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The Existence of a Multiple Spider Martingale in the Natural Filtration of a Certain Diffusion in the Plane

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Introduction

The notion of spider martingales (martingales-araignées) with n rays, $n = 2, 3, \dots, \infty$, has been introduced by Yor ([Y]) by generalizing Walsh's Brownian motions. A spider martingale with 2 rays is essentially a continuous local martingale and, on the other hand, a non-trivial spider martingale with $n \geq 3$ rays is called a *multiple spider martingale*. By the recent works by Tsirelson ([T]) and Barlow, Emery, Knight, Song and Yor ([BEKSY]), it has been recognized that a multiple spider martingale plays an important role in distinguishing a filtration from a Brownian filtration or, more generally, from a filtration which is homomorphic to a Brownian filtration; that is, any filtration which is homomorphic to a Brownian filtration can not contain a multiple spider martingale. In other words, as a noise generating the randomness of probability models, multiple spider martingales could sometimes provide us with a more useful information than a usual martingale could do. So it seems important to study, for a given stochastic process, if there exists a multiple spider martingale or not in its natural filtration. For the convenience of readers, we give in Section 1 a brief survey on the isomorphism problem of filtrations in connection with spider martingales.

The filtration of a smooth diffusion, "smooth" in the sense that it can be obtained as a strong solution of an Itô's stochastic differential equation (SDE) driven by a Wiener process, can not contain a multiple spider martingale because, as a strong solution of SDE, the natural filtration of the diffusion is homomorphic to the Brownian filtration generated by the driving Wiener process. On the other hand, the natural filtration of a Walsh's Brownian motion on $n \geq 3$ rays is a typical (and, indeed, a trivial) example of a filtration containing a multiple spider martingale. In Section 2, as a main purpose of this note, we give a less trivial example of a diffusion process on the plane \mathbf{R}^2 whose natural filtration contains a multiple spider martingale. This diffusion process has been studied by Ikeda and Watanabe ([IW 1] or [IW 2]) as an example of diffusions whose infinitesimal generators are not differential operators in the classical sense.

1 The isomorphism problem of filtrations

As we said in Introduction, we give here a summary of recent important results on the isomorphism problem of filtrations by Tsirelson ([T]) and Barlow, Emery, Knight, Song and Yor ([BEKSY]). No proofs are given. The reader is recommended to refer to [T] and [BEKSY] for proofs and more details.

As usual, by a *filtration* $\mathbf{F} = (F(t))_{t \in [0, \infty)}$ on a complete probability space (Ω, F, P) , we mean an increasing family of sub σ -fields of F satisfying the usual conditions, that is,

it is right-continuous and $F(0)$ contains all P -null sets. We set $F(\infty) = \bigvee_{t \in [0, \infty)} F(t)$. Let \mathbf{F} be a filtration on (Ω, F, P) and \mathbf{F}' be another filtration on (Ω', F', P') .

Definition 1.1. By a morphism π from \mathbf{F} to \mathbf{F}' , denoted by $\pi : \mathbf{F} \rightarrow \mathbf{F}'$, we mean a map

$$\pi_* : L^0(\Omega'; F'(\infty)) \longrightarrow L^0(\Omega; F(\infty))$$

satisfying the following conditions (i)~(iii). ($L^0(\Omega; F(\infty))$, or $L^0(F(\infty))$, stands for the real vector space formed of all $F(\infty)$ -measurable real random variables on Ω .)

(i) For any $X_1, \dots, X_n \in L^0(\Omega'; F'(\infty))$,

$$([X_1, \dots, X_n], P') \stackrel{d}{=} ([\pi_*(X_1), \dots, \pi_*(X_n)], P).$$

(ii) For any $X_1, \dots, X_n \in L^0(\Omega'; F'(\infty))$ and $f : \mathbf{R}^n \rightarrow \mathbf{R}$ which is Borel measurable,

$$\pi_*[f(X_1, \dots, X_n)] = f(\pi_*(X_1), \dots, \pi_*(X_n)).$$

(iii) For any $X \in L^1(\Omega'; F'(\infty))$, (then, obviously $\pi_*(X) \in L^1(\Omega; F(\infty))$),

$$\pi_*[E(X|F'(t))] = E[\pi_*(X)|F(t)], \quad \text{for all } t \geq 0.$$

A morphism is also called a homomorphism; we say that \mathbf{F}' is homomorphic to \mathbf{F} if there exists a morphism π from \mathbf{F} to \mathbf{F}' .

Note that the map π_* is obviously one-to-one and *non-anticipative* in the sense that, for every $t \geq 0$, $\pi_*(X)$ is $F(t)$ -measurable if X is $F'(t)$ -measurable.

Definition 1.2. A morphism π from \mathbf{F} to \mathbf{F}' is called an isomorphism from \mathbf{F} to \mathbf{F}' if $\pi_* : L^0(\Omega'; F'(\infty)) \longrightarrow L^0(\Omega; F(\infty))$ is onto. We say that \mathbf{F}' is isomorphic to \mathbf{F} or, \mathbf{F} and \mathbf{F}' are isomorphic, if there exists an isomorphism π from \mathbf{F} to \mathbf{F}' .

Remark 1.1. Since the map π_* is one-to-one, we can say equivalently as follows: \mathbf{F} and \mathbf{F}' are isomorphic if and only if there exists a morphism π from \mathbf{F} to \mathbf{F}' and a morphism π' from \mathbf{F}' to \mathbf{F} such that $\pi' \circ \pi = \text{id}$ and $\pi \circ \pi' = \text{id}$, i.e., $\pi_* \circ \pi'_* = \text{id}_*$ on $L^0(\Omega; F(\infty))$ and $\pi'_* \circ \pi_* = \text{id}_*$ on $L^0(\Omega'; F'(\infty))$.

Generally, for two filtrations $\mathbf{F} = (F(t))$ and $\mathbf{G} = (G(t))$ on the same probability space (Ω, F, P) , we denote $\mathbf{G} \subset \mathbf{F}$ if $G(t) \subset F(t)$ for all $t \geq 0$.

A probability space (Ω, F, P) endowed with a filtration \mathbf{F} is called a *filtered probability space* and is denoted by $\{(\Omega, F, P), \mathbf{F}\}$.

Definition 1.3. For an \mathbf{F} -adapted stochastic process $X = (X(t))$ on $\{(\Omega, F, P), \mathbf{F}\}$ and an \mathbf{F}' -adapted stochastic process $Y = (Y(t))$ on $\{(\Omega', F', P'), \mathbf{F}'\}$, we say that Y has a canonical representation by X if the natural filtration $\mathbf{F}^Y(\subset \mathbf{F}')$ of Y is homomorphic to the natural filtration $\mathbf{F}^X(\subset \mathbf{F})$ of X .

Definition 1.4. If, in Def. 1.3, \mathbf{F}^X and \mathbf{F}^Y are isomorphic, then we say that Y has a properly canonical representation by X .

We denote by $\mathcal{M}(\mathbf{F})$ the space of all locally square-integrable \mathbf{F} -martingales $M = (M(t))$ with $M(0) = 0$.

Proposition 1.1. (1) Y has a canonical representation by X if and only if

$$\exists Y' \stackrel{d}{=} Y \quad \text{such that} \quad \mathbf{F}^{Y'} \subset \mathbf{F}^X \quad \text{and} \quad \mathcal{M}(\mathbf{F}^{Y'}) \subset \mathcal{M}(\mathbf{F}^X).$$

(2) Y has a properly canonical representation by X if and only if

$$\exists Y' \stackrel{d}{=} Y \quad \text{such that} \quad \mathbf{F}^{Y'} = \mathbf{F}^X, \quad (\text{then, obviously, } \mathcal{M}(\mathbf{F}^{Y'}) = \mathcal{M}(\mathbf{F}^X).)$$

Remark 1.2. By Proposition 1.1, we can see clearly that the notions of the canonical and properly canonical representations exactly correspond to Hida's ([H]) (in the case of linear representations of Gaussian processes by a Wiener process) and Nisio's ([N]) (in the case of nonlinear representations of general stochastic processes by a Wiener process).

Remark 1.3. We say that a map $\pi_* : L^0(F'(\infty)) \longrightarrow L^0(F(\infty))$ is a morphism in the weak sense if it satisfies the same conditions as in Def. 1.1 in which (iii) is replaced by: (iii)' If $X \in L^1(F'(\infty))$ is $F'(t)$ -measurable, then $\pi_*(X)$ is $F(t)$ -measurable for every $t \geq 0$.

The existence of a weak morphism corresponds, in the case of stochastic processes, to the condition: $\exists Y' \stackrel{d}{=} Y$ such that $\mathbf{F}^{Y'} \subset \mathbf{F}^X$. In such a case, we say that Y has a non-anticipative representation by X . However, this notion is very weak compared to that of canonical or properly canonical representation. Indeed, denoting by $BM^0(m)$ an m -dimensional standard Brownian motion starting at 0, if $X = BM^0(m)$ and $Y = BM^0(n)$, then a non-anticipative representation of Y by X exists for any m and n . However, a canonical representation of Y by X exists if and only if $n \leq m$, and a properly canonical representation of Y by X exists if and only if $n = m$. (These facts follow immediately from Theorem 1.1 and its Corollary given below since the multiplicity of the natural filtration of $X = BM^0(m)$ is m .)

In the problem of existence or non-existence of canonical and properly canonical representations for stochastic processes, or more generally, existence and non-existence of homomorphisms and isomorphisms for filtrations, a useful and well-known invariant is the *multiplicity* or the *rank* of filtrations (cf. Davis-Varaiya ([DV]), Skorohod [S], cf. also, Motoo-Watanabe ([MW]), Kunita-Watanabe ([KW])).

Let $\mathbf{F} = (F(t))$ be a filtration on (Ω, \mathcal{F}, P) . We assume that the filtration is separable in the sense that the Hilbert space $L_2(\Omega, F(\infty), P)$ is separable.

Theorem 1.1.

(1) There exist $M_1, M_2, \dots \in \mathcal{M}(\mathbf{F})$ such that

$$\langle M_i, M_j \rangle = 0 \quad \text{if } i \neq j, \quad \langle M_1 \rangle \gg \langle M_2 \rangle \gg \dots$$

and every $M \in \mathcal{M}(\mathbf{F})$ can be represented as a sum of stochastic integrals for some \mathbf{F} -predictable processes Φ_i as

$$M = \sum_i \int \Phi_i dM_i.$$

If N_1, N_2, \dots is another such sequence, then

$$\langle M_1 \rangle \approx \langle N_1 \rangle, \langle M_2 \rangle \approx \langle N_2 \rangle, \dots$$

Such a system M_1, M_2, \dots is called a basis of $\mathcal{M}(\mathbf{F})$. Here as usual, $\langle M, N \rangle$ for $M, N \in \mathcal{M}^2(\mathcal{F})$ is the quadratic covariational process, $\langle M \rangle = \langle M, M \rangle$ and, \gg and \approx denote the absolute continuity and the equivalence of increasing processes, respectively.

In particular, the cardinal of a basis is an invariant of the filtration \mathbf{F} which we call the multiplicity of \mathbf{F} and denote by $\text{mult}(\mathbf{F})$.

- (2) Let $\mathbf{F}' \subset \mathbf{F}$ be a sub-filtration of \mathbf{F} and suppose $\mathcal{M}(\mathbf{F}') \subset \mathcal{M}(\mathbf{F})$. Let $\{M_i\}$ and $\{M'_i\}$ be the basis of $\mathcal{M}(\mathbf{F})$ and $\mathcal{M}(\mathbf{F}')$, respectively. Then

$$\langle M'_1 \rangle \ll \langle M_1 \rangle, \langle M'_2 \rangle \ll \langle M_2 \rangle, \dots$$

In particular, $\text{mult}(\mathbf{F}') \leq \text{mult}(\mathbf{F})$.

Corollary 1.1. If \mathbf{F}' is homomorphic to \mathbf{F} , then $\text{mult}(\mathbf{F}') \leq \text{mult}(\mathbf{F})$.

Let $\mathcal{M}^c(\mathbf{F})$ be the totality of continuous elements of $\mathcal{M}(\mathbf{F})$. Then the property that $\mathcal{M}^c(\mathbf{F}) = \mathcal{M}(\mathbf{F})$, is an invariant for the existence of homomorphisms: If $\mathcal{M}^c(\mathbf{F}) = \mathcal{M}(\mathbf{F})$ and \mathbf{F}' is homomorphic to \mathbf{F} , then we must have $\mathcal{M}^c(\mathbf{F}') = \mathcal{M}(\mathbf{F}')$.

The notion of the multiplicity of filtrations is useful to distinguish various filtrations. However, it is by no means complete; in fact, we have several examples of filtrations \mathbf{F} such that $\mathcal{M}^c(\mathbf{F}) = \mathcal{M}(\mathbf{F})$ with a single base M_1 such that $\langle M_1 \rangle(t) = t$ and \mathbf{F} is not a natural filtration of $BM^0(1)$. An example was given by Dubins, Feldman, Smorodinsky and Tsirelson ([DFST]) and recently, a conjecture of Barlow, Pitman and Yor ([BPY]) that the natural filtration of a Walsh's Brownian motion on $n \geq 3$ rays is not a Brownian filtration has been finally settled affirmatively by Tsirelson ([T]). For this, Tsirelson introduced another invariant notion for filtrations, the notion of *cosiness* of filtrations. We would formulate this notion as follows:

Definition 1.5. A family $\mathbf{F}_\alpha = (F_\alpha(t)), \alpha \in [0, 1]$, of filtrations on (Ω, \mathcal{F}, P) is called a T-system (Tsirelson system) if it satisfies the following properties:

- (1) There exists a filtration $\hat{\mathbf{F}}$ such that, for every $\alpha \in [0, 1]$, $\mathbf{F}_\alpha \subset \hat{\mathbf{F}}$ and $\mathcal{M}(\mathbf{F}_\alpha) \subset \mathcal{M}(\hat{\mathbf{F}})$ so that the injection $i_* : L^0(F_\alpha(\infty)) \rightarrow L^0(\hat{\mathbf{F}}(\infty))$ satisfies the conditions (i)~(iii) of Def. 1.1.
- (2) For every $\alpha \in (0, 1]$, there exists $0 < \rho(\alpha) < 1$ such that, for all $M \in \mathcal{M}(\mathbf{F}_0) \subset \mathcal{M}(\hat{\mathbf{F}})$ and $N \in \mathcal{M}(\mathbf{F}_\alpha) \subset \mathcal{M}(\hat{\mathbf{F}})$, the following holds:

$$|\langle M, N \rangle(t)| \leq \rho(\alpha) \sqrt{\langle M \rangle(t) \cdot \langle N \rangle(t)}, \quad \forall t \geq 0, \text{ a.s.}$$

- (3) $\forall \alpha \in (0, 1], \exists$ isomorphism $\pi_\alpha : \mathbf{F}_\alpha \rightarrow \mathbf{F}_0$, i.e., $(\pi_\alpha)_* : L^0(F_0(\infty)) \rightarrow L^0(F_\alpha(\infty))$ such that

$$\|X - (\pi_\alpha)_*(X)\|_2 \rightarrow 0 \quad \text{as } \alpha \rightarrow 0$$

for all $X \in L^2(F_0(\infty))$. Note that, by (1), $X \in L^2(F_0(\infty)) \subset L^2(\hat{\mathbf{F}}(\infty))$ and $(\pi_\alpha)_*(X) \in L^2(F_\alpha(\infty)) \subset L^2(\hat{\mathbf{F}}(\infty))$.

A typical example is that induced from a family of Brownian filtrations as follows:

Example 1.1. On a suitable probability space (Ω, F, P) , we set up a $BM^0(2d)$ as $B = (B(t)) = (B_1(t), B_2(t))$ where B_1, B_2 are two mutually independent $BM^0(d)$'s. Set

$$W_\alpha(t) = \sqrt{1 - \alpha^2} B_1(t) + \alpha B_2(t), \quad \alpha \in [0, 1].$$

Then W_α is a $BM^0(d)$ for all α . Set $\hat{\mathbf{F}} = \mathbf{F}^B$ and $\mathbf{F}_\alpha = \mathbf{F}^{W_\alpha}$. Then, $\{(\Omega, F, P), \{\mathbf{F}_\alpha\}, \hat{\mathbf{F}}\}$ is a T -system.

Indeed, if $(\mathbf{W}^d, F(\mathbf{W}^d), \mu^d)$ is the d -dimensional Wiener space, i.e.,

$$\mathbf{W}^d = \{w \in \mathcal{C}([0, \infty) \rightarrow \mathbf{R}^d) \mid w(0) = 0\}$$

and μ^d is the d -dimensional Wiener measure defined on the σ -field $F(\mathbf{W}^d)$ of μ^d -measurable sets, then for every $X \in L^0(F_0(\infty))$, there exists a unique $\tilde{X} \in L^0(F(\mathbf{W}^d))$ such that $X(\omega) = \tilde{X}(W_0(\omega))$. Define $(\pi_\alpha)_* : L^0(F_0(\infty)) \rightarrow L^0(F_\alpha(\infty))$ by $(\pi_\alpha)_*(X) = \tilde{X}(W_\alpha(\omega))$. (2) can be deduced, by taking $\rho(\alpha) = \sqrt{1 - \alpha^2}$, from the martingale representation theorem for $\mathcal{M}(\mathbf{F}_\alpha)$ and the relation $\langle W_0, W_\alpha \rangle(t) = \sqrt{1 - \alpha^2} \cdot t \cdot I$, I being $d \times d$ -identity matrix. Finally, (3) can be deduced from the relation

$$\|X - (\pi_\alpha)_*(X)\|_2 = \|\tilde{X} - T_t \tilde{X}\|_{L^2(\mu^d)}, \quad X \in L^2(F_0(\infty)).$$

Here, $1 - \alpha^2 = e^{-2t}$ and T_t is the Ornstein-Uhlenbeck semigroup on the Wiener space.

Definition 1.6. A filtration \mathbf{F} on (Ω, F, P) is said to be cosy if there exist a T -system $\{(\Omega', F', P'), \{\mathbf{F}'_\alpha\}, \hat{\mathbf{F}}'\}$ and a morphism $\pi : \mathbf{F}'_0 \rightarrow \mathbf{F}$; that is, \mathbf{F} is homomorphic to \mathbf{F}'_0 .

From this definition and Example 1.1, we can easily deduce the following proposition:

Proposition 1.2. (1) The Brownian filtration, i.e., the natural filtration of a $BM^0(d)$, for any dimension d , is cosy.

(2) If \mathbf{F} is cosy and \mathbf{F}' is homomorphic to \mathbf{F} , then \mathbf{F}' is cosy.

The notion of spider martingales (martingales-araignées) has been introduced by Yor ([Y], p.110). We follow the definition given in [BEKSY]: Before proceeding, we give some notions and notations. Let $n \geq 2$ and \mathbf{E} be a real vector space of $n - 1$ dimension. Let $\mathbf{U} = \{u_1, \dots, u_n\}$ be a set of n nonzero vectors in \mathbf{E} such that \mathbf{U} spans the whole space and $\sum_{k=1}^n u_k = 0$. Let

$$\mathbf{T}(= \mathbf{T}(\mathbf{U})) := \bigcup_{k=1}^n \{\lambda u_k \mid \lambda \in [0, \infty)\} \subset \mathbf{E}.$$

\mathbf{T} is called a *web* (une toile d'araignée) of n -rays. When $n = 2$, then $\mathbf{U} = \{u_1, u_2 = -u_1\}$ and

$$\mathbf{T} = \{\lambda u_1 \mid \lambda \in [0, \infty)\} \cup \{\lambda u_1 \mid \lambda \in (-\infty, 0]\} = \{\lambda u_1 \mid \lambda \in \mathbf{R}\} \cong \mathbf{R},$$

so that a web of 2 rays is essentially a real line.

Definition 1.7. A spider martingale is a $\mathbf{T}(\subset \mathbf{E})$ -valued continuous local martingale $M = (M(t))$ with $M(0) = 0$ for some web \mathbf{T} in \mathbf{E} . If \mathbf{T} is a web of n rays, then M is said a spider martingale with n rays.

Thus a spider martingale with 2 rays is essentially a continuous local martingale M with $M(0) = 0$.

Definition 1.8. A nontrivial spider martingale with $n \geq 3$ rays is called a multiple spider martingale.

When martingales are referred to a filtration \mathbf{F} , we say an \mathbf{F} -spider martingale.

We denote by $WBM^x(n; p_1, \dots, p_n)$ a Walsh's Brownian motion (cf. [BPY]) on n rays from the origin with the rate of excursions at the origin given by p_1, \dots, p_n , ($p_i > 0$, $\sum_{i=1}^n p_i = 1$) and starting at x . A Walsh's Brownian motion $WBM^0(n; 1/n, \dots, 1/n)$ is a typical example of a spider martingale with n rays and, indeed, a spider martingale with n rays is essentially obtained from a $WBM^0(n; 1/n, \dots, 1/n)$ by a time change.

Theorem 1.2. (Tsirolson 1996, cf. [T], [BEKSY]). If there exists a multiple \mathbf{F} -spider martingale in a filtration \mathbf{F} , then \mathbf{F} is not cosy. In particular, such a filtration can not be homomorphic to a Brownian filtration in any dimension. In other words, when a filtration \mathbf{F} is the natural filtration of a stochastic process $X = (X(t))$ and a multiple \mathbf{F} -spider martingale exists, then X can not have a canonical representation by a Wiener process in any dimension.

It is not difficult to see that a Walsh's Brownian motion $WBM^x(n; p_1, \dots, p_n)$ contains a multiple spider martingale with n rays in its natural filtration. Also, the following result was obtained by Barlow et al. (cf. [BEKSY]).

Theorem 1.3. If \mathbf{F} is a filtration which is homomorphic to the natural filtration of a $WBM^x(n; p_1, \dots, p_n)$ and if $m > n$, there does not exist any multiple \mathbf{F} -spider martingale with m rays.

We recall the operation of *time change* on filtrations (cf. [IW 3], p.102). Given a filtration \mathbf{F} on (Ω, \mathcal{F}, P) , we mean, by a *process of time change* with respect to \mathbf{F} , an \mathbf{F} -adapted increasing process $A = (A(t))$ such that, $A(0) = 0$, $t \rightarrow A(t)$ is continuous, strictly increasing and $\lim_{t \uparrow \infty} A(t) = \infty$, almost surely. If A is a process of time change with respect to \mathbf{F} , then, for each $t \geq 0$, $A^{-1}(t) = \inf\{u | A(u) = t\}$ is an \mathbf{F} -stopping time and the σ -field $\mathcal{F}(A^{-1}(t)) := \mathcal{F}^{(A)}(t)$ is defined as usual. Then, we have a filtration $\mathbf{F}^{(A)} = (\mathcal{F}^{(A)}(t))$ and, with respect to which, the increasing process $A^{-1} = (A^{-1}(t))$ is a process of time change. We can easily see that $\mathbf{F} = \{\mathcal{F}^{(A)}\}^{(A^{-1})}$. More generally, if A is a process of time change with respect to \mathbf{F} and B is a process of time change with respect to $\mathbf{F}^{(A)}$, then $C = B \circ A = \{C(t) := B(A(t))\}$ is a process of time change with respect to \mathbf{F} and $\mathbf{F}^{(C)} = \{\mathcal{F}^{(A)}\}^{(B)}$. Also, the following proposition can be easily deduced:

Proposition 1.3. If $\pi : \mathbf{F} \rightarrow \mathbf{F}'$ is a morphism (i.e., \mathbf{F}' is homomorphic to \mathbf{F}) and $A' = (A'(t))$ is a process of time change with respect to \mathbf{F}' , then the process $A = (A(t))$ defined by $A(t) = \pi_*(A'(t))$ is a process of time change with respect to \mathbf{F} and the same map $\pi_* : L^0(\mathbf{F}'(\infty)) \rightarrow L^0(\mathbf{F}(\infty))$ induces the homomorphism $\pi : \mathbf{F}^{(A)} \rightarrow \mathbf{F}'^{(A')}$.

Remark 1.4. The property of cosiness of filtrations is not invariant under the time change; indeed, as we shall see, the filtration of a sticky Brownian motion, which is a time change of a Brownian filtration, is not cosy.

The following strengthens a little Theorem 1.2.

Theorem 1.4. If there exists an \mathbf{F} -multiple spider martingale in a filtration \mathbf{F} , then \mathbf{F} can not be homomorphic to any time change of a cosy filtration, in particular, to any time change of a Brownian filtration in any dimension.

Indeed, if \mathbf{F} is homomorphic to a time change $\mathbf{G}^{(A)}$ of a cosy filtration \mathbf{G} , then $\mathbf{G}^{(A)}$ contains a $\mathbf{G}^{(A)}$ -multiple spider martingale and hence, \mathbf{G} contains a \mathbf{G} -multiple spider martingale. However, this contradicts with Theorem 1.2.

Finally, we introduce the notion of the *direct product* of filtrations. Given filtrations $\mathbf{F}^{(i)}$ on $(\Omega^{(i)}, \mathcal{F}^{(i)}, P^{(i)})$, $i = 1, \dots, m$, define their direct product $\otimes_{i=1}^m \mathbf{F}^{(i)}$, as a filtration on the product probability space $(\Pi_{i=1}^m \Omega^{(i)}, \otimes_{i=1}^m \mathcal{F}^{(i)}, \otimes_{i=1}^m P^{(i)})$, by

$$\otimes_{i=1}^m \mathbf{F}^{(i)} = (\otimes_{i=1}^m \mathcal{F}^{(i)}(t))_{t \geq 0},$$

where $\otimes_{i=1}^m \mathcal{F}^{(i)}(t)$ is the usual product σ -field. Then the following propositions are easily deduced:

Proposition 1.4. If $\pi_i : \mathbf{F}^{(i)} \rightarrow \mathbf{F}'^{(i)}$, $i = 1, \dots, m$ are morphisms, then there exists a unique morphism

$$\otimes_{i=1}^m \pi_i : \otimes_{i=1}^m \mathbf{F}^{(i)} \rightarrow \otimes_{i=1}^m \mathbf{F}'^{(i)}$$

such that $(\otimes_{i=1}^m \pi_i)_* (\otimes_{i=1}^m X_i) = \otimes_{i=1}^m [(\pi_i)_* (X_i)]$ for $X_i \in L^0(\Omega'^{(i)}; \mathcal{F}'^{(i)}(\infty))$.

Here, as usual, $[\otimes_{i=1}^m X_i](\omega'_1, \dots, \omega'_m) = \Pi_{i=1}^m X_i(\omega'_i)$, $(\omega'_1, \dots, \omega'_m) \in \Pi_{i=1}^m \Omega'^{(i)}$.

Proposition 1.5. If $\{(\Omega^{(i)}, \mathcal{F}^{(i)}, P^{(i)}), \{\mathbf{F}_\alpha^{(i)}\}, \widehat{\mathbf{F}}^{(i)}\}$, $i = 1, \dots, m$, are T -systems, then $\{(\Pi_{i=1}^m \Omega^{(i)}, \otimes_{i=1}^m \mathcal{F}^{(i)}, \otimes_{i=1}^m P^{(i)}), \{\otimes_{i=1}^m \mathbf{F}_\alpha^{(i)}\}, \otimes_{i=1}^m \widehat{\mathbf{F}}^{(i)}\}$ is a T -system.

Corollary 1.2. If $\mathbf{F}^{(i)}$ are cosy for all i , then their direct product $\otimes_{i=1}^m \mathbf{F}^{(i)}$ is cosy.

2 The existence of a multiple spider martingale in a diffusion on the plane

In the plane \mathbf{R}^2 , let L_1, \dots, L_n be n different straight half-lines (rays) starting at the origin 0. Let $\mathbf{e}^{(k)} = (e_1^k, e_2^k)$, $k = 1, \dots, n$, be the unit direction vector of L_k , respectively. Let \mathcal{D}_0 be the space of all C^∞ -functions on the plane \mathbf{R}^2 with a compact support and vanishing also in a neighborhood of the origin 0. Define the following bilinear form for $f, g \in \mathcal{D}_0$;

$$\mathcal{E}(f, g) = \frac{1}{2} \int_{\mathbf{R}^2} \sum_{i=1}^2 \frac{\partial f(x)}{\partial x^i} \frac{\partial g(x)}{\partial x^i} dx + \sum_{k=1}^n \int_{L_k} \sum_{i,j=1}^2 e_i^k e_j^k \frac{\partial f(x)}{\partial x^i} \frac{\partial g(x)}{\partial x^j} d\mu_k(x) \quad (1)$$

where $d\mu_k$ is the (one-dimensional) Lebesgue measure on the half-line L_k . Then, setting $D = \mathbf{R}^2 \setminus \{0\}$, $\mathcal{E}(f, g)$ with domain \mathcal{D}_0 is a closable Markovian form on $L^2(D; dx)$ and its closure is a regular and local Dirichlet form. Hence, by the general theory of

Dirichlet forms ([FOT]), there corresponds a unique diffusion process $\mathbf{X} = \{X(t), P_x\}$ on D . Locally, the sample paths of this diffusion can be constructed by a skew product of two mutually independent $BM(1)$'s so that the diffusion is precisely defined for every starting point in D . As is proved in [IW 1] or [IW 2], we have that $P_x(\zeta < \infty) = 1$ and $P_x(\lim_{t \uparrow \zeta} X(t) = 0) = 1$ for every $x \in D$, where ζ is the lifetime of \mathbf{X} and the terminal point (cemetery) can be identified with the origin 0. \mathbf{X} is a symmetric diffusion with respect to the Lebesgue measure dx on D and it possesses the continuous transition density $p(t, x, y)$, $(t, x, y) \in (0, \infty) \times D \times D$, so that $p(t, x, y) = p(t, y, x)$.

We say that a continuous function $u(x)$ on D is \mathbf{X} -harmonic if $u(x) = E_x[u(X(\sigma_{U^c}))]$ for every $x \in D$ and every bounded neighborhood U of x such that $\bar{U} \subset D$, where $\sigma_{U^c} = \inf\{t | X(t) \notin U\}$. $u(x)$ is \mathbf{X} -harmonic if and only if, writing $L_k^o = L_k \cap D$,

- (i) $u(x)$ is continuous in D ,
- (ii) $u(x)$ is harmonic in the usual sense in the open set $D \setminus \{\bigcup_{k=1}^n L_k\}$,
- (iii) for each $k = 1, \dots, n$

$$u|_{L_k^o} \in C^2(L_k^o) \quad \text{and} \quad 2 \frac{\partial^2 u}{\partial \xi^2}(\xi, 0) = \frac{\partial u}{\partial \eta}(\xi, 0+) - \frac{\partial u}{\partial \eta}(\xi, 0-)$$

where we introduce a local coordinate (ξ, η) of $y \in U$, in a sufficiently small neighborhood U of $x \in L_k^o$, by $y - x = \xi \cdot \mathbf{e}^{(k)} + \eta \cdot \mathbf{e}^{(k)\perp}$; $(\mathbf{e}^{(k)\perp} = (-e_2^k, e_1^k)$: the unit vector perpendicular to $\mathbf{e}^{(k)}$.)

It was shown in [IW 1] or [IW 2] that, for each $k = 1, \dots, n$, there exists a unique bounded \mathbf{X} -harmonic function $u_k(x)$ such that

$$\lim_{x \rightarrow 0, x \in L_j^o} u_k(x) = \delta_{jk}, \quad j = 1, \dots, n.$$

It satisfies $0 < u_k(x) < 1$ and, furthermore, every bounded \mathbf{X} -harmonic function $u(x)$ can be expressed as

$$u(x) = \sum_{k=1}^n c_k u_k(x), \quad c_k \in \mathbf{R},$$

the expression being unique because $c_k = \lim_{x \rightarrow 0, x \in L_k^o} u(x)$. In particular,

$$\sum_{k=1}^n u_k(x) \equiv 1.$$

If we set

$$\Xi^{(k)} = \{X(t) \rightarrow 0 \text{ as } t \uparrow \zeta \text{ tangentially along } L_k\}, \quad k = 1, \dots, n,$$

(for the precise meaning of "tangentially along," cf. [IW 1] or [IW 2]), then

$$u_k(x) = P_x(\Xi^{(k)}), \quad x \in D, \quad k = 1, \dots, n.$$

For each $k = 1, \dots, n$, $u_k(x)$ is an \mathbf{X} -excessive function and we can define the u_k -subprocess $\mathbf{X}^{(k)} = (X(t), P_x^{(k)})$, i.e., the diffusion on D obtained from \mathbf{X} by the transformation by the multiplicative functional (cf. [FOT], Chap. 6.3),

$$M(t) = l_{\{\zeta > t\}} \cdot \frac{u_k(X(t))}{u_k(X(0))}.$$

This process satisfies

$$P_x^{(k)}(\Xi^{(k)}) = 1 \quad (2)$$

for all $x \in D$. For $j = 1, \dots, n$, $u_j(x)u_k(x)^{-1}$ is an $\mathbf{X}^{(k)}$ -excessive function and, by the symmetry of \mathbf{X} , the measure $u_j(x)u_k(x)dx$ is $\mathbf{X}^{(k)}$ -excessive measure. Then, we can construct the $\mathbf{X}^{(k)}$ -Markovian measure \mathbf{N}_{jk} , called also the *approximate process* or *quasi-process*, associated to $\mathbf{X}^{(k)}$ -excessive measure $u_j(x)u_k(x)dx$, cf. e.g. Weil ([We]): \mathbf{N}_{jk} is a σ -finite measure on the path space

$$\mathcal{W} = \{ w \in \mathcal{C}([0, \infty) \rightarrow \mathbf{R}^2) \mid w(0) = 0, \exists \sigma(w) \in (0, \infty) \text{ such that } \\ w(t) \in \mathbf{R}^2 \setminus \{0\} \text{ for } t \in (0, \sigma(w)) \text{ and } w(t) = 0 \text{ for } t \geq \sigma(w) \}$$

endowed with σ -field $\mathcal{B}(\mathcal{W})$ generated by Borel cylinder sets, uniquely determined by the following properties:

$$(i) \quad \int_0^\infty dt \int_{\mathcal{W}} f(w(t)) 1_{\{\sigma(w) > t\}} \mathbf{N}_{jk}(dw) = \int_D f(x) u_j(x) u_k(x) dx, \quad f \in \mathcal{C}_0(D),$$

$$(ii) \text{ for } t > 0, E \in \mathcal{B}(D) \text{ and } U \in \mathcal{B}(\mathcal{W}),$$

$$\mathbf{N}_{jk}(\{ w \mid w(t) \in E, \theta_t(w) \in U \}) = \int_{\mathcal{W}} P_{w(t)}^{(k)}(X \in U) 1_{\{w(t) \in E\}} \mathbf{N}_{jk}(dw), \quad (3)$$

where $\theta_t(w)$ is the shifted path: $\theta_t(w)(s) = w(t+s)$.

Since \mathbf{X} is symmetric, we can deduce the following property under the time reversal:

$$\mathbf{N}_{jk}\{T^{-1}(U)\} = \mathbf{N}_{kj}(U), \quad U \in \mathcal{B}(\mathcal{W}), \quad j, k = 1, \dots, n \quad (4)$$

where $T: \mathcal{W} \rightarrow \mathcal{W}$ is the time reversal operator:

$$(Tw)(t) = \begin{cases} w(\sigma(w) - t), & 0 \leq t \leq \sigma(w), \\ 0, & t \geq \sigma(w). \end{cases}$$

If we set, for $k = 1, \dots, n$,

$$\tilde{\Xi}^{(k)} = \{w \in \mathcal{W} \mid w(t) \rightarrow 0 \text{ as } t \uparrow \sigma(w) \text{ tangentially along } L_k\}$$

and

$$\tilde{\Pi}^{(k)} = T^{-1}(\tilde{\Xi}^{(k)}) = \{w \in \mathcal{W} \mid w(t) \text{ starts at } 0 \text{ tangentially along } L_k\},$$

then, obviously, $\tilde{\Xi}^{(1)}, \dots, \tilde{\Xi}^{(n)}$ are mutually disjoint and so are also $\tilde{\Pi}^{(1)}, \dots, \tilde{\Pi}^{(n)}$. From (2), (3) and (4), we can deduce the following:

Proposition 2.1.

$$N_{jk}(\mathcal{W} \setminus \{\tilde{\Pi}^{(j)} \cup \tilde{\Xi}^{(k)}\}) = 0, \quad j, k = 1, \dots, n.$$

If we set

$$N_j = \sum_{k=1}^n N_{jk}, \quad j = 1, \dots, n, \quad (5)$$

then N_j is the \mathbf{X} -Markovian measure associated to \mathbf{X} -excessive measure $u_j(x)dx$.

Now, the possible extension of \mathbf{X} to a diffusion on the whole plane can be obtained by applying Itô's theory of excursion point processes (cf. [I]).

Theorem 2.1. ([IW 1] or [IW 2].) An extension $\widehat{\mathbf{X}} = (X(t), \widehat{P}_x)$, for which the origin 0 is not a trap, is determined by the nonnegative parameters p_1, \dots, p_n and m such that $\sum_{k=1}^n p_k = 1$. $m = 0$ if and only if $\int_0^\infty 1_{\{0\}}(X(t))dt = 0$ a.s. with respect to \widehat{P}_x for every $x \in \mathbf{R}^2$. $\widehat{\mathbf{X}}$ is symmetric with respect to some measure on \mathbf{R}^2 if and only if $p_1 = \dots = p_n = 1/n$ and, then, a symmetrizing measure is given by $m(dx) = dx + m \cdot \delta_{\{0\}}(dx)$. In this case, the corresponding Dirichlet form is the closure on $L^2(m(dx))$ of the $\mathcal{E}(f, g)$ given by (1) with the domain $C_0^\infty(\mathbf{R}^2)$.

The sample paths of $\widehat{\mathbf{X}}$ starting at the origin 0 can be constructed as follows. Let $\mathbf{N} = \sum_{k=1}^n p_k \mathbf{N}_k$ which is a σ -finite measure on $(\mathcal{W}, \mathcal{B}(\mathcal{W}))$ with infinite total mass. We set up a Poisson point process p on the state space \mathcal{W} with the characteristic measure \mathbf{N} (cf. [I] or [IW 3], p.43 and p.123-130). Note that each sample of p is a point function $p: \mathbf{D}_p \in t \mapsto p_t \in \mathcal{W}$, where the domain \mathbf{D}_p of p is a countable subset of $(0, \infty)$. Set

$$A(t) = mt + \sum_{s \in \mathbf{D}_p, s \leq t} \sigma(p_s).$$

Then, it is a càdlàg increasing process with stationary independent increments and $A(0) = 0$. Since $\mathbf{N}(\mathcal{W}) = \infty$, $t \mapsto A(t)$ is strictly increasing and $\lim_{t \uparrow \infty} A(t) = \infty$, a.s. Hence, for each $t \geq 0$, there exists unique $s \geq 0$ such that $A(s-) \leq t \leq A(s)$. $s \in \mathbf{D}_p$ if and only if $A(s) > A(s-)$. Set, for each $t \in [0, \infty)$,

$$X(t) = \begin{cases} p_s(t - A(s-)), & s \in \mathbf{D}_p, A(s-) \leq t \leq A(s), \\ 0, & t = A(s-) = A(s). \end{cases}$$

Then, $\widehat{\mathbf{X}} = (X(t))$ is the sample path starting at 0 of the diffusion which is the extension of \mathbf{X} corresponding to the parameters p_1, \dots, p_n and m .

Let $\mathbf{A} = \{k \in \{1, \dots, n\} \mid p_k > 0\}$. Let $\mathbf{F} = \mathbf{F}^{\widehat{\mathbf{X}}}$ be the natural filtration of the diffusion $\widehat{\mathbf{X}}$ constructed above.

Theorem 2.2. Assume that the set \mathbf{A} contains $l \geq 2$ elements. Then there exists a non-trivial \mathbf{F} -spider martingale with l rays. Hence, if \mathbf{A} contains $l \geq 3$ elements, there exists a multiple \mathbf{F} -spider martingale so that the filtration \mathbf{F} is not cosy.

Proof. Let $\widehat{\mathbf{X}} = (X(t))$ be the diffusion constructed above and $\mathbf{F} = \{F(t)\}$ be the natural filtration of $\widehat{\mathbf{X}}$. For $t > 0$, set

$$g(t) = \sup\{s \in [0, t] \mid X(s) = 0\}.$$

Then, $g(t)$ is an **F**-honest time and, by Proposition 2.1 and an excursion theory, we deduce

$$F(g(t)+) = F(g(t)) \bigvee \{\Theta_k, k \in \mathbf{A}\},$$

where

$$\Theta_k = [\theta_{g(t)}X \in \tilde{\Pi}^{(k)}], \quad k \in \mathbf{A}.$$

The existence of a non-trivial **F**-spider martingale with l rays follows from a general result in [BEKSY].

Or, we can give a more direct construction of a non-trivial **F**-spider martingale by piecing out some part of each excursion by the method given in [Wat], (the collection of excursions is the point process p from which we have constructed the process $\widehat{\mathbf{X}}$).

Multidimensional extensions of Theorem 2.2 are of course possible. We give a typical example in the case of a three dimensional diffusion process. We define a diffusion \mathbf{X} on $D = \mathbf{R}^3 \setminus \{0\}$ similarly as above by the following Dirichlet form: Let $\Pi_j, j = 1, \dots, m$, be m different planes in \mathbf{R}^3 , each passing through the origin 0 , and let $L_k, k = 1, \dots, n$, be n different half lines, each, starting at the origin and lying on some plane Π_j . Let \mathcal{D}_0 be, as above, the space of all C^∞ -functions with a compact support and vanishing also in a neighborhood of origin. Let $D(u, v)$, $u, v \in \mathcal{D}_0$, be the usual Dirichlet integral on \mathbf{R}^3 , $D_{\Pi_j}(u, v)$ be the two-dimensional Dirichlet integral for $u|_{\Pi_j}, v|_{\Pi_j}$ on Π_j (by regarding Π_j as two-dimensional Euclidean space by the imbedding) and $D_{L_k}(u, v)$ be the one-dimensional Dirichlet integral for $u|_{L_k}, v|_{L_k}$ on L_k . For positive constants $\mu_j, j = 1, \dots, m$ and $\nu_k, k = 1, \dots, n$, define a bilinear form on \mathcal{D}_0 by

$$\mathcal{E}(u, v) = \frac{1}{2}D(u, v) + \sum_{j=1}^m \mu_j D_{\Pi_j}(u, v) + \sum_{k=1}^n \nu_k D_{L_k}(u, v), \quad u, v \in \mathcal{D}_0.$$

Then, it is a closable Markovian form on $L^2(D; dx)$ and its closure is a regular Dirichlet form. Therefore, there corresponds a unique diffusion \mathbf{X} on D with a finite life time. We can obtain similar results as above: the space of bounded \mathbf{X} -harmonic functions are n -dimensional and the possible extensions of \mathbf{X} as diffusions on \mathbf{R}^3 are determined in exactly the same way as Theorem 2.1. Also, Theorem 2.2 is valid in the same way: Namely, if an extension $\widehat{\mathbf{X}}$ which corresponds to nonnegative parameters p_1, \dots, p_n and m , is such that $\#\{k \mid p_k > 0\} = l$, then the natural filtration of $\widehat{\mathbf{X}}$ contains a multiple spider martingale with l rays.

3 An application to sticky Brownian motions

Here we apply Theorem 2.2 to show the non-cosiness of the filtration of one-dimensional Brownian motion which is sticky at the origin 0 .

For given $c \geq 0$, $\rho \geq 0$ with $c + \rho > 0$, consider the following stochastic differential equation for a continuous **F**-semimartingale $X = (X(t))$ on \mathbf{R} on a filtered probability space $\{(\Omega, \mathcal{F}, P), \mathbf{F}\}$:

$$d[X(t) \vee 0] = 1_{\{X(t) > 0\}} \cdot dB(t) + c \cdot d\phi(t), \quad d[X(t) \wedge 0] = 1_{\{X(t) < 0\}} \cdot dB(t) - c \cdot d\phi(t) \quad (6)$$

where $B(t)$ is an \mathbf{F} -Wiener process with $B(0) = 0$, $\phi(t)$ is an \mathbf{F} -adapted, continuous increasing process with $\phi(0) = 0$ such that

$$\phi(t) = \int_0^t 1_{\{X(s)=0\}} \cdot d\phi(s), \quad \int_0^t 1_{\{X(s)=0\}} \cdot ds = \rho\phi(t). \quad (7)$$

Given $x \in \mathbf{R}$, a solution $X = (X(t))$ with $X(0) = x$ exists on a suitable filtered probability space and it is unique in the law sense ([IW 3]). When $c = 0$, then the solution $X(t)$ is the Brownian motion $x + B(t)$ stopped at $\sigma_0 = \min\{u \mid x + B(u) = 0\}$. When $\rho = 0$, the solution is irrelevant to c and coincides with the Brownian motion $x + B(t)$. In these two extreme cases, the natural filtration \mathbf{F}^X of the solution X is either homomorphic or isomorphic to the Brownian filtration \mathbf{F}^B so that it is a cosy filtration.

So we assume $c > 0, \rho > 0$ and, replacing $c\phi$ by ϕ and ρ by ρ/c , we can always assume $c = 1$ in the equation (6). In the following, we assume that $X(0) = 0$, for simplicity, and denote the solution by X_ρ .

Theorem 3.1. *The natural filtration \mathbf{F}^{X_ρ} of X_ρ is not cosy.*

Proof. Without loss of generality, we may assume $\rho = 1$. Let $X^{(1)}$ and $X^{(2)}$ be independent copies of X_1 and define a diffusion $\mathbf{X} = (X(t))$ in the plane \mathbf{R}^2 by $\mathbf{X} = (X^{(1)}, X^{(2)})$. Then, $\mathbf{F}^{\mathbf{X}} = \mathbf{F}^{X_1} \otimes \mathbf{F}^{X_1}$. Let L_1, L_2, L_3, L_4 be the positive part of x -axis, the positive part of y -axis, the negative part of x -axis and the negative part of y -axis, respectively, so that $L = L_1 \cup L_2 \cup L_3 \cup L_4$ coincides with the union of x - and y -axes. Let

$$A(t) = \int_0^t 1_{\{\mathbf{R}^2 \setminus L\}}(X(s)) ds.$$

Then, we easily deduce that $t \mapsto A(t)$ is strictly increasing and $\lim_{t \uparrow \infty} A(t) = \infty$, a.s. so that $A = (A(t))$ is a process of time change. Define $\widehat{X}(t) = X(A^{-1}(t))$ and $\widehat{\mathbf{X}} = ((\widehat{X}(t)))$. Then, the natural filtration $\mathbf{F}^{\widehat{\mathbf{X}}}$ is just the time change $(\mathbf{F}^{\mathbf{X}})^{(A)}$. A key observation is that the diffusion process $\widehat{\mathbf{X}}$ is a particular case of diffusions given in Theorem 2.1: It is the case of $n = 4$ with L_1, L_2, L_3, L_4 given above and, $m = 0$, $p_1 = p_2 = p_3 = p_4 = 1/4$, cf. [IW 1], p.118. Hence, by Theorem 2.2, the filtration $\mathbf{F}^{\widehat{\mathbf{X}}}$ contains an $\mathbf{F}^{\widehat{\mathbf{X}}}$ -multiple spider martingale with 4 rays. Since the filtration $\mathbf{F}^{\mathbf{X}}$ is obtained from the filtration $\mathbf{F}^{\widehat{\mathbf{X}}}$ by a time change as $\mathbf{F}^{\mathbf{X}} = \{\mathbf{F}^{\widehat{\mathbf{X}}}\}^{(A^{-1})}$, it also contains an $\mathbf{F}^{\mathbf{X}}$ -multiple spider martingale with 4 rays. By Theorem 1.2, we can conclude that the filtration $\mathbf{F}^{\mathbf{X}} = \mathbf{F}^{X_1} \otimes \mathbf{F}^{X_1}$ is not cosy. Now, the non-cosiness of the filtration \mathbf{F}^{X_1} follows from Corollary 1.2.

Remark 3.1. Recently, J. Warren ([War]) proved directly that the natural filtration of a reflecting sticky Brownian motion is not cosy. His result is stronger than ours because there is a homomorphism from the filtration of a bilateral sticky Brownian motion to a reflecting sticky Brownian motion. However, we can give the following argument from which we can also deduce that the natural filtration of a reflecting sticky Brownian motion is not cosy.

Let $X^{(1)}, X^{(2)}$ and $X^{(3)}$ be independent copies of X_1 and define a diffusion process $\mathbf{X} = (X(t))$ in \mathbf{R}^3 by $\mathbf{X} = (X^{(1)}, X^{(2)}, X^{(3)})$. Let Π_1, Π_2, Π_3 be coordinate planes in \mathbf{R}^3 and let Π be their union. Let

$$A(t) = \int_0^t 1_{\{\mathbf{R}^3 \setminus \Pi\}}(X(s)) ds \quad \text{and} \quad \widehat{X}(t) = X(A^{-1}(t)).$$

Then, the process $\widehat{\mathbf{X}} = ((\widehat{X}(t)))$ is exactly a kind of diffusions discussed in Section 2 as multi-dimensional extensions of Theorem 2.2: It is the case that $m = 3$, Π_1, Π_2, Π_3 are coordinate planes as above and, $n = 6$, L_1, \dots, L_6 are six half coordinate axes each starting at the origin. Furthermore, $\mu_j = \nu_k \equiv 1$, and $\widehat{\mathbf{X}}$ corresponds to parameters $p_1 = \dots = p_6 = 1/6$ and $m = 0$. Hence, we can conclude that the natural filtration $\mathbf{F}^{\widehat{\mathbf{X}}}$ of $\widehat{\mathbf{X}}$ contains a multiple $\mathbf{F}^{\widehat{\mathbf{X}}}$ -spider martingale with 6 rays. Then, the natural filtration $\mathbf{F}^{\mathbf{X}} = \mathbf{F}^{X_1} \otimes \mathbf{F}^{X_1} \otimes \mathbf{F}^{X_1}$ of \mathbf{X} contains also a multiple $\mathbf{F}^{\mathbf{X}}$ -spider martingale with 6 rays because $\mathbf{F}^{\mathbf{X}}$ is obtained from $\mathbf{F}^{\widehat{\mathbf{X}}}$ by a time change. From this, we can deduce that the filtration $\mathbf{F} = \mathbf{F}' \otimes \mathbf{F}' \otimes \mathbf{F}'$, where \mathbf{F}' is the natural filtration of $|X_1| = (|X_1(t)|)$, has a multiple \mathbf{F} -spider martingale with 3 rays. Therefore, \mathbf{F} is not cosy by Theorem 1.2 and hence, by Corollary 1.2, the filtration \mathbf{F}' , which is the natural filtration of the reflecting sticky Brownian motion $|X_1|$, is not cosy.

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