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Homogeneous diffusions on the Sierpinski gasket

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ABSTRACT: We prove that certain diffusions on the Sierpinski gasket may be characterized, up to a multiplicative constant in the time scale, by a parameter $\alpha \in [0, \frac{1}{4}]$. The diffusions considered have the Feller property and certain natural symmetry properties, but they are not necessarily scale invariant. The case $\alpha = 0$ corresponds roughly speaking to one-dimensional Brownian motion and the case $\alpha = \frac{1}{4}$ corresponds to Brownian motion on the Sierpinski gasket.

1 Introduction

In the last few years diffusions on fractals have emerged as an area of probability theory in which an intensive research activity has taken place. Diffusions on the Sierpinski gasket in particular have attracted great attention, perhaps because of their remarkable accessibility.

The construction of Brownian motion on the Sierpinski gasket is due to Goldstein [3], Kusuoka [8] and Barlow and Perkins [1] (in the following referred to as B.-P.). These authors used different methods to obtain their respective results.

Possibly one of the most remarkable properties of Brownian motion on the Sierpinski gasket is the invariance under natural symmetries of the gasket, to be precise under bijective isometries of open subsets of the gasket. Here the distance between two points in an open subset is the length of a shortest path in the set which connects the two points, or ∞ if there is no connecting path. This property sets Brownian motion apart from all other diffusions on the Sierpinski gasket. It also implies another important property of Brownian motion, the so called scale invariance.

Kumagai [7] succeeded in constructing diffusions on the Sierpinski gasket which fulfill less stringent symmetry requirements than Brownian motion, but which are still scale invariant. Hattori, K., Hattori, T. and Watanabe [4] on the other hand constructed interesting diffusions on the Sierpinski gasket which lack both types of invariance.

Apart from the metric mentioned, there is another natural metric, namely the one inherited from $\mathbb{R}^2$. This metric is fairly natural if one is interested in forces, such as gravitational forces, which act through the space surrounding the gasket rather than through the gasket itself.

If $\phi : U \rightarrow V$ is a bijective isometry with respect to the second metric, then for every path in U the image of the path under ϕ is a path in V of the same length. Hence ϕ is also an isometry with respect to the first metric.

Moreover, we shall see that the bijective isometries related to the second metric form a proper subset of the bijective isometries related to the first metric.

In the present paper we shall construct a class of diffusions, which, though generally not scale invariant, are invariant under isometries relative to the metric on the gasket inherited from $\mathbb{R}^2$.

Our construction follows the pattern of the construction given by B.-P. That construction is based on the convergence of certain random walks with scaled time. In that case the choice of the time scaling factors is rather natural because of an underlying spatial scale invariance. In our construction without spatial scale invariance, however, establishing the existence of time scaling factors presents a major problem. Our way to overcome this problem is to apply a perturbation result on matrix powers (see [5]). This perturbation result seems interesting in itself. We shall use it here to obtain convergence results for fairly general multi-type branching processes with varying environment – convergence results which play the key role in the construction of our diffusions.

To explain our main result, we need some notations and definitions:
1. We recall the definition of the Sierpinski gasket G as a subset of $\mathbb{R}^2$. To this end, let$F_0 := \{(0,0),(1,0),(\frac{1}{2},\frac{\sqrt{3}}{2})\}$, $F_{n+1} := F_n \cup [(2^n,0)+F_n] \cup [(2^{n-1},2^{n-1}\sqrt{3})+F_n]$, $\hat{G}^{(0)} := \bigcup_{n=0}^{\infty} F_n$. We denote by $G^{(0)}$ the union of $\hat{G}^{(0)}$ and $\hat{G}^{(0)}$, reflected at the y-axis. The sequence $G^{(n)} := 2^{-n}G^{(0)}, n \in \mathbb{N}_0$ is increasing, and if we let $G^{(\infty)} := \bigcup_{n=0}^{\infty} G^{(n)}$, then the Sierpinski gasket is $G := \text{closure}(G^{(\infty)})$. The Sierpinski gasket is endowed with the metric and the topology inherited from $\mathbb{R}^2$.

For the reader's convenience, we include an illustration of a section of the Sierpinski gasket (Figure 1).
2. By a symmetry of the Sierpinski gasket G we understand a bijective isometry, say $\psi : U \to V$, between two open subsets of G. We note that obviously ψ can be extended uniquely to a bijective isometry (also called ψ) between the closures of U and V.

If we denote by $\mu : \mathbb{R}^2 \to \mathbb{R}^2$ the reflection at the y-axis and let $\nu : \mathbb{R}^2 \to \mathbb{R}^2$ be the affine transformation of $\mathbb{R}^2$, with $\nu(0,0) := (\frac{3}{4},\frac{\sqrt{3}}{4})$, $\nu(-\frac{1}{2},0) := (1,0)$ and $\nu(-\frac{1}{4},\frac{\sqrt{3}}{4}) := (\frac{1}{2},0)$, then the restrictions of ν and $\nu \circ \mu$ to sufficiently small open neighborhoods of $(0,0)$ are symmetries of the Sierpinski gasket (see figure 1).

In the same way one can easily obtain for all $x, y \in G^{(\infty)}$ two affine transformations of $\mathbb{R}^2$, say $\phi_{x,y}$ and $\lambda_{x,y}$, with $\phi_{x,y}(x) = \lambda_{x,y}(x) = y$ such that $\phi_{x,y}$ preserves the orientation and $\lambda_{x,y}$ reverses the orientation and the restrictions of $\phi_{x,y}$ and $\lambda_{x,y}$ to sufficiently small open neighborhoods of x are symmetries. It is not very hard to see that conversely every symmetry $\phi : U \to V$ of the Sierpinski gasket is the restriction to U of such an affine transformation $\phi_{x,y}$ or $\lambda_{x,y}$.
3. A diffusion on G is a family of probability measures $(P_x;\ x \in G)$ on $C([0,\infty),G)$ which are connected by the strong Markov property and satisfy

$P_x\{\omega(0) = x\} = 1$.
4. For every set $A \subset G$ and a continuous path ω in G we denote by $T_A(\omega)$ the first time ω leaves A.
5. We call a diffusion $(P_x; x \in G)$ homogeneous, if for every symmetry $\psi : U \to V$ and every $x \in U$ the P_x-distribution of $\psi \circ \omega(\cdot \wedge T_U(\omega))$ equals the $P_{\psi(x)}$-distribution of $\omega(\cdot \wedge T_V(\omega))$. A diffusion on G is thus, roughly speaking, homogeneous, if it is invariant under the symmetries of G.

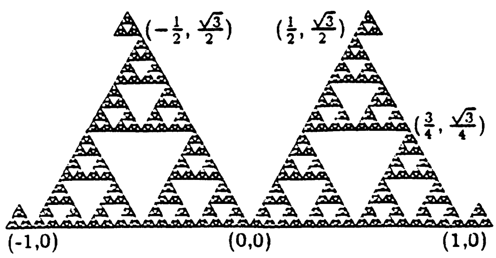

Figure 1

The purpose of the present paper is the description of homogeneous diffusions on G in terms of one real parameter, which may be interpreted as a certain exit probability. We shall explain briefly how this description is achieved.

If we denote by N the interior in G of the set of all points in G, which lie in one of the two closed triangles $\triangle((0,0),(-1,0),(-\frac{1}{2},\frac{\sqrt{3}}{2}))$ and $\triangle((0,0),(1,0),(\frac{1}{2},\frac{\sqrt{3}}{2}))$, then the open set N contains the point $(0,0)$, and its boundary in G consists of the two collinear boundary points $(-1,0), (1,0)$ and the two non-collinear boundary points $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$, $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. A continuous path starting at $(0,0)$ can leave N only through one of these four points. Excluding the trivial case $P_x\{\omega(t) = x, \text{ for all } t \geq 0\} = 1$ for all $x \in G$, we shall see, that for a homogeneous diffusion on G with Feller property, the exit time from N, i.e. T_N, is finite $P_{(0,0)}$-almost everywhere. In this case the $P_{(0,0)}$-exit-distribution coincides for the two non-collinear boundary points of N as well as for the two collinear boundary points of N, as the reflection ν of $\mathbb{R}^2$ at the y-axis defines a symmetry $\nu|_N : N \to N$ with $\nu((0,0)) = (0,0)$. In particular, the $P_{(0,0)}$-exit-distribution on the boundary of N is uniquely determined by the probability $\alpha = P_{(0,0)}\{\omega \text{ leaves } N \text{ through } (\frac{1}{2}, \frac{\sqrt{3}}{2})\}$.

The following theorem states that this parameter α characterizes completely the homogeneous diffusions with Feller property and that the parameter set is the interval $[0, \frac{1}{4}]$.

1.1 Theorem

a) *For every non trivial homogeneous diffusion on G, say $(P_x; x \in G)$, with Feller property, the probability $\alpha = P_{(0,0)}\{\omega$ leaves N through $(\frac{1}{2}, \frac{\sqrt{3}}{2})\}$ is a number in $[0, \frac{1}{4}]$.*

b) *Conversely, for each $\alpha \in [0, \frac{1}{4}]$ there exists a homogeneous diffusion $(P_x; x \in G)$ with Feller property, such that $P_{(0,0)}\{\omega$ leaves N through $(\frac{1}{2}, \frac{\sqrt{3}}{2})\} = \alpha$. This diffusion is uniquely determined up to a linear scaling of time.*

Remark *a) For $\alpha = \frac{1}{4}$ the diffusion of Theorem 1.1 is Brownian motion as constructed by B.-P. and for $\alpha = 0$ it is a modification of the ordinary one-dimensional Brownian motion. With the exception of these two values of α, the diffusions of Theorem 1.1 are not scale-invariant.*

b) It should be noted that a diffusion on G is already homogeneous, if it is invariant under symmetries of G of a certain local nature.

c) If $(P_x;\, x \in G)$ is a non trivial diffusion on G with Feller property which not only is homogeneous, but also is invariant under bijective isometries related to the metric of the length of a shortest path, then Theorem 1.1 implies that $(P_x;\, x \in G)$ is Brownian motion, as constructed by B.-P.

Indeed, if we denote by $r : \mathbb{R}^2 \to \mathbb{R}^2$ the reflection at the axis through $(0,0)$ and $(\frac{3}{4}, \frac{\sqrt{3}}{4})$ and define $\phi : G \to G$ by

$$\phi(x) := \begin{cases} x & : \text{if the first coordinate of } x \text{ is negative} \\ r(x) & : \text{otherwise} \end{cases},$$

then obviously ϕ is not a symmetry, but it is an isometry with respect to the metric of the length of a shortest path. Now the invariance under ϕ and under the reflection at the y-axis implies that the $P_{(0.0)}$-exit-distribution for the open set N coincides for all four boundary points and hence $\alpha = \frac{1}{4}$, i.e. $(P_x;\, x \in G)$ is Brownian motion. The corresponding uniqueness result for Brownian motion in B.-P. is obviously stronger since Barlow and Perkins do not assume that the diffusion has the Feller property.

2 Multi-type branching processes with varying environment

We start with limit results for a class of supercritical multi-type branching processes with varying environment, which we will present as Theorem 2.1 below.

For $d \in \mathbb{N}$ we consider a d-type branching process with varying environment which we shall denote by $(Z^{\langle n \rangle};\, n \in \mathbb{N}_0)$. This process is completely described by the distribution of the initial generation $Z^{\langle 0 \rangle}$ and the generating functions $(F^{\langle n \rangle})_{n \in \mathbb{N}_0}$, which determine the branching mechanism for each generation $Z^{\langle n \rangle}$, $n \in \mathbb{N}_0$.

We shall assume that

(A) the distribution of $Z^{\langle 0 \rangle}$ has finite second moments,

and that the generating functions $F^{\langle n \rangle}$, $n \in \mathbb{N}_0$ fulfill the following conditions:

(B) $M_1 := \sup\limits_{1 \le i,j \le d,\, n \in \mathbb{N}_0} \dfrac{\partial^2 F_i^{\langle n \rangle}}{\partial {x_j}^2}(1, \ldots, 1) < \infty.$

(C) $a_{i,j}^{\langle n \rangle} := \dfrac{\partial F_i^{\langle n \rangle}}{\partial x_j}(1, \ldots, 1) > 0$ for all $1 \le i, j \le d$ and $n \in \mathbb{N}_0$.

(D) $a_{i,j} := \lim_{n\to\infty} a_{i,j}^{\langle n\rangle}$ exists for all $1 \le i,j \le d$ and the matrix $A := (a_{i,j})_{i,j=1,\dots,d}$ is invertible.

(E) There exists a real eigenvalue λ_A of A such that:

(α) $\lambda_A > 1$.

(β) $|\mu| < \lambda_A$ for all eigenvalues μ of A, different from λ_A.

(γ) λ_A has multiplicity one.

(δ) The entries of every non zero eigenvector of A corresponding to λ_A are either all strictly positive or all strictly negative.

If there exists $n \in \mathrm{N}$ such that all entries of A^n are strictly positive, then by the Perron–Frobenius Theorem assumption (E) reduces to:

(E′) There exists an eigenvalue ν of A with $|\nu| > 1$.

The probabilistic meaning of $a_{i,j}^{\langle n\rangle}$ is the expected number of j-type descendants of a i-type individual of the n'th generation and according to (D) this expected number stabilizes if n tends to ∞. In the case of branching processes with non varying environment, i.e. if all $F^{\langle n\rangle}$ coincide, our assumption (E·α) just describes the supercritical case.

In order to formulate Theorem 2.1 we introduce the following notations: For a matrix A satisfying (E) we denote by c_A the eigenvector of A corresponding to λ_A which has strictly positive entries and Euclidean norm 1. If in addition there exists $n \in \mathrm{N}$ such that all entries of A^n are strictly positive, then A^t, the transpose of A, also satisfies (E′) and hence (E) and we let $\tilde{c}_A := \frac{1}{c_A^t c_{A^t}} c_{A^t}$. Here the row vector c_A^t is the transpose of the column vector c_A.

2.1 Theorem *Let $(Z^{\langle n\rangle}; n \in \mathrm{N}_0)$ be a d-type branching process with varying environment which satisfies conditions* (A) *to* (E).
Then there exists a sequence $(a_n)_{n\in\mathrm{N}_0}$ of strictly positive numbers converging to 0 *such that:*

a) $W^{\langle\infty\rangle} := \lim_{n\to\infty} a_n\, c_A^t Z^{\langle n\rangle}$ *exists* a.e. *and in* L^2.

b) $0 < E(W^{\langle\infty\rangle}) < \infty$.

c) For a.a. ω: $W^{\langle\infty\rangle}(\omega) = 0$ *iff* $Z^{\langle n\rangle}(\omega) = (0,\dots,0)^t$ *for all sufficiently large n.*

d) If there exists $n \in \mathrm{N}$ such that all entries of A^n are strictly positive, then

$$\lim_{n\to\infty} a_n Z^{\langle n\rangle} = W^{\langle\infty\rangle}\tilde{c}_A \quad \text{a.e.}$$

Remark *The sequence $(a_n)_{n\in\mathbb{N}_0}$ in the preceding Theorem 2.1 is obviously determined up to asymptotic equivalence by a) and b) in Theorem 2.1.*

Moreover, any sequence $(a_n)_{n\in\mathbb{N}_0}$ of Theorem 2.1 satisfies

$$\lim_{n\to\infty} a_n{}^{\frac{1}{n}} = \frac{1}{\lambda_A}.$$

For the proof of Theorem 2.1 we need two Lemmas.
These two Lemmas are results on convergent sequences of matrices, which apply in particular to the sequence $\left((a_{i,j}^{\langle n\rangle})_{i,j=1,\dots,d}\right)_{n\in\mathbb{N}_0}$ introduced in (C) above.

We let $\mathbb{R}_+ := \{x \in \mathbb{R};\, x > 0\}$ and denote by $\|\quad\|$ the Euclidean norm on $\mathbb{R}^d$.

2.2 Lemma *Let $A, A^{\langle n\rangle}, n \in \mathbb{N}_0$ be invertible $d \times d$–matrices, such that $A = \lim_{n\to\infty} A^{\langle n\rangle}$ and there exists an eigenvalue ν of A with multiplicity 1 such that $|\mu| < |\nu|$ for all eigenvalues μ of A different from ν.*
Then there exists a sequence $(c_n)_{n\in\mathbb{N}_0}$ of vectors in $\mathbb{R}^d$, satisfying:

a) $c^{\langle n\rangle} = A^{\langle n\rangle} c^{\langle n+1\rangle}$ for all $n \in \mathbb{N}_0$.

b) All limit points of $(\frac{c^{\langle n\rangle}}{\|c^{\langle n\rangle}\|})_{n\in\mathbb{N}_0}$ are contained in the eigenspace of A corresponding to ν.

The sequence $(c^{\langle n\rangle})_{n\in\mathbb{N}_0}$ is unique up to a multiplicative constant.

Lemma 2.2 is an immediate consequence of Theorem 1.1 in [5]. □

2.3 Lemma *Let $A^{\langle n\rangle} := (a_{i,j}^{\langle n\rangle})_{i,j=1,\dots,d}$, $n \in \mathbb{N}_0$ be matrices with strictly positive entries, satisfying (D) and (E). Then there exists a sequence $(c^{\langle n\rangle})_{n\in\mathbb{N}_0}$ of vectors in $\mathbb{R}_+^d$ satisfying:*

$$c^{\langle n\rangle} = A^{\langle n\rangle} \cdot c^{\langle n+1\rangle} \text{ for all } n \in \mathbb{N}_0. \tag{2.1a}$$

$\lim_{n\to\infty} \frac{c^{\langle n\rangle}}{\|c^{\langle n\rangle}\|} = c_A$, *where c_A is the eigenvector of the limit matrix A introduced previously.*

(2.1b)

The sequence $(c^{\langle n\rangle})_{n\in\mathbb{N}_0}$ is unique up to a strictly positive multiplicative constant.

Remark *It is easy to see that under the assumptions of Lemma 2.3 we obtain*

$$\lim_{n\to\infty} \| c^{\langle n\rangle} \|^{\frac{1}{n}} = \tfrac{1}{\lambda_A} \tag{2.1c}$$

from (2.1a) and (2.1b). Moreover, Theorem 1.1 in [5] implies that any sequence $(c^{\langle n\rangle})_{n\in\mathbb{N}_0}$ of vectors in $\mathbb{R}_+^d$ satisfying (2.1a) and (2.1c) also satisfies (2.1b).

Proof of Lemma 2.3: By (D) there exists $n_0 \in \mathbb{N}_0$, such that $A^{\langle n\rangle}$ is invertible for all $n \geq n_0$. Using (D) and (E), Lemma 2.2 implies that there exists $c^{\langle n\rangle}; n \geq n_0$ which satisfy the equation in (2.1a) for all $n \geq n_0$ and for which c_A and $-c_A$ are the only possible limit points of $(\frac{c^{\langle n\rangle}}{\|c^{\langle n\rangle}\|})_{n\geq n_0}$. Lemma 2.2 also implies, that this sequence is unique up to a multiplicative constant. It follows that c_A is a limit point of $(\frac{c^{\langle n\rangle}}{\|c^{\langle n\rangle}\|})_{n\geq n_0}$ or of $(\frac{-c^{\langle n\rangle}}{\|-c^{\langle n\rangle}\|})_{n\geq n_0}$. Without loss of generality we may assume that the former is true.

If all entries of $c^{\langle k\rangle}$ are strictly positive for a $k \geq n_0$ then all entries of $c^{\langle n\rangle}$ are strictly positive for $n_0 \leq n \leq k$ by (2.1a) and the fact that all entries of $A^{\langle n\rangle}$ are strictly positive. Since c_A is a limit point of $(\frac{c^{\langle n\rangle}}{\|c^{\langle n\rangle}\|})_{n\geq n_0}$ and c_A has strictly positive entries, we conclude that all entries of $c^{\langle n\rangle}$ are strictly positive for infinitely many $n \geq n_0$ and hence for all $n \geq n_0$. This implies (2.1b), since c_A and $-c_A$ are the only possible limit points of $(\frac{c^{\langle n\rangle}}{\|c^{\langle n\rangle}\|})_{n\geq n_0}$. For $n < n_0$ we define $c^{\langle n\rangle}$ by (2.1a). □

Proof of Theorem 2.1: Let $(c^{\langle n\rangle})_{n\in\mathbb{N}_0}$ be a sequence of vectors in $\mathbb{R}^d_+$ which satisfies (2.1a) and (2.1b) and let $p \in (\frac{1}{\lambda_A}, 1)$. Furthermore, let $W^{\langle n\rangle} := c^{\langle n\rangle t} Z^{\langle n\rangle}$ for $n \in \mathbb{N}_0$.

Obviously (2.1a) implies for $n, l \in \mathbb{N}$

$$\frac{\| c^{\langle n+l\rangle} \|}{\| c^{\langle n\rangle} \|} = \prod_{k=1}^{l} \frac{\| c^{\langle n+k\rangle} \|}{\| c^{\langle n+k-1\rangle} \|} = \prod_{k=1}^{l} \left\| A^{\langle n+k-1\rangle} \frac{c^{\langle n+k\rangle}}{\| c^{\langle n+k\rangle} \|} \right\|^{-1}.$$

This together with (D) and (2.1b) implies

$$\lim_{n\to\infty} \frac{\| c^{\langle n+l\rangle} \|}{\| c^{\langle n\rangle} \|} = \lambda_A^{-l} \quad \text{for all } l \in \mathbb{N} \tag{2.2}$$

and for sufficiently large $n \in \mathbb{N}$: $\quad \frac{\|c^{\langle n+l\rangle}\|}{\|c^{\langle n\rangle}\|} \leq p^l \quad$ for all $l \in \mathbb{N}$.

Observing that $\inf\limits_{n\in\mathbb{N}_0, 1\leq i\leq d} \frac{c_i^{\langle n\rangle}}{\|c^{\langle n\rangle}\|} > 0$ by (2.1b), we conclude that there exists $M_2 > 0$, such that

$$\sup_{n\in\mathbb{N}_0, 1\leq i,j\leq d} \frac{c_i^{\langle n+l\rangle}}{c_j^{\langle n\rangle}} \leq M_2 p^l \quad \text{for all } l \in \mathbb{N}_0. \tag{2.3}$$

Obviously $(W^{\langle n\rangle}; n \in \mathbb{N}_0)$ is a positive martingale. A simple but lengthy computation using (B), (C), (2.1a) and (2.3) shows L^2 boundedness of this martingale, to be more precise that

$$E[(W^{\langle n\rangle})^2] \leq d^2\,(M_1+1)\,M_2^2\,\| c^{\langle 0\rangle} \|\, E(W^{\langle 0\rangle})\,\frac{p^2}{1-p} + E[(W^{\langle 0\rangle})^2].$$

Hence the martingale convergence theorem implies:

$$W^{\langle\infty\rangle} := \lim_{n\to\infty} W^{\langle n\rangle} \quad \text{exists a.e. and in } L^2. \tag{2.4a}$$

$$E(W^{\langle\infty\rangle}) = E(W^{\langle 0\rangle}). \tag{2.4b}$$

$$E[(W^{\langle\infty\rangle})^2] \le d^2\,(M_1+1)\,M_2^2\,\|\,c^{\langle 0\rangle}\,\|\,E(W^{\langle 0\rangle})\,\frac{p^2}{1-p} + E[(W^{\langle 0\rangle})^2]. \tag{2.4c}$$

Letting $a_n := \|c^{\langle n\rangle}\|$, we obtain by (2.1b) $|a_n c_A^t Z^{\langle n\rangle} - W^{\langle n\rangle}| \le \varepsilon W^{\langle n\rangle}$ for $\varepsilon > 0$ and n sufficiently large. Thus part a) and b) of Theorem 2.1 are easy consequences of (2.4a) to (2.4c).

For the proof of part c) of Theorem 2.1 let $(X^{\langle n\rangle}(i,j,r);\ n \in \mathrm{N}_0)$, $1 \le j \le d, i, r \in \mathrm{N}$ be independent branching processes with generating functions $(F^{\langle n+r\rangle})_{n\in\mathrm{N}_0}$ such that $X^{\langle 0\rangle}(i,j,r)_m$ equals 1 for $m = j$ and 0 otherwise.

Letting $W^{\langle n\rangle}(i,j,r) := \frac{1}{c_j^{\langle r\rangle}} c^{\langle n+r\rangle t} X^{\langle n\rangle}(i,j,r)$, we conclude from equations (2.4a) to (2.4c) that $W^{\langle\infty\rangle}(i,j,r) := \lim_{n\to\infty} W^{\langle n\rangle}(i,j,r)$ exists a.e. and that $E(W^{\langle\infty\rangle}(i,j,r)) = 1$ and $E[W^{\langle\infty\rangle}(i,j,r)^2] \le d^3(M_1+1)M_2^3\frac{p^2}{1-p} + 1 =: \rho$. By Hölder's inequality we conclude that

$$\begin{aligned} 1 &= (E[W^{\langle\infty\rangle}(i,j,r)])^2 \le E[W^{\langle\infty\rangle}(i,j,r)^2]P\{W^{\langle\infty\rangle}(i,j,r) > 0\} \\ &\le \rho P\{W^{\langle\infty\rangle}(i,j,r) > 0\} \end{aligned}$$

and that hence $P\{W^{\langle\infty\rangle}(i,j,r) = 0\} \le 1 - \frac{1}{\rho}$ for $1 \le j \le d, i, r \in \mathrm{N}$.

Since $(Z^{\langle n\rangle};\ n \in \mathrm{N}_0)$ is a branching process, we have

$$P\{W^{\langle\infty\rangle} = 0 | Z^{\langle r\rangle}, \ldots, Z^{\langle 0\rangle}\}(\tilde\omega) = P\left\{\sum_{j=1}^{d}\sum_{i=1}^{Z_j^{\langle r\rangle}(\tilde\omega)} W^{\langle\infty\rangle}(i,j,r) = 0\right\}$$

$$= \prod_{j=1}^{d}\prod_{i=1}^{Z_j^{\langle r\rangle}(\tilde\omega)} P\{W^{\langle\infty\rangle}(i,j,r) = 0\} \le (1-\frac{1}{\rho})^{\sum_{j=1}^{d} Z_j^{\langle r\rangle}(\tilde\omega)} \qquad \text{a.e.}$$

Since $1_{\{W^{\langle\infty\rangle}=0\}}$ is measurable with respect to $\sigma(Z^{\langle n\rangle}, n \in \mathrm{N}_0)$, the well known martingale convergence theorem implies

$$\lim_{r\to\infty} P\{W^{\langle\infty\rangle} = 0 | Z^{\langle r\rangle}, \ldots, Z^{\langle 0\rangle}\} = P\{W^{\langle\infty\rangle} = 0 | Z^{\langle n\rangle}, n \in \mathrm{N}_0\} = 1_{\{W^{\langle\infty\rangle}=0\}}.$$

Hence $1_{\{W^{\langle\infty\rangle}=0\}} \le (1-\frac{1}{\rho})^{\limsup_{r\to\infty}\sum_{j=1}^{d} Z_j^{\langle r\rangle}}$ a.e.

Thus, for a.a. ω: $\quad W^{\langle\infty\rangle}(\omega) = 0$ implies $\limsup_{n\to\infty}\sum_{j=1}^{d} Z_j^{\langle n\rangle}(\omega) = 0$.

The converse implication is obvious. Since $\sum_{j=1}^{d} Z_j^{\langle n\rangle} \in \mathrm{N}_0$, we conclude part c).

The proof of part d) of Theorem 2.1 is similar to that of Lemma 8.2 in [10]. For the reader's convenience we shall sketch the proof.

Let $\lambda := \lambda_A$, $c := c_A$ and $\tilde c := \tilde c_A$. If we define $B := A - \lambda c\tilde c^t$, then $A^k = \lambda^k c\tilde c^t + B^k$ for $k \in \mathrm{N}$. Moreover, it can be shown that for some $q \in (0, \lambda)$ and $M_3 > 0$

$$\max_{1\le i,j\le d} |(B^k)_{i,j}| < M_3\, q^k \quad \text{for } k \in \mathrm{N}. \tag{2.5}$$

For details – under somewhat weaker assumptions – we refer to Karlin [6], Appendix 2.

Let $1 \leq j \leq d$. For $\varepsilon > 0$ let $l \in \mathbf{N}$ such that $M_3 \left(\frac{q}{\lambda}\right)^l \leq \varepsilon$. Considering $\| c^{\langle n \rangle} \| Z^{\langle n \rangle} - W^{\langle \infty \rangle} \tilde{c}$ for $n > l$, we obtain for its j'th entry the estimate

$$\begin{aligned} & \left| \| c^{\langle n \rangle} \| \, Z_j^{\langle n \rangle} - W^{\langle \infty \rangle} \tilde{c}_j \right| \\ \leq & \left| \| c^{\langle n \rangle} \| (Z^{\langle n-l \rangle \mathrm{t}} A^l)_j - W^{\langle \infty \rangle} \tilde{c}_j \right| + \left| \| c^{\langle n \rangle} \| \left(Z_j^{\langle n \rangle} - (Z^{\langle n-l \rangle \mathrm{t}} A^l)_j \right) \right|. \end{aligned} \tag{2.6}$$

Substituting $A^l = \lambda^l c \tilde{c}^{\mathrm{t}} + B^l$ we have

$$\begin{aligned} & \| c^{\langle n \rangle} \| (Z^{\langle n-l \rangle \mathrm{t}} A^l)_j \\ = & \left(\| c^{\langle n-l \rangle} \| Z^{\langle n-l \rangle \mathrm{t}} c \right) \tilde{c}_j \left(\lambda^l \frac{\| c^{\langle n \rangle} \|}{\| c^{\langle n-l \rangle} \|} \right) + \left(\| c^{\langle n \rangle} \| (Z^{\langle n-l \rangle \mathrm{t}} B^l)_j \right). \end{aligned} \tag{2.7}$$

The first summand on the right sight of (2.7) tends to $W^{\langle \infty \rangle} \tilde{c}_j$ a.e. by part a) of Theorem 2.1 and (2.2). For the second summand on the right sight of (2.7) we obtain the asymptotic estimate

$$\begin{aligned} & \limsup_{n \to \infty} \left| \| c^{\langle n \rangle} \| (Z^{\langle n-l \rangle \mathrm{t}} B^l)_j \right| \\ = & \limsup_{n \to \infty} \left| \left((\| c^{\langle n-l \rangle} \| Z^{\langle n-l \rangle})^{\mathrm{t}} \left(\frac{\| c^{\langle n \rangle} \|}{\| c^{\langle n-l \rangle} \|} B^l \right) \right)_j \right| \leq \max_{1 \leq i \leq d} \left(\frac{1}{c_i} \right) W^{\langle \infty \rangle} \varepsilon \text{ a.e.,} \end{aligned}$$

by (2.2), (2.5) and part a) of Theorem 2.1. We therefore obtain from (2.7) for the first summand on the right sight of (2.6) the estimate

$$\limsup_{n \to \infty} \left| \| c^{\langle n \rangle} \| (Z^{\langle n-l \rangle \mathrm{t}} A^l)_j - W^{\langle \infty \rangle} \tilde{c}_j \right| \leq \max_{1 \leq i \leq d} \left(\frac{1}{c_i} \right) W^{\langle \infty \rangle} \varepsilon \text{ a.e.}$$

Thus it remains to prove that the second summand of the right sight of (2.6) tends to zero. Indeed, using (D) and part a) of Theorem 2.1 we see that

$$\begin{aligned} & \limsup_{n \to \infty} \left| \| c^{\langle n \rangle} \| \left(Z_j^{\langle n \rangle} - (Z^{\langle n-l \rangle \mathrm{t}} A^l)_j \right) \right| \\ \leq & \quad \limsup_{n \to \infty} \left| \| c^{\langle n \rangle} \| \left(Z_j^{\langle n \rangle} - (Z^{\langle n-l \rangle \mathrm{t}} A^{\langle n-l \rangle} \cdots A^{\langle n-1 \rangle})_j \right) \right| \\ & + \limsup_{n \to \infty} \| c^{\langle n \rangle} \| \sum_{k=1}^{d} Z_k^{\langle n-l \rangle} \max_{1 \leq k,j \leq d} |(A^{\langle n-l \rangle} \cdots A^{\langle n-1 \rangle})_{k,j} - (A^l)_{k,j}| \\ = & \limsup_{n \to \infty} \left| \| c^{\langle n \rangle} \| \left(Z_j^{\langle n \rangle} - (Z^{\langle n-l \rangle \mathrm{t}} A^{\langle n-l \rangle} \cdots A^{\langle n-1 \rangle})_j \right) \right| \qquad a.e. \end{aligned}$$

We obtain for $n > l$

$$\begin{aligned} & E \left[\left(Z_j^{\langle n \rangle} - (Z^{\langle n-l \rangle \mathrm{t}} A^{\langle n-l \rangle} \cdots A^{\langle n-1 \rangle})_j \right)^2 \right] \| c^{\langle n \rangle} \|^2 \\ \leq & E \left[\left(\sum_{r=0}^{l-1} (Z^{\langle n-r \rangle \mathrm{t}} A^{\langle n-r \rangle} \cdots A^{\langle n-1 \rangle} - Z^{\langle n-r-1 \rangle \mathrm{t}} A^{\langle n-r-1 \rangle} \cdots A^{\langle n-1 \rangle})_j \right)^2 \right] \| c^{\langle n \rangle} \|^2 \end{aligned}$$

$$
\begin{aligned}
&\le l\sum_{r=0}^{l-1} E\left[\left(\left[Z^{(n-r)\,t} - Z^{(n-r-1)\,t}A^{(n-r-1)}\right] A^{(n-r)}\cdots A^{(n-1)}\right)_j^2\right] \|c^{(n)}\|^2 \\
&\le l(1 + \sup_{\substack{r\in\{0,\ldots,l\}\\ k\in\mathbb{N}_0, 1\le i,j\le d}} (A^{(k)}\cdots A^{(k+r)})_{i,j}^2) \\
&\quad \times \sum_{r=0}^{l-1} E\left[\left(\sum_{k=1}^{d}\left[Z_k^{(n-r)} - \sum_{m=1}^{d} Z_m^{(n-r-1)} a_{m,k}^{(n-r-1)}\right]\right)^2\right] \| c^{(n)} \|^2
\end{aligned}
$$

Here we used the convention $(A^{(i)}\cdots A^{(j)}) =: Id$ for $i > j$.

Keeping in mind that $(Z^{(n)}; n \in \mathbb{N}_0)$ is a branching process for which the matrices $A^{(n)}$ are the matrices of the first moments, a lengthy but standard computation using (B) and (2.3) shows that the last term is majorized by

$$
l(1 + \sup_{\substack{r\in\{0,\ldots,l\}\\ k\in\mathbb{N}_0, 1\le i,j\le d}} (A^{(k)}\cdots A^{(k+r)})_{i,j}^2) d^3(M_1+1)M_2^2 \| c^{(0)} \| E(W^{(0)}) \frac{p^n}{1-p}.
$$

Since the supremum in the last majorant is finite by (D) it follows that the last majorant is summable. The fact that the second summand on the right sight of (2.6) tends to 0 a.e. now follows by Chebyshev's inequality and the Borel-Cantelli Lemma. □

With a_n and $W^{(n)}$ as in the preceding proof, we conclude from (2.1b) that $|a_n c_A^t Z^{(n)} - W^{(n)}| \le \varepsilon a_n c_A^t Z^{(n)}$ for $\varepsilon > 0$ and n sufficiently large. Hence, Theorem 2.1 implies the following corollary.

2.4 Corollary *Under the assumptions of Theorem 2.1, a) to c) in Theorem 2.1 hold with $a_n c_A^t Z^{(n)}$ replaced by $c^{(n)t} Z^{(n)}$, where $(c^{(n)})_{n\in\mathbb{N}_0}$ is any sequence in $\mathbb{R}_+^d$ satisfying (2.1a) and (2.1b).*

3 Consistent random walks

For every $k \in \mathbb{N}_0$ and $x \in G^{(k)}$ there are exactly four points in $G^{(k)}$ which have distance 2^{-k} from x. We call the two points, whose connecting line contains x collinear k-neighbors of x and the remaining two points non-collinear k-neighbors of x.

For $k \in \mathbb{N}_0$ let

$$
\Omega^{(k)} := \{\omega : \mathbb{N}_0 \to G^{(k)};\ \text{with } \|\omega_{i+1} - \omega_i\| \le 2^{-k} \text{ for all } i \in \mathbb{N}_0\}.
$$

Given $\alpha \in [0, \frac{1}{2}]$, a Markov process $(P_x^{(k)}; x \in G^{(k)})$ on $\Omega^{(k)}$ is an (α, k)-random walk if

$$
P_x^{(k)}\{\omega_1 = y\} = \begin{cases} \alpha & ;\ \text{if } y \text{ is an non-collinear } k\text{-neighbor of } x \\ \frac{1}{2} - \alpha & ;\ \text{if } y \text{ is a collinear } k\text{-neighbor of } x \\ 0 & ;\ \text{otherwise} \end{cases}.
$$

In the following let $k, l \in \mathbb{N}_0$ such that $k \ge l$.

For $\omega \in \Omega^{\langle k\rangle}$ let $T_0^{\langle l,k\rangle}(\omega)$ be the first time ω hits $G^{\langle l\rangle}$ and let $T^{\langle l,k\rangle}(\omega)$ be the first time ω hits $G^{\langle l\rangle}\setminus\{\omega_0\}$. Starting with the stopping time $T_0^{\langle l,k\rangle}(\omega)$ we define inductively for $i \in \mathrm{N}_0$ the stopping times $T_{i+1}^{\langle l,k\rangle}(\omega)$

$$T_{i+1}^{\langle l,k\rangle}(\omega) := \begin{cases} T^{\langle l,k\rangle}(\omega_{T_i^{\langle l,k\rangle}+\cdot}) + T_i^{\langle l,k\rangle}(\omega) & ; \text{if } T_i^{\langle l,k\rangle}(\omega) < \infty \\ \infty & ; \text{otherwise} \end{cases} .$$

For $\omega \in \Omega^{\langle k\rangle}$ and $i \in \mathrm{N}_0$ we define $\Psi^{\langle l,k\rangle}(\omega)_i$ by

$$\Psi^{\langle l,k\rangle}(\omega)_i := \begin{cases} \omega_{T_i^{\langle l,k\rangle}} & ; \text{if } T_i^{\langle l,k\rangle}(\omega) < \infty \\ \Psi^{\langle l,k\rangle}(\omega)_{i-1} & ; \text{if } T_i^{\langle l,k\rangle}(\omega) = \infty \end{cases} .$$

Here we use the convention $\Psi^{\langle l,k\rangle}(\omega)_{-1} := \mathcal{O} := (0,0)$.
Obviously $\|\Psi^{\langle l,k\rangle}(\omega)_i - \Psi^{\langle l,k\rangle}(\omega)_{i-1}\| \le 2^{-l}$ for $i \in \mathrm{N}$, so that $\Psi^{\langle l,k\rangle}$ maps $\Omega^{\langle k\rangle}$ into $\Omega^{\langle l\rangle}$.
For $\omega \in C([0,\infty), G)$ we obtain in an analogous way stopping times $T^{\langle l\rangle}$ and $T_i^{\langle l\rangle}$ and functions $\Psi^{\langle l\rangle}$.
Let $(\alpha^{\langle n\rangle})_{n\in\mathrm{N}_0}$ be a sequence of real numbers in $[0,\frac{1}{2}]$ and for each $n \in \mathrm{N}_0$ let $(P_x^{\langle n\rangle};\ x \in G^{\langle n\rangle})$ be an $(\alpha^{\langle n\rangle}, n)$-random walk.
We call the sequence $(P_x^{\langle n\rangle};\ x \in G^{\langle n\rangle})_{n\in\mathrm{N}_0}$ consistent, if

$$P_x^{\langle l\rangle} = P_x^{\langle k\rangle} \circ \Psi^{\langle l,k\rangle -1} \text{ for all } x \in G^{\langle l\rangle},\ k,l \in \mathrm{N}_0 \text{ with } k \ge l, \tag{3.8}$$

where $P_x^{\langle k\rangle} \circ \Psi^{\langle l,k\rangle -1}$ denotes the image of $P_x^{\langle k\rangle}$ under $\Psi^{\langle l,k\rangle}$.
We shall remark, that this condition is very natural in the following sense:

Remark *We shall see later, that for a non trivial homogeneous diffusion $(P_x;\ x \in G)$ with Feller property, we obtain random walks $(P_x \circ \Psi^{\langle n\rangle -1};\ x \in G^{\langle n\rangle})$ for all $n \in \mathrm{N}_0$. Since*

$$\Psi^{\langle l,k\rangle} \circ \Psi^{\langle k\rangle} = \Psi^{\langle l\rangle} \tag{3.9}$$

the sequence $(P_x \circ \Psi^{\langle n\rangle -1};\ x \in G^{\langle n\rangle})_{n\in\mathrm{N}_0}$ is consistent.

In the following we shall use the indicator functions χ_1 and χ_2 on $G^{\langle\infty\rangle} \times G^{\langle\infty\rangle}$

$$\chi_i(x,y) := \begin{cases} 1 & ; \text{if } i = 1 \text{ and } y \text{ is a non-collinear } k\text{-neighbor of } x \text{ for some } k \\ & \quad \text{or if } i = 2 \text{ and } y \text{ is a collinear } k\text{-neighbor of } x \text{ for some } k\,. \\ 0 & ; \text{otherwise} \end{cases}$$

For $m \in \{1,2\}$, $i \in \mathrm{N}_0$ and $\omega \in \Omega^{\langle k\rangle}$ let $R_i^{\langle l,k\rangle}(m,\omega) := \sum_{j=1}^{T_i^{\langle l,k\rangle}(\omega)} \chi_m(\omega_{j-1},\omega_j)$ denote the number of 'steps to non-collinear neighbors' ($m = 1$), respectively the number of 'steps to collinear neighbors' ($m = 2$), up to the time $T_i^{\langle l,k\rangle}(\omega)$.
For $k \in \mathrm{N}$, $x \in G^{\langle k-1\rangle}$, $i \in \{1,2\}$, $\alpha \in [0,\frac{1}{2}]$ and $a, b \ge 0$ we denote by

$$g_{\alpha,i}^{\langle k\rangle,x}(a,b) := E_x^{\langle k\rangle}\left[a^{R_1^{\langle k-1,k\rangle}(1)}\, b^{R_1^{\langle k-1,k\rangle}(2)};\ \chi_i(x, \Psi^{\langle k-1,k\rangle}(\cdot)_1) = 1, T_1^{\langle k-1,k\rangle} < \infty\right]$$

where $E_x^{\langle k\rangle}$ denotes the expectation relative to $P_x^{\langle k\rangle}$ for the (α,k)-random walk $(P_x^{\langle k\rangle};\ x \in G^{\langle k\rangle})$.

3.1 Lemma *For all $a, b \in [0,1)$ we have*

$$g^{\langle k\rangle,x}_{\alpha,1}(a,b) = g_{\alpha,1}(a,b) := \frac{2b\alpha a(1-2\alpha)[(1-2\alpha)b+1]}{2(1-2\alpha a)(1+\alpha a) - b(1-2\alpha)[2\alpha a + (1-2\alpha)b]},$$

$$g^{\langle k\rangle,x}_{\alpha,2}(a,b) = g_{\alpha,2}(a,b) := \frac{b(1-2\alpha)[(1-2\alpha)b+4\alpha^2a^2]}{2(1-2\alpha a)(1+\alpha a) - b(1-2\alpha)[2\alpha a + (1-2\alpha)b]}.$$

In the case $\alpha \neq \frac{1}{2}$, the equations are valid for all $a, b \in [0,1]$.

The proof is similar to the proof of Lemma 2.2 in B.-P. □

Since $R_i^{\langle k-1,k\rangle} - R_{i-1}^{\langle k-1,k\rangle}$, $i \in \mathbb{N}$ have the same distribution by Lemma 3.1 and the strong Markov property, Lemma 3.1 implies for $\alpha < \frac{1}{2}$, $k, i \in \mathbb{N}$ and $x \in G^{\langle k-1\rangle}$ that the random variables

$$R_i^{\langle k-1,k\rangle} \text{ and hence } T_i^{\langle k-1,k\rangle} \text{ are } P_x^{\langle k\rangle}\text{-square integrable.} \tag{3.10}$$

Let $f(\alpha) := \frac{2\alpha(1-\alpha)}{1+2\alpha}$. By the strong Markov property and Lemma 3.1 we obtain:

3.2 Lemma *For $k \in \mathbb{N}$ let $(P_x^{\langle k\rangle}\,;\, x \in G^{\langle k\rangle})$ be an (α, k)-random walk and let $l < k$.*

a) For $\alpha \neq \frac{1}{2}$: $(P_x^{\langle k\rangle} \circ \Psi^{\langle l,k\rangle^{-1}};\, x \in G^{\langle l\rangle})$ is a $(\underbrace{f\circ\cdots\circ f}_{k-l \text{ times}}(\alpha), l)$-random walk.

b) For $\alpha = \frac{1}{2}$: $\Psi^{\langle l,k\rangle}(\omega) \equiv \omega_0$ $P_x^{\langle k\rangle}$-a.s. for all $x \in G^{\langle l\rangle}$.

□

3.3 Proposition *For $k \in \mathbb{N}_0$ let $\alpha^{\langle k\rangle} \in [0, \frac{1}{2}]$ and let $(P_x^{\langle k\rangle};\, x \in G^{\langle k\rangle})$ be an $(\alpha^{\langle k\rangle}, k)$-random walk. The sequence $(P_x^{\langle k\rangle};\, x \in G^{\langle k\rangle})_{k\in\mathbb{N}_0}$ is consistent, iff $\alpha^{\langle 0\rangle} \in [0, \frac{1}{4}]$ and*

$$\alpha^{\langle k+1\rangle} = \delta(\alpha^{\langle k\rangle})\alpha^{\langle k\rangle} \text{ for all } k \in \mathbb{N}_0 \tag{3.11}$$

with $\delta(\alpha) := (1-\alpha+\sqrt{(1-\alpha)^2-2\alpha})^{-1}$. In this case

$$\begin{array}{ll} \alpha^{\langle\cdot\rangle} \equiv \alpha^{\langle 0\rangle} & ;\text{ if } \alpha^{\langle 0\rangle} = 0 \text{ or } \frac{1}{4}. \\ 2^{-k}\alpha^{\langle 0\rangle} \leq \alpha^{\langle k\rangle} \leq \delta(\alpha^{\langle 0\rangle})^k\alpha^{\langle 0\rangle} \text{ with } \delta(\alpha^{\langle 0\rangle}) < 1 & ;\text{ if } \alpha^{\langle 0\rangle} \in (0, \frac{1}{4}). \end{array}$$

Remark *We remark that for $\alpha^{\langle 0\rangle} \in [0, \frac{1}{4}]$ equation (3.11) defines a sequence of real numbers in $[0, \frac{1}{4}]$.*

<u>Proof:</u> Let $(P_x^{\langle k\rangle};\, x \in G^{\langle k\rangle})_{k\in\mathbb{N}_0}$ be consistent. By Lemma 3.2 b) $\alpha^{\langle k\rangle} \neq \frac{1}{2}$ for all $k \in \mathbb{N}$. By Lemma 3.2 a) consistency is then equivalent to $f(\alpha^{\langle k+1\rangle}) = \alpha^{\langle k\rangle}$ for $k \in \mathbb{N}_0$, i.e. $\alpha^{\langle k+1\rangle} \in f^{-1}\{\alpha^{\langle k\rangle}\}$. The set on the right sight consists of the (possibly complex) points $\frac{1}{2}\left(1-\alpha^{\langle k\rangle}+\sqrt{(1-\alpha^{\langle k\rangle})^2-2\alpha^{\langle k\rangle}}\right)$ and $\frac{1}{2}\left(1-\alpha^{\langle k\rangle}-\sqrt{(1-\alpha^{\langle k\rangle})^2-2\alpha^{\langle k\rangle}}\right)$.

We shall prove that consistency is equivalent to

$$\alpha^{\langle k+1\rangle} = \frac{1}{2}\left(1 - \alpha^{\langle k\rangle} - \sqrt{(1-\alpha^{\langle k\rangle})^2 - 2\alpha^{\langle k\rangle}}\right) \qquad \text{for } k \in \mathbb{N}_0.$$

Since $\alpha^{\langle k+1\rangle}$ has to be real, consistency is equivalent to $\alpha^{\langle k+1\rangle} \in f^{-1}\{\alpha^{\langle k\rangle}\}$ and $\alpha^{\langle k\rangle} \in [0, 2-\sqrt{3}]$ for all $k \in \mathbb{N}_0$.
But for $\alpha^{\langle k\rangle} \in [0, 2-\sqrt{3}]$, $\frac{1}{2}\left(1 - \alpha^{\langle k\rangle} + \sqrt{(1-\alpha^{\langle k\rangle})^2 - 2\alpha^{\langle k\rangle}}\right) > 2-\sqrt{3}$. Hence consistency is equivalent to
$\alpha^{\langle k+1\rangle} = \frac{1}{2}\left(1 - \alpha^{\langle k\rangle} - \sqrt{(1-\alpha^{\langle k\rangle})^2 - 2\alpha^{\langle k\rangle}}\right) = \delta(\alpha^{\langle k\rangle})\alpha^{\langle k\rangle}$ for $k \in \mathbb{N}_0$.

Using strict monotony of $\delta|_{[0,2-\sqrt{3})}$ and $\delta(0) = \frac{1}{2}, \delta(\frac{1}{4}) = 1$ the remaining part of the Proposition follows now easily. □

4 Consistent velocities

For the following we fix a consistent sequence $(P_x^{\langle k\rangle}; x \in G^{\langle k\rangle})_{k\in\mathbb{N}_0}$ of $(\alpha^{\langle k\rangle}, k)$-random walks on $G^{\langle k\rangle}$, where $(\alpha^{\langle k\rangle})_{k\in\mathbb{N}_0}$ is a sequence of real numbers in $[0, \frac{1}{4}]$ which satisfies equation (3.11).

In order to embed the sample spaces $\Omega^{\langle k\rangle}$ of the random walks into $C([0,\infty), G)$ we fix velocities $c^{\langle k\rangle} \in \mathbb{R}_+^2$ and define for $\omega \in \Omega^{\langle k\rangle}$ the function $Y_{c^{\langle k\rangle}}^{\langle k\rangle}(\omega) \in C([0,\infty), G)$ by $Y_{c^{\langle k\rangle}}^{\langle k\rangle}(\omega)_t = \omega_i$ for $t = c_1^{\langle k\rangle} R_i^{\langle k,k\rangle}(1,\omega) + c_2^{\langle k\rangle} R_i^{\langle k,k\rangle}(2,\omega), i \in \mathbb{N}_0$, and by linear interpolation for all other times t.

For $k \in \mathbb{N}_0$, $x \in G^{\langle k\rangle}$ let the measure $\tilde{P}_x^{\langle k\rangle}$ on $C([0,\infty), G)$ be the image measure of $P_x^{\langle k\rangle}$ on $\Omega^{\langle k\rangle}$ under the embedding $Y_{c^{\langle k\rangle}}^{\langle k\rangle}$.

In turns of this measures $(\tilde{P}_x^{\langle k\rangle}; x \in G^{\langle k\rangle})_{k\in\mathbb{N}_0}$ the previously discussed consistency of the random walks $(P_x^{\langle k\rangle}; x \in G^{\langle k\rangle})_{k\in\mathbb{N}_0}$ translates into

$$\tilde{P}_x^{\langle k\rangle}\{X_{T^{\langle k-1\rangle}} \in \cdot\} = \tilde{P}_x^{\langle k-1\rangle}\{X_{T^{\langle k-1\rangle}} \in \cdot\}$$

for $k \in \mathbb{N}$ and $x \in G^{\langle k-1\rangle}$, where $X_t(\omega) = \omega_t$.

The sequence of velocities $(c^{\langle k\rangle})_{k\in\mathbb{N}_0}$ underlying the measures $\tilde{P}_x^{\langle k\rangle}$, $x \in G^{\langle k\rangle}, k \in \mathbb{N}_0$ is called consistent if

$$\tilde{E}_x^{\langle k\rangle}(T^{\langle k-1\rangle} \mid X_{T^{\langle k-1\rangle}}) = \tilde{E}_x^{\langle k-1\rangle}(T^{\langle k-1\rangle} \mid X_{T^{\langle k-1\rangle}}) \qquad \text{for } k \in \mathbb{N} \text{ and } x \in G^{\langle k-1\rangle},$$

where $\tilde{E}_x^{\langle k\rangle}$ denotes integration with respect to $\tilde{P}_x^{\langle k\rangle}$.

4.1 Proposition *For every consistent sequence of random walks $(P_x^{\langle k\rangle}; x \in G^{\langle k\rangle})_{k\in\mathbb{N}_0}$ there exists a consistent sequence of velocities $(c^{\langle k\rangle})_{k\in\mathbb{N}_0}$. For two consistent sequences of velocities $(c^{\langle k\rangle})_{k\in\mathbb{N}_0}$ and $(c^{\langle k\rangle\prime})_{k\in\mathbb{N}_0}$ the resulting measures $\tilde{P}_x^{\langle k\rangle}$ and $\tilde{P}_x^{\langle k\rangle\prime}$ are related in a simple way: There exists $c > 0$ such that $\tilde{P}_x^{\langle k\rangle}\{\omega(\cdot) \in \star\} = \tilde{P}_x^{\langle k\rangle\prime}\{\omega(c\,\cdot) \in \star\}$ for $k \in \mathbb{N}_0$, $x \in G^{\langle k\rangle}$.*

For the proof of the Proposition we shall use Lemma 4.2 below. This Lemma will again be used later on and is therefore formulated slightly more general than would be necessary in the present context. To formulate this Lemma we introduce the variables $V_i^{\langle l,k\rangle} := R_i^{\langle l,k\rangle} - R_{i-1}^{\langle l,k\rangle}$, $i \in \mathbb{N}$ and $k, l \in \mathbb{N}_0, k \geq l$.

4.2 Lemma *For $x \in G^{\langle k-1\rangle}$, $k, i \in \mathbb{N}$*

$$\begin{aligned} & P_x^{\langle k\rangle}\{V_i^{\langle k-1,k\rangle} \in \star | X^{\langle k\rangle}_{\cdot \wedge T_{i-1}^{\langle k-1,k\rangle}}, X^{\langle k\rangle}_{T_i^{\langle k-1,k\rangle}+\cdot}\} \\ = \; & P_x^{\langle k\rangle}\{V_i^{\langle k-1,k\rangle} \in \star | V_i^{\langle k-1,k-1\rangle} \circ \Psi^{\langle k-1,k\rangle}\} \end{aligned}$$

where $X_i^{\langle k\rangle}(\omega) = \omega_i$ for $\omega \in \Omega^{\langle k\rangle}$. Furthermore,

$$\begin{aligned} & E_x^{\langle k\rangle}(a^{V_i^{\langle k-1,k\rangle}(1)} b^{V_i^{\langle k-1,k\rangle}(2)} | V_i^{\langle k-1,k-1\rangle} \circ \Psi^{\langle k-1,k\rangle}) \\ = \; & \frac{g_{\alpha^{\langle k\rangle},1}(a,b)}{g_{\alpha^{\langle k\rangle},1}(1,1)} V_i^{\langle k-1,k-1\rangle}(1) \circ \Psi^{\langle k-1,k\rangle} + \frac{g_{\alpha^{\langle k\rangle},2}(a,b)}{g_{\alpha^{\langle k\rangle},2}(1,1)} V_i^{\langle k-1,k-1\rangle}(2) \circ \Psi^{\langle k-1,k\rangle} \end{aligned}$$

for $a, b \in [0,1]$, where $\frac{g_{0,1}(a,b)}{g_{0,1}(1,1)} := 1$.

As the proof of Lemma 4.2 is similar to the proof of Lemma 2.5 in B.-P. we omit the proof. Notation: $G_\alpha(a,b) := \left(\frac{g_{\alpha,i}(a,b)}{g_{\alpha,i}(1,1)}\right)_{i=1,2}$.

Proof of Proposition 4.1: We observe, that $T^{\langle k-1\rangle}(Y^{\langle k\rangle}_{c^{\langle k\rangle}}(\omega)) = c^{\langle k\rangle t} V_1^{\langle k-1\rangle}(\omega)$ for $\omega \in \Omega^{\langle k\rangle}$ with $\omega_0 \in G^{\langle k-1\rangle}$.

Hence, if $\alpha^{\langle 0\rangle} = 0$, i.e. $\alpha^{\langle\cdot\rangle} \equiv 0$, Lemma 4.2 implies that consistency of the sequence of velocities $(c^{\langle k\rangle})_{k\in\mathbb{N}_0}$ is equivalent to

$$c_2^{\langle k\rangle} = \frac{\partial G_{0,2}}{\partial x_2}(1,1)\, c_2^{\langle k+1\rangle} = 4c_2^{\langle k+1\rangle}, \tag{4.12}$$

which in turn is equivalent to $c_2^{\langle k\rangle} = (\frac{1}{4})^k c_2^{\langle 0\rangle}$, for all $k \in \mathbb{N}_0$. This implies in particular for two consistent sequences of velocities $(c^{\langle k\rangle})_{k\in\mathbb{N}_0}$ and $(c^{\langle k\rangle\prime})_{k\in\mathbb{N}_0}$ that there exists $c > 0$ such that $c\, c_2^{\langle k\rangle} = c_2^{\langle k\rangle\prime}$ for all $k \in \mathbb{N}_0$.

Moreover, since the assumption $\alpha^{\langle\cdot\rangle} \equiv 0$ implies that $R_i^{\langle k,k\rangle}(1) = 0$ $P_x^{\langle k\rangle}$-a.s., we conclude that

$$Y^{\langle k\rangle}_{c^{\langle k\rangle}}(\omega)_t = Y^{\langle k\rangle}_{c^{\langle k\rangle\prime}}(\omega)_{ct} \qquad \text{for } P_x^{\langle k\rangle}\text{-a.a. } \omega$$

which implies the statement in the Proposition regarding the measures $\tilde{P}_x^{\langle k\rangle}$ and $\tilde{P}_x^{\langle k\rangle\prime}$.

In the case $\alpha^{\langle 0\rangle} \in (0, \frac{1}{4}]$ Lemma 4.2 implies that consistency of the sequence of velocities $(c^{\langle k\rangle})_{k\in\mathbb{N}_0}$ is equivalent to

$$c^{\langle k\rangle} = \left(\frac{\partial G_{\alpha^{\langle k+1\rangle},i}}{\partial x_j}(1,1)\right)_{i,j=1,2} \cdot c^{\langle k+1\rangle}. \tag{4.13}$$

Since the conditions (D) and (E) hold for $a_{i,j}^{\langle n\rangle} := \frac{\partial G_{\alpha^{\langle n+1\rangle},i}}{\partial x_j}(1,1) > 0$, Lemma 2.3 implies the existence of a sequence $(c^{\langle k\rangle})_{k\in\mathbb{N}_0}$ which satisfies (2.1a) and (2.1b) and hence (4.13).

For the remaining part of Proposition 4.1 it suffices to prove, that the sequence $(c^{\langle k\rangle})_{k\in\mathbb{N}_0}$ is uniquely determined by (4.13) up to a multiplicative constant $c > 0$. In the case that $\alpha^{\langle 0\rangle} = \frac{1}{4}$ this is easily seen by Corollary 1.1 in [5] and in the case that $\alpha^{\langle 0\rangle} \in (0, \frac{1}{4})$ it follows from Lemma 2.3 and Lemma 4.3 below. □

4.3 Lemma *For $\alpha^{\langle 0\rangle} \in (0, \frac{1}{4})$ define $(\alpha^{\langle k\rangle})_{k\in\mathbb{N}_0}$ by (3.11) and let $c^{\langle k\rangle}$, $k \in \mathbb{N}_0$ be vectors in $\mathbb{R}^2_+$, satisfying (4.13) for all $k \in \mathbb{N}_0$. Then the sequence $(c^{\langle k\rangle})_{k\in\mathbb{N}_0}$ satisfies (2.1b) with $A^{\langle n\rangle} := (\frac{\partial G_{\alpha^{\langle n+1\rangle},i}}{\partial x_j}(1,1))_{i,j=1,2}$.*

Proof: $\left(\frac{\partial G_{\alpha,i}}{\partial x_j}(1,1)\right)_{i,j=1,2}$ is invertible for $\alpha \in (0, \frac{1}{4})$ and

$$\left(\frac{\partial G_{\alpha,i}}{\partial x_j}(1,1)\right)^{-1}_{i,j=1,2} = \begin{pmatrix} \frac{2-3\alpha+4\alpha^2}{2(1+\alpha)(1-2\alpha-2\alpha^2)} & -\frac{(7-6\alpha-4\alpha^2)(1+2\alpha+4\alpha^2)}{8(1+\alpha)(1-2\alpha-2\alpha^2)(1-\alpha)} \\ -\frac{\alpha(1-\alpha)(1+2\alpha+4\alpha^2)}{2(1+\alpha)(1-2\alpha-2\alpha^2)(1-2\alpha)} & \frac{(1+4\alpha)(1-2\alpha+4\alpha^2)}{4(1+\alpha)(1-2\alpha-2\alpha^2)(1-2\alpha)} \end{pmatrix}.$$

Let $(b^{\langle k\rangle}_{i,j})_{i,j=1,2} := \left(\frac{\partial G_{\alpha^{\langle k\rangle},i}}{\partial x_j}(1,1)\right)^{-1}_{i,j=1,2}$.
Since $\alpha^{\langle k\rangle} \to 0$ by Proposition 3.3, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$

$$b^{\langle k\rangle}_{11} + b^{\langle k\rangle}_{12}\frac{1}{10} > \frac{9}{10}, \quad \frac{\alpha^{\langle k\rangle}}{20} < -b^{\langle k\rangle}_{21} \text{ and } \frac{1}{5} < b^{\langle k\rangle}_{22} < \frac{1}{3}. \tag{4.14}$$

Another way of saying that the sequence $(c^{\langle k\rangle})_{k\in\mathbb{N}_0}$ satisfies (4.13) is to say that it satisfies (2.1a).

We assume now, that the sequence did not satisfy (2.1b). We know by Proposition 2.3 in [5] that $\frac{c^{\langle n\rangle}}{\|c^{\langle n\rangle}\|} \to (1,0)^t$ or equivalently $\gamma^{\langle n\rangle} := \frac{c_2^{\langle n\rangle}}{c_1^{\langle n\rangle}} \to 0$. Obviously $c^{\langle n\rangle} \in \mathbb{R}^2_+$ implies $\gamma^{\langle n\rangle} > 0$.

If $k \geq k_0$ and $\gamma^{\langle k\rangle} < \frac{1}{10}$ then by (4.14)

$$\gamma^{\langle k+1\rangle} = \frac{b^{\langle k+1\rangle}_{21} + b^{\langle k+1\rangle}_{22}\gamma^{\langle k\rangle}}{b^{\langle k+1\rangle}_{11} + b^{\langle k+1\rangle}_{12}\gamma^{\langle k\rangle}} \leq \frac{b^{\langle k+1\rangle}_{22}}{b^{\langle k+1\rangle}_{11} + b^{\langle k+1\rangle}_{12}\gamma^{\langle k\rangle}}\gamma^{\langle k\rangle} \leq \frac{10}{27}\gamma^{\langle k\rangle}. \tag{4.15}$$

Inductively we obtain for $l \in \mathbb{N}$ and some $k_1 \geq k_0$

$$\gamma^{\langle k_1+l\rangle} \leq \frac{1}{10}\left(\frac{10}{27}\right)^l. \tag{4.16}$$

As by (4.14) and Proposition 3.3 $b^{\langle k\rangle}_{21} < -\frac{\alpha^{\langle 0\rangle}}{20}2^{-k}$ for $k \geq k_0$ we conclude from (4.14) and (4.16) that $b^{\langle k_1+l+1\rangle}_{21} + b^{\langle k_1+l+1\rangle}_{22}\gamma^{\langle k_1+l\rangle} < 0$ for l sufficiently large. Using this inequality in combination with (4.14) and (4.15) we obtain that $\gamma^{\langle k_1+l\rangle} < 0$ for l sufficiently large. This contradicts $c^{\langle n\rangle} \in \mathbb{R}^2_+$ for all $n \in \mathbb{N}_0$. □

5 Proof of Theorem 1.1

In this section we shall conclude the proof of Theorem 1.1, of which part b) is by far more delicate than part a). The essential tools for the proof of part b) have already been developed in sections 2, 3 and 4 and for the conclusion of the proof of part b) we may follow arguments used in B.-P. Thus we shall only sketch this part of the proof.

We remind the reader that we shall refer to isometries of open subsets of G, relative to the Euclidean metric, simply as symmetries. We introduce first a special class of symmetries. For $k \in \mathbb{N}_0$ and $x \in G^{\langle k \rangle}$ consider the two closed equilateral triangles, which have as vertices x, one collinear and one non-collinear k-neighbor of x. We denote by $N^{\langle k \rangle}(x)$ the interior (relative to the topology of G) of the set of all points in G, which lie in one of those two triangles. It is not hard to see that $\{N^{\langle k \rangle}(x); x \in G^{\langle k \rangle}, k \in \mathbb{N}_0\}$ is a base for the topology of G. For $x, y \in G^{\langle k \rangle}$ there exist exactly two isometries of $\mathbb{R}^2$, which map x onto y and $N^{\langle k \rangle}(x)$ onto $N^{\langle k \rangle}(y)$. One of these isometries, say $\phi_{x,y}$, preserves the orientation the other isometry, say $\lambda_{x,y}$, reverses the orientation. The restrictions of $\phi_{x,y}$ and $\lambda_{x,y}$ to $N^{\langle k \rangle}(x)$ are symmetries of the gasket and shall be denoted by $\phi_{x,y}^{\langle k \rangle}$ and $\lambda_{x,y}^{\langle k \rangle}$..

For $A \subset G$ let T_A be the first exit time from A. It is not hard to see, that for $\omega \in C([0,\infty), G)$ with $\omega_0 \in G^{\langle k \rangle}$ we have $T_1^{\langle k \rangle}(\omega) = T^{\langle k \rangle}(\omega) = T_{N^{\langle k \rangle}(\omega_0)}(\omega)$.

In the following two lemmas $(Q_x; x \in G)$ will denote a non trivial, homogeneous diffusion on G with Feller property.

5.1 Lemma *There exists $k_0 \in \mathbb{N}_0$, such that*

a) $\sup_{x \in G^{\langle k_0 \rangle}} \sup_{y \in N^{\langle k_0 \rangle}(x)} Q_y\{T_{N^{\langle k_0 \rangle}(x)} > 1\} < 1$,

b) $T_{N^{\langle k \rangle}(x)}$ *is Q_x-square integrable for all $x \in G^{\langle k \rangle}, k \geq k_0$.*

<u>Proof:</u> Homogeneity of the diffusion $(Q_x\,;\, x \in G)$ implies for all $x \in G^{\langle k \rangle}, k \in \mathbb{N}_0$ and $y \in N^{\langle k \rangle}(x)$

$$Q_y\{\phi_{x,\mathcal{O}}(\omega(\cdot \wedge T_{N^{\langle k \rangle}(x)})) \in \star\} = Q_{\phi_{x,\mathcal{O}}(y)}\{\omega(\cdot \wedge T_{N^{\langle k \rangle}(\mathcal{O})}) \in \star\}. \tag{5.17}$$

The invariance identity (5.17) and the fact that the diffusion is non trivial imply that $Q_\mathcal{O}$ is non trivial, i.e. $Q_\mathcal{O}\{\omega(t) = \mathcal{O} \text{ for all } t \geq 0\} < 1$. Indeed, if $Q_\mathcal{O}$ were trivial the invariance identity (5.17) would imply that Q_x is trivial for all $x \in G^{\langle \infty \rangle}$, and since $G \backslash G^{\langle \infty \rangle}$ is totally disconnected this would imply that the diffusion $(Q_x\,;\, x \in G)$ is trivial.

The invariance identity (5.17) also implies that for all $y \in N^{\langle k \rangle}(x)$, $x \in G^{\langle k \rangle}, k \in \mathbb{N}_0$:

$$Q_y\{T_{N^{\langle k \rangle}(x)} \in \cdot\} = Q_{\phi_{x,\mathcal{O}}^{\langle k \rangle}(y)}\{T_{N^{\langle k \rangle}(\mathcal{O})} \in \cdot\}. \tag{5.18}$$

As $Q_\mathcal{O}$ is non trivial and $\{N^{\langle k \rangle}(\mathcal{O}); k \in \mathbb{N}_0\}$ is a base for the neighborhood system of $\mathcal{O}$, the strong Markov property implies, that there exists $k_1 \in \mathbb{N}_0$ with

$$Q_\mathcal{O}\{T_{N^{\langle k_1 \rangle}(\mathcal{O})} > 1\} < 1. \tag{5.19}$$

Let $\varphi : G \to [0,1]$ be a continuous function which vanishes on the closed set $\overline{N^{\langle k_1+1 \rangle}(\mathcal{O})}$ and is strictly positive on its complement. By the Feller property the function $H : G \to [0,1]$ defined by

$$H(\star) = \int \int_0^1 \varphi(\omega_t) dt \, dQ_\star$$

is continuous. Moreover, $H(\mathcal{O}) > 0$ by (5.19).

If assertion a) did not hold, (5.18) would imply, that for all $k \in \mathbf{N}$ there exists $y_k \in N^{\langle k \rangle}(\mathcal{O})$ such that $\lim_{k\to\infty} Q_{y_k}\{T_{N^{\langle k \rangle}(\mathcal{O})} > 1\} = 1$. Hence we would have $\lim_{k\to\infty} H(y_k) = 0$.

On the other hand $y_k \in N^{\langle k \rangle}(\mathcal{O})$ implies $\|y_k\| \leq 2^{-k}$ and by the continuity of H $\lim_{k\to\infty} H(y_k) = H(\mathcal{O}) > 0$.

As for assertion b) we observe first that $T_{N^{\langle k \rangle}(x)}$ decreases in k. By a standard argument involving the strong Markov property we conclude from assertion a) that

$$\sup_{x \in G^{\langle k_0 \rangle}} Q_x\{T_{N^{\langle k_0 \rangle}(x)} > k\} \leq \Big(\sup_{x \in G^{\langle k_0 \rangle}} \sup_{y \in N^{\langle k_0 \rangle}(x)} Q_y\{T_{N^{\langle k_0 \rangle}(x)} > 1\}\Big)^k.$$

This implies immediately assertion b). □

5.2 Lemma

a) For each $k \in \mathrm{N}_0$ $(Q_x \circ \Psi^{\langle k \rangle^{-1}}\,;\, x \in G^{\langle k \rangle})$ is a $(\beta^{\langle k \rangle}, k)$-random walk on $G^{\langle k \rangle}$, where $\beta^{\langle k \rangle} := \frac{1}{2} Q_{\mathcal{O}}\{\chi_1(\mathcal{O}, \omega_{T^{\langle k \rangle}}) = 1\}$.
Moreover, $(Q_x \circ \Psi^{\langle k \rangle^{-1}}\,;\, x \in G^{\langle k \rangle})_{k \in \mathrm{N}_0}$ is a consistent sequence of random walks.

b) For all $k \in \mathrm{N}_0$ we have that

$$T_i^{\langle k \rangle} \textit{ is } Q_x\textit{-square integrable for all } x \in G^{\langle k \rangle} \textit{ and } i \in \mathbf{N}. \tag{5.20}$$

<u>Proof:</u> Fix $k_0 \in \mathrm{N}_0$ according to Lemma 5.1. Then the invariance identity (5.17), Lemma 5.1 part b) and the strong Markov property imply that $(Q_x \circ \Psi^{\langle k \rangle^{-1}}\,;\, x \in G^{\langle k \rangle})$ is a $(\beta^{\langle k \rangle}, k)$-random walk on $G^{\langle k \rangle}$, for indices $k \geq k_0$. For indices $k < k_0$ this follows from (3.9), Lemma 3.2 and the fact that $\beta^{\langle k_0 \rangle} = f(\beta^{\langle k_0+1 \rangle}) \in [0, \frac{1}{2})$.

In order to conclude part a) we observe that consistency of the sequence of random walks is an easy consequence of (3.9).

As for part b) we observe that for indices $k \geq k_0$ assertion (5.20) follows by (5.17), Lemma 5.1 and the strong Markov property.

It remains to verify (5.20) for indices $k < k_0$. To this end we shall prove that if (5.20) holds for some $k \in \mathbf{N}$, it also holds for $k-1$.

Arguments similar to those in the proof of B.-P. Lemma 8.2 a) show that for $i \in \mathbf{N}$ and $x \in G^{\langle k-1 \rangle}$

$$\begin{aligned} & Q_x\{T_i^{\langle k \rangle} - T_{i-1}^{\langle k \rangle} \in \star \mid X_{\cdot \wedge T_{i-1}^{\langle k \rangle}}, X_{T_i^{\langle k \rangle} + \cdot}\} \\ = \; & Q_x\{T_i^{\langle k \rangle} - T_{i-1}^{\langle k \rangle} \in \star \mid \chi_j(\Psi^{\langle k \rangle}(\cdot)_{i-1}, \Psi^{\langle k \rangle}(\cdot)_i), j = 1,2\} \end{aligned} \tag{5.21}$$

and that for $j \in \{1,2\}$

$$\begin{aligned} & Q_x\{T_i^{\langle k\rangle} - T_{i-1}^{\langle k\rangle} \in \star \mid \chi_j(\Psi^{\langle k\rangle}(\cdot)_{i-1}, \Psi^{\langle k\rangle}(\cdot)_i) = 1\} \\ = \; & Q_O\{T_1^{\langle k\rangle} - T_0^{\langle k\rangle} \in \star \mid \chi_j(\Psi^{\langle k\rangle}(\cdot)_0, \Psi^{\langle k\rangle}(\cdot)_1) = 1\}. \end{aligned} \tag{5.22}$$

Moreover, for $\omega \in C([0,\infty), G)$ with $\omega_0 \in G^{\langle k\rangle}$

$$T_i^{\langle k-1\rangle}(\omega) = \sum_{j=1}^{T_i^{\langle k-1,k\rangle}(\Psi^{\langle k\rangle}(\omega))} T_j^{\langle k\rangle}(\omega) - T_{j-1}^{\langle k\rangle}(\omega). \tag{5.23}$$

Now Q_x-square integrability of $T_i^{\langle k\rangle}, i \in \mathrm{N}_0$ implies Q_x-square integrability of $T_i^{\langle k-1\rangle}, i \in \mathrm{N}_0$ by (3.10), (5.21) and (5.22). □

Remark *Lemma 5.2 part a) and Proposition 3.3 imply* $\beta^{\langle k\rangle} \in [0, \frac{1}{4}]$. *Moreover, as* $\beta^{\langle 0\rangle} = Q_O\{\omega$ *leaves* N *through* $(\frac{1}{2}, \frac{\sqrt{3}}{2})\}$ *we have proved part a) of Theorem 1.1.*

For the proof of part b) of Theorem 1.1 fix $\alpha \in [0, \frac{1}{4}]$ and define the sequence $(\alpha^{\langle k\rangle})_{k\in\mathrm{N}_0}$ by (3.11) with $\alpha^{\langle 0\rangle} = \alpha$. Let $(P_x^{\langle k\rangle};\ x \in G^{\langle k\rangle})_{k\in\mathrm{N}_0}$ be the sequence of $(\alpha^{\langle k\rangle}, k)$-random walks. By Proposition 3.3 this sequence is consistent. By Bochner's version of Kolomogorov's Extension Theorem (Theorem 5.1.1. in [2]) it is possible to show that there exist a probability space $(\Omega, \mathfrak{F}, P)$ and random functions $\tilde{X}(k,x) : \Omega \to \Omega^{\langle k\rangle}$, $k \in \mathrm{N}_0$, $x \in G^{\langle k\rangle}$, such that for all $k, l \in \mathrm{N}_0, l \geq k$ and $x \in G^{\langle k\rangle}$:

a) $P \circ \tilde{X}(k,x)^{-1} = P_x^{\langle k\rangle}$.

b) $\tilde{X}(k,x) = \Psi^{\langle k,l\rangle} \circ \tilde{X}(l,x)$.

c) The σ-algebras $\mathfrak{F}_y := \sigma\{\tilde{X}(k,y); k \in \mathrm{N}_0$ with $y \in G^{\langle k\rangle}\}$, $y \in G^{\langle\infty\rangle}$ are independent.

The consistency of our $(\alpha^{\langle k\rangle}, k)$-random walks is just the consistency condition in Bochner's Version of Kolmogorov's Theorem.

Starting from the functions $\tilde{X}(n,x)$ we construct inductively for each $k \in \mathrm{N}_0$ random functions $X(k+l,x) : \Omega \to \Omega^{\langle k+l\rangle}$, $l \in \mathrm{N}_0$ and $x \in G^{\langle k\rangle}$:
For $x \in G^{\langle 0\rangle}$ let $X(l,x) := \tilde{X}(l,x)$ for all $l \in \mathrm{N}_0$.
If $X(k,x), X(k+1,x), \dots$ are defined for all $x \in G^{\langle k\rangle}$, we define for $i \in \mathrm{N}_0$, $x \in G^{\langle k+1\rangle} \setminus G^{\langle k\rangle}$ and $l \geq k+1$

$$X(l,x)_i := \begin{cases} \widetilde{X}(l,x)_i & ;\text{ if } i < T^{\langle k,l\rangle}(\widetilde{X}(l,x)) \\ X(l, \widetilde{X}(l,x)_{T^{\langle k,l\rangle}})_{i - T^{\langle k,l\rangle}(\widetilde{X}(l,x))} & ;\text{ if } i \geq T^{\langle k,l\rangle}(\widetilde{X}(l,x)) \end{cases}.$$

We remark that the strong Markov property implies that $X(k,x)$ is $P_x^{\langle k\rangle}$-distributed.

Let $(c^{\langle k\rangle})_{k\in\mathbb{N}_0}$ be a consistent sequence of velocities for the consistent sequence of random walks under consideration, whose existence is assured by Proposition 4.1, and define $Y(k,x) := Y^{\langle k\rangle}_{c^{\langle k\rangle}}(X(k,x))$. Furthermore, we denote by $L^0(C([0,\infty),G))$ the space of $C([0,\infty),G)$-valued measurable functions on Ω endowed with the topology of convergence in probability.

5.3 Proposition

a) $Y(x) := \lim_{k\to\infty} Y(k,x)$ *exists P-a.e. in* $C([0,\infty),G)$ *for all* $x \in G^{\langle\infty\rangle}$.

b) The function $Y : G^{\langle\infty\rangle} \to L^0(C([0,\infty),G))$ *is uniformly continuous on bounded subsets of* $G^{\langle\infty\rangle}$, *and hence there exists a unique continuous extension of Y to all of G (also called Y).*

c) $(P\circ Y(x)^{-1};\ x\in G)$ *is a diffusion with Feller property and* $P\circ Y(\mathcal{O})^{-1}\{\omega$ *leaves N through* $(\frac{1}{2},\frac{\sqrt{3}}{2})\} = \alpha^{\langle 0\rangle} = \alpha$.

Proposition 5.3 corresponds to Theorem 2.8, Proposition 2.13 and Theorem 2.15 in B.-P. and the proofs can easily be adapted, with one exception.

For the proof of part a) we have to show that $\lim_{k\to\infty} T_i^{\langle l\rangle}(Y(k,x))$ exists and that this limit is strictly increasing as a function of i P-a.e. for $l \in \mathbb{N}_0$ and $x \in G^{\langle l\rangle}$.

In B.-P. the corresponding result is derived from the classical limit theorem for single-type branching processes with constant environment. In the present case we may apply our limit result on multi-type branching processes with varying environment. Indeed, using Lemma 4.2 it is not hard to see, that $\{V_i^{\langle l,k+l\rangle}(X(l+k,x))\,;\,k\in\mathbb{N}_0\}$ is a two-type branching process with varying environment with generating functions $F^{\langle k\rangle} = G_{\alpha^{\langle k+l+1\rangle}}$ and that this process does not die out a.s. Since $T_i^{\langle l\rangle}(Y(l+k,x)) - T_{i-1}^{\langle l\rangle}(Y(l+k,x)) = c^{\langle l+k\rangle t}V_i^{\langle l,l+k\rangle}(X(l+k,x))$ and $T_0^{\langle k\rangle}(Y(l+k,x)) = 0$ P-a.e. we may conclude that $\lim_{k\to\infty} T_i^{\langle l\rangle}(Y(k,x))$ exists and is strictly increasing as a function of i a.e., by Corollary 2.4.

Next we will show that the diffusion of Proposition 5.3 part c) is homogeneous. In the following we shall use the notation $P_x = P\circ Y(x)^{-1}$.

We remark first that the operator Θ_A mapping the path ω into the stopped path $\Theta_A(\omega) := \omega_{\cdot\wedge T_A}$ is continuous in P_x-a.a. paths $\omega\in C([0,\infty),G)$, for $A\subset G$ open with $\partial A\subset G^{\langle\infty\rangle}$ finite. This follows from an argument which is essentially the same as that for Lemma 2.16 in B.-P. and shall be omitted here.

Applying this result to the sets $A = N^{\langle k\rangle}(x)$ we conclude by Proposition 5.3, part a) and the corresponding properties of $(P_x^{\langle l\rangle};\ x\in G^{\langle l\rangle})$ that for $l\geq k$

$$P_x\{\phi_{x,y}\circ\Theta_{N^{\langle k\rangle}(x)}(\omega)\in\cdot\} = P_x\{\lambda_{x,y}\circ\Theta_{N^{\langle k\rangle}(x)}(\omega)\in\cdot\} = P_y\{\Theta_{N^{\langle k\rangle}(y)}(\omega)\in\cdot\} \tag{5.24}$$

for $x,y\in G^{\langle k\rangle}, k\in\mathbb{N}_0$. If $\eta : U\to V$ is a symmetry with domain

$$U = \cup_{i=1}^n N^{\langle k\rangle}(x_i) \tag{5.25}$$

for some $k \in \mathbb{N}_0$, $n \in \mathbb{N}$ and $x_i \in G^{\langle k \rangle}$, then $\eta|_{N^{\langle k \rangle}(x_i)}$ equals $\phi^{\langle k \rangle}_{x_i,\eta(x_i)}$ or $\lambda^{\langle k \rangle}_{x_i,\eta(x_i)}$ and thus by the strong Markov property and (5.24) we obtain

$$P_x\{\eta \circ \Theta_U(\omega) \in \cdot\} = P_{\eta(x)}\{\Theta_V(\omega) \in \cdot\} \tag{5.26}$$

for $x \in \{x_1, \ldots, x_n\}$.

Now for U of the form (5.25) we have $U = \cup_{y \in G^{\langle l \rangle} \cap U} N^{\langle l \rangle}(y)$ and $G^{\langle l \rangle} \cap U$ is finite for all $l \geq k$. Therefore (5.26) holds for all $x \in G^{\langle l \rangle} \cap U$, $l \geq k$ and hence all $x \in G^{\langle \infty \rangle} \cap U$. By the continuity of Θ_U Proposition 5.3 b) implies (5.26) for all $x \in U$.

Finally let $\eta : U \to V$ be a symmetry with an arbitrary open set $U \subsetneq G$ as domain. There exists a sequence $U_n \subset G$, increasing to U, such that each U_n is of the form (5.25). Since (5.26) holds for $\eta|_{U_n} : U_n \to \eta(U_n)$ and $x \in U_n$ it follows by a continuity argument that (5.26) also holds for $\eta : U \to V$ and $x \in U$. This proves that the diffusion $(P_x\,;\, x \in G)$ is homogeneous.

In order to verify uniqueness of the diffusion we assume that $(Q_x\,;\, x \in G)$ is another homogeneous diffusion with Feller property, for which

$$\alpha = Q_{\mathcal{O}}\{\omega \text{ leaves } N \text{ through } (\frac{1}{2}, \frac{\sqrt{3}}{2})\}.$$

Lemma 5.2 a) and Proposition 3.3 imply

$$Q_x \circ \Psi^{\langle k \rangle -1} = P^{\langle k \rangle}_x \text{ for all } x \in G^{\langle k \rangle}, k \in \mathbb{N}_0. \tag{5.27}$$

If we denote by E^{Q_x} integration with respect to Q_x, we obtain by Lemma 5.2 b) that $E^{Q_{\mathcal{O}}}(T^{\langle k \rangle}_1) < \infty$ for $k \in \mathbb{N}_0$. Therefore

$$d^{\langle k \rangle}_i := E^{Q_{\mathcal{O}}}[T^{\langle k \rangle}_1 | \chi_i(\mathcal{O}, X_{T^{\langle k \rangle}_1}) = 1] \quad i = 1, 2,$$

are finite. In the case $\alpha = 0$ we have $Q_{\mathcal{O}}\{\chi_1(\mathcal{O}, X_{T^{\langle k \rangle}_1}) = 1\} = 0$ and we let $d^{\langle k \rangle}_1 = 1$.

We now conclude from (5.21), (5.22) and (5.23) that for $x \in G^{\langle k-1 \rangle}$, $k \in \mathbb{N}$

$$\begin{aligned}
& E^{Q_x}(T^{\langle k-1 \rangle}_1(Y^{\langle k \rangle}_{d^{\langle k \rangle}}(\Psi^{\langle k \rangle})) \mid \Psi^{\langle k-1 \rangle}(.)_1) \\
= \; & E^{Q_x}(\sum_{i=1}^{2} R^{\langle k-1,k \rangle}_1(i, \Psi^{\langle k \rangle}) d^{\langle k \rangle}_i \mid \Psi^{\langle k-1 \rangle}(.)_1) \\
= \; & E^{Q_x}(\sum_{i=1}^{2} \sum_{j=1}^{T^{\langle k-1,k \rangle}_1(\Psi^{\langle k \rangle})} 1_{\{\chi_i(\omega_{T^{\langle k \rangle}_j}, \omega_{T^{\langle k \rangle}_{j-1}})=1\}} E^{Q_x}(T^{\langle k \rangle}_j - T^{\langle k \rangle}_{j-1} | \Psi^{\langle k \rangle}) \mid \Psi^{\langle k-1 \rangle}(.)_1) \\
= \; & E^{Q_x}(\sum_{j=1}^{T^{\langle k-1,k \rangle}_1(\Psi^{\langle k \rangle})} T^{\langle k \rangle}_j - T^{\langle k \rangle}_{j-1} \mid \Psi^{\langle k-1 \rangle}(.)_1) \\
= \; & E^{Q_x}(T^{\langle k-1 \rangle} \mid \Psi^{\langle k-1 \rangle}(.)_1).
\end{aligned}$$

In view of (5.27) this implies that $(d^{\langle k \rangle})_{k \in \mathbb{N}_0}$ is a consistent sequence of velocities for the consistent sequence of $(\alpha^{\langle k \rangle}, k)$-random walks $(P_x^{\langle k \rangle}; x \in G^{\langle k \rangle})$. By Proposition 4.1, Proposition 5.3, a) and (5.27) there exists $c > 0$ such that

$$\lim_{k \to \infty} Q_x \circ \Psi^{\langle k \rangle^{-1}} \circ Y_{d^{\langle k \rangle}}^{\langle k \rangle^{-1}} = P_x \circ X_{c \cdot}^{-1} \quad \text{for all } x \in G^{\langle \infty \rangle}. \tag{5.28}$$

Moreover,

$$\lim_{k \to \infty} Y_{d^{\langle k \rangle}}^{\langle k \rangle}(\Psi^{\langle k \rangle}(\omega)) = \omega \quad Q_x\text{-a.e.} \quad \text{for all } x \in G^{\langle \infty \rangle}. \tag{5.29}$$

The proof of (5.29) uses Lemma 5.1 and 5.2 and is similar to the proof of Theorem 8.1 in B.-P. Details shall be omitted.

Now, (5.28) and (5.29) imply that $Q_x = P_x \circ X_{c \cdot}^{-1}$ for all $x \in G^{\langle \infty \rangle}$. Hence $Q_x = P_x \circ X_{c \cdot}^{-1}$ for all $x \in G$ by the Feller property. □.

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