times [0,1)$, $(x,y) \mapsto (\{x\}, y + [x]/4)$. Further we define, for any probability measure Q on $[0,4) \times [0,1/4)$, the Markov kernels $\bar{K}(Q)$ and, for any $b \in \mathbb{R}$, $\bar{L}^b(Q)$ from $[0,1/4)$ to $[0,4)$ by

$$\bar{K}(Q)(y,A) := (K^y Q^{\pi_1|\pi_2=y})(A) = \int K^y(x,A)\, Q^{\pi_1|\pi_2=y}(dx),$$

$$\bar{L}(Q)(y,A) := (L^{y,b} Q^{\pi_1|\pi_2=y})(A) = \int L^{y,b}(x,A)\, Q^{\pi_1|\pi_2=y}(dx)$$

for any $y \in [0,1/4)$, $A \in \mathcal{B}([0,4))$. One easily checks that $\bar{K}(Q)$, $\bar{L}^b(Q)$ are indeed Markov kernels. For any $k \in -\mathbb{N}$, we denote by μ_k the distribution of $\psi(\{\eta_{t_k}\}, \{\tilde{\eta}_{t_k} - \eta_{t_k}\})$.

Lemma 3.6 *For any* $k \in -\mathbb{N}$, *we have* $\mu_k = \bar{K}(\mu_{k-1}) \otimes \mu_{k-1}^{\pi_2}$.

Proof. For $k \in -\mathbb{N}$ let $(U_k, V_k) := \psi(\{\eta_{t_k}\}, \{\tilde{\eta}_{t_k} - \eta_{t_k}\})$. Then we have

$$\begin{aligned}
(U_k, V_k) &= \psi(\{\eta_{t_k}\}, \{\tilde{\eta}_{t_k} - \eta_{t_k}\})\\
&= \psi(\{\varepsilon_{t_k} + \alpha(\{\eta_{t_{k-1}}\})\}, \{\alpha(\{\tilde{\eta}_{t_{k-1}}\}) - \alpha(\{\eta_{t_{k-1}}\})\})\\
&= (\{\varepsilon_{t_k} + \alpha(\{U_{k-1}\})\} + [4\{\alpha(\{U_{k-1} + V_{k-1} + [U_{k-1}]/4\}) - \alpha(\{U_{k-1}\})\}],\\
&\qquad \{4\{\alpha(\{\tilde{\eta}_{t_{k-1}}\}) - \alpha(\{\eta_{t_{k-1}}\})\}\}/4)\\
&= (\{\varepsilon_{t_k} + \alpha(\{U_{k-1}\})\} + [S^{V_{k-1}}(U_{k-1})], \{4\{\tilde{\eta}_{t_{k-1}} - \eta_{t_{k-1}}\}\}/4)\\
&= (\{\varepsilon_{t_k} + \alpha(\{U_{k-1}\})\} + [S^{V_{k-1}}(U_{k-1})], V_{k-1}). \qquad (3.1)
\end{aligned}$$

Since ε_{t_k} is independent of $\mathcal{F}_{t_{k-1}}$ and $N(0,1)$-distributed, we have for any $A \in \mathcal{B}([0,4) \times [0,1/4))$

$$\begin{aligned}
\mu_k(A) &= \iint 1_A(\{w + \alpha(\{u\})\} + [S^v(u)], v)\, \mu_{k-1}(d(u,v))\phi(w)\, dw\\
&= \iint 1_A(\{w'\} + [S^v(u)], v)\phi(w' - \alpha(\{u\}))\, \mu_{k-1}(d(u,v))\, dw'\\
&= \iint_{[0,1)} \sum_{n\in\mathbb{Z}} 1_A(w' + [S^v(u)], v)\phi(w' + n - \alpha(\{u\}))\, dw'\, \mu_{k-1}(d(u,v))\\
&= \iint_{[[S^v(u)],[S^v(u)]+1)} \sum_{n\in\mathbb{Z}} 1_A(w'', v)\phi(\{w''\} + n - \alpha(\{u\}))\, dw''\, \mu_{k-1}(d(u,v))\\
&= \iiint_{\mathbb{R}} 1_A(w'', v) \sum_{n\in\mathbb{Z}} \phi(\{w''\} + n - \alpha(\{u\}))1_{\{0\}}([w''] - [S^v(u)])\, dw''\, \mu_{k-1}(d(u,v))\\
&= \iiint_{[0,4)} 1_A(w'', v)\kappa^v(u, w'')\, dw''\, \mu_{k-1}^{\pi_1|\pi_2=v}(du)\, \mu_{k-1}^{\pi_2}(dv)\\
&= (\bar{K}(\mu_{k-1}) \otimes \mu_{k-1}^{\pi_2})(A).
\end{aligned}$$

□

Lemma 3.7 *Fix $k \in -\mathbb{N}$. Then we have:*

1. *$\mu_k^{\pi_2}$ does not depend on k. We denote this distribution by ρ.*

2. *$\mu_k^{\pi_1|\pi_2=y} = \frac{1}{4}\lambda|_{[0,4)}$ ρ-a.s. for $y \in (0,1/4)$. There are constants $c_1, c_2 \geq 0$ with $c_1 + c_2 = 1$ and such that $\mu_k^{\pi_1|\pi_2=0} = c_1\lambda|_{[0,1)} + c_2\frac{1}{3}\lambda|_{[1,4)}$ ρ-a.s. In addition, c_1, c_2 are independent of k.*

3. *μ_k does not depend on k. We write $\mu := \mu_k$.*

Proof.

1. This follows by induction from Lemma 3.6.

2. Since, by Lemma 3.6, $\mu_{k'}^{\pi_1|\pi_2=y}(A) = \int_A \int \kappa^y(x,x')\,\mu_{k'-1}^{\pi_1|\pi_2=y}(dx)\,dx'$ for ρ-almost all $y \in [0,1/4)$, $A \in \mathcal{B}([0,4))$, we conclude that $\mu_{k'}^{\pi_1|\pi_2=y}$ has a Lebesgue density $g_{k'}^y \in \mathcal{D}_{[0,4)}$ for any $k' \in -\mathbb{N}$.

   It suffices to show:

   $$\|g_k^y - 1/4\|_{L^1([0,4))} = 0 \text{ for } \rho\text{-a.a. } y \in (0,1/4)$$

   and, if $\rho(\{0\}) > 0$, then there are $c_1, c_2 \geq 0$ with $c_1 + c_2 = 1$ such that

   $$\left\|g_k^0 - \left(c_1 1_{[0,1)} + c_2\frac{1}{3}1_{[1,4)}\right)\right\|_{L^1([0,4))} = 0.$$

   By Lemma 3.6 and induction, one has that for any $l \in \mathbb{N}$:

   $$g_k^y = (\kappa^y)^l g_{k-l}^y \ \lambda\text{-a.s. for } \rho\text{-a.a. } y \in (0,1/4)$$

   and

   $$g_k^0(\cdot) = (\kappa^y)^l g_{k-l}^0 = (\kappa_1)^l g_{k-l}^0|_{[0,1)} 1_{[0,1)}(\cdot) + (\kappa_2)^l g_{k-l}^0|_{[1,4)} 1_{[1,4)}(\cdot)\ \lambda\text{-a.s.},$$

   hence

   $$g_k^0(\cdot)|_{[0,1)} = (\kappa_1)^l g_{k-l}^0|_{[0,1)}\ \lambda\text{-a.s.},\ g_k^0(\cdot)|_{[1,4)} = (\kappa_2)^l g_{k-l}^0|_{[1,4)}\ \lambda\text{-a.s.} \tag{3.2}$$

   Let $\varepsilon > 0$ and choose $l \in \mathbb{N}$ big enough to ensure

   $$\sup_{g \in \mathcal{D}_{[0,4)}} \|(\kappa^y)^l g - 1/4\|_{L^1([0,4))} < \varepsilon \text{ for } y \in (0,1/4)$$

   and

   $$\sup_{g \in \mathcal{D}_{[0,1)}} \|(\kappa_1)^l g - 1\|_{L^1([0,1))} < \varepsilon, \quad \sup_{g \in \mathcal{D}_{[1,4)}} \|(\kappa_2)^l g - 1/3\|_{L^1([1,4))} < \varepsilon.$$

   Then we obtain

   $$\|g_k^y - 1/4\|_{L^1([0,4))} = \|(\kappa^y)^l g_{k-l}^y - 1/4\|_{L^1([0,4))} < \varepsilon \text{ for } \rho\text{-a.a. } y \in (0,1/4).$$

We define real numbers $c_1 := \mu^0_{k-l}([0,1)) = \int_{[0,1)} g^0_{k-l}(x)\,dx$ and $c_2 := \mu^0_{k-l}([1,4)) = \int_{[1,4)} g^0_{k-l}(x)\,dx$. By Equation (3.2), c_1, c_2 are independent of l. Since $\frac{1}{c_1} g^0_{k-l}|_{[0,1)} \in \mathcal{D}_{[0,1)}$ and $\frac{1}{c_2} g^0_{k-l}|_{[1,4)} \in \mathcal{D}_{[1,4)}$, we obtain

$$
\begin{aligned}
&\left\| g^0_k - \left(c_1 1_{[0,1)} + c_2 \frac{1}{3} 1_{[1,4)}\right)\right\|_{L^1([0,4))} \\
&= \left\| g^0_k - c_1 \right\|_{L^1([0,1))} + \left\| g^0_k - c_2 \frac{1}{3}\right\|_{L^1([1,4))} \\
&= c_1 \left\| (\kappa_1)^l \left(\frac{1}{c_1} g^0_{k-l}\right) - 1 \right\|_{L^1([0,1))} + c_2 \left\| (\kappa_2)^l \left(\frac{1}{c_2} g^0_{k-l}\right) - \frac{1}{3} \right\|_{L^1([1,4))} \\
&< c_1 \varepsilon + c_2 \varepsilon = \varepsilon.
\end{aligned}
$$

3. This follows immediately from 1. and 2. □

Corollary 3.8 *ρ, c_1, c_2 satisfy Statement 2 in Section 2.*

Proof. Fix $k \in -\mathbb{N}$. By ν we denote the distribution of $(\{\eta_{t_k}\}, \{\tilde{\eta}_{t_k} - \eta_{t_k}\})$, i.e., $\nu = \psi^{-1}(\mu)$. Then we have for any $A \in \mathcal{B}([0,1) \times [0,1))$:

$$
\begin{aligned}
\nu(A) &= \mu(\psi(A)) \\
&= \mu(A \cap ([0,1) \times [0,1/4))) + \mu((A \cap ([0,1) \times [1/4,1/2))) + (1,-1/4)) \\
&\quad + \mu((A \cap ([0,1) \times [1/2,3/4))) + (2,-1/2)) \\
&\quad + \mu((A \cap ([0,1) \times [3/4,1))) + (3,-3/4)) \\
&= \left(\frac{1}{4}\lambda|_{[0,1)} \otimes \mu^{\pi_2}|_{(0,1/4)}\right)(A) + \left(c_1 \lambda|_{[0,1)} \otimes \mu^{\pi_2}|_{\{0\}}\right)(A) \\
&\quad + \left(\frac{1}{4}\lambda|_{[0,1)} \otimes (\mu^{\pi_2}|_{(0,1/4)} * \epsilon_{1/4})\right)(A) + \left(c_2 \frac{1}{3}\lambda|_{[0,1)} \otimes (\mu^{\pi_2}|_{\{0\}} * \epsilon_{1/4})\right)(A) \\
&\quad + \left(\frac{1}{4}\lambda|_{[0,1)} \otimes (\mu^{\pi_2}|_{(0,1/4)} * \epsilon_{1/2})\right)(A) + \left(c_2 \frac{1}{3}\lambda|_{[0,1)} \otimes (\mu^{\pi_2}|_{\{0\}} * \epsilon_{1/2})\right)(A) \\
&\quad + \left(\frac{1}{4}\lambda|_{[0,1)} \otimes (\mu^{\pi_2}|_{(0,1/4)} * \epsilon_{3/4})\right)(A) + \left(c_2 \frac{1}{3}\lambda|_{[0,1)} \otimes (\mu^{\pi_2}|_{\{0\}} * \epsilon_{3/4})\right)(A) \\
&= \lambda|_{[0,1]} \otimes \Bigg(\rho|_{(0,1/4)} * \frac{1}{4}(\epsilon_0 + \epsilon_{1/4} + \epsilon_{1/2} + \epsilon_{3/4}) \\
&\quad + c_1 \rho|_{\{0\}} + c_2 \rho|_{\{0\}} * \frac{1}{3}(\epsilon_{1/4} + \epsilon_{1/2} + \epsilon_{3/4}) \Bigg)(A).
\end{aligned}
$$

□

Proof of Statement 3.
Fix $k \in -\mathbb{N}$. It suffices to show that $\psi(\{\eta_{t_k}\}, \{\tilde{\eta}_{t_k} - \eta_{t_k}\})$ is independent of $(B_t - B_{t_{k-l}})_{t_{k-l} < t \le 1}$ for any $l \in \mathbb{N}$. This follows from

Lemma 3.9 *For any $l \in \mathbb{N}$, μ is a version of $P^{\psi(\{\eta_{t_k}\}, \{\tilde{\eta}_{t_k} - \eta_{t_k}\}) | (B_t - B_{t_{k-l}})_{t_{k-l} < t \le 1}}$.*

Proof. We prove the lemma by induction on l. For $l = 0$ this holds, because $(B_t - B_{t_k})_{t_k < t \le 1}$ is independent of $\mathcal{F}_{t_k}$. Assume that the lemma holds for $l - 1 \in \mathbb{N}$ (regardless of k). By Equation (3.1), by assumption, and by Lemmas 3.7.2 and 3.2, we have for any $A \in \mathcal{B}([0,4) \times [0,1/4))$:

$$
\begin{aligned}
&P^{\psi(\{\eta_{t_k}\},\{\tilde{\eta}_{t_k}-\eta_{t_k}\})|(B_t-B_{t_{k-l}})_{t_{k-l}<t\leq 1}}(A) \\
&= \int 1_A(\{\varepsilon_{t_k} + \alpha(\{u\})\} + [S^v(u)], v)\, P^{\psi(\{\eta_{t_{k-1}}\},\{\tilde{\eta}_{t_{k-1}}-\eta_{t_{k-1}}\})|(B_t-B_{t_{k-l}})_{t_{k-l}<t\leq 1}}(d(u,v)) \\
&= \int 1_A(\{\varepsilon_{t_k} + \alpha(\{u\})\} + [S^v(u)], v)\, \mu(d(u,v)) \\
&= \iiint 1_A(u', v)\, \epsilon_{\{\varepsilon_{t_k}+\alpha(\{u\})\}+[S^v(u)]}(du')\, \mu^{\pi_1|\pi_2=v}(du)\, \mu^{\pi_2}(dv) \\
&= (\bar{L}^{\varepsilon_{t_k}}(\mu) \otimes \mu^{\pi_2})(A) \\
&= \mu(A)
\end{aligned}
$$

□

Proof of Statement 4.
This follows easily from 3. and the fact that the distribution of $\{\eta_{t_k}\}$ is $\lambda|_{[0,1)}$ and hence not degenerate.

Proof of Statement 5.
We have

$$\{\tilde{\eta}_{t_k}\} = E(\{\tilde{\eta}_{t_k}\}|\mathcal{F}_1^X) = E(\{\tilde{\eta}_{t_k}\}|\sigma(\{\eta_{t_k}\}) \vee \mathcal{F}_1^B) = E(\{\tilde{\eta}_{t_k}\}|\sigma(\{\eta_{t_k}\}))\ \ P\text{-a.s.},$$

where the third equality follows from Statement 3 (cf. Bauer (1991), Satz 15.5).

Proof of Statement 6.
Observe that

$$
\begin{aligned}
\epsilon_{\{\beta(\{\eta_{t_k}\})-\{\eta_{t_k}\}\}} &= P^{\{\beta(\{\eta_{t_k}\})-\{\eta_{t_k}\}\}|\{\eta_{t_k}\}} \\
&= P^{\{\tilde{\eta}_{t_k}-\eta_{t_k}\}|\{\eta_{t_k}\}} \\
&= \Big(\rho|_{(0,1/4)} * \frac{1}{4}(\epsilon_0 + \epsilon_{1/4} + \epsilon_{1/2} + \epsilon_{3/4}) \\
&\qquad + c_1\rho|_{\{0\}} + c_2\rho|_{\{0\}} * \frac{1}{3}(\epsilon_{1/4} + \epsilon_{1/2} + \epsilon_{3/4})\Big).
\end{aligned}
$$

This is only possible if $\rho|_{(0,1/4)} = 0, c_2 = 0$ and hence $\{\tilde{\eta}_{t_k}\} = \beta(\{\eta_{t_k}\}) = \{\eta_{t_k}\}$ P-a.s. By Equations (2.3), (2.2), and according equations for $\tilde{X}$, we conclude that $\{\tilde{\eta}_{t_l}\} = \{\eta_{t_l}\}$ P-a.s. for any $l \in -\mathbb{N}$ and therefore $\tilde{X}_t = X_t$ P-a.s. for any $t \geq 0$.

Proof of Statement 7.
X^a is obviously a solution to SDE (2.1). By 4., it is not strong. For any solution $\tilde{X}^a$ starting at a, the process $\tilde{X}^a - a$ is a solution starting at 0 and hence indistinguishable from X. Thus, $\tilde{X}^a$ is indistinguishable from X^a.

Proof of the remark.
The whole proof works analogously if all processes are restricted to $[0,T]$ for any $T > 0$.

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