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Closedness of some spaces of stochastic integrals

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Abstract

We consider an R^d-valued continous semimartingale $(X_t)_{t\in[0,T]}$, the space of processes $\mathcal{G}^p = \{\theta \cdot X \,|\, \theta \cdot X$ is a semimartingale in $\mathcal{S}^p\}$ and the space of their terminal values $\mathcal{G}^p_T$. We give necessary and sufficient conditions for completeness of $\mathcal{G}^p$ in the norm $||(\theta \cdot X)^*||_p$ and closedness of $\mathcal{G}^p_T$ in L^p. These results are related to some problems in mathematical finance and have been given for $p = 2$ in [DMSSS].

1 Introduction

As the results of our paper are given in [DMSSS] for the case $p = 2$, we have tried to be as short as possible and refer for a motivating section on the financial interpretation to [DMSSS] (see also [DM] especially for the discontinuous case).

By construction, the stochastic integral with respect to a square integrable martingale M is an isometry. The space

$$\left\{\int_0^T \theta dM | \int_0^\bullet \theta dM \text{ is a square integrable martingale}\right\}$$

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is therefore closed in L^2. Quite recently a characterization of the closedness of the space of stochastic integrals with respect to a continuous *semimartingale* has been given in an L^2-setting [DMSSS]. The aim of this paper is to generalize some of these results to the case L^p $(p \neq 2)$.

After some definitions in section 2 we consider in section 3 the space of processes $\mathcal{G}^p = \{(\theta \cdot X)_\bullet | \theta \in \Theta^p\}$ for a fixed continuous semimartingale X and an appropriately chosen space of integrands Θ^p. We give necessary and sufficient conditions for the closedness of this space with respect to the norm $||H||_{\mathcal{R}^p} = ||H^*||_{L^p}$. The idea of the proof of Theorem 3.1 comes from [DMSSS]. We have simplified parts $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iv)$. It is remarkable that if this space is closed for some $p > 1$, then it is closed for all $p > 1$.

In section 4 we consider $\mathcal{G}^p_T = \{(\theta \cdot X)_T | \theta \in \Theta^p\}$, the terminal values of the space of processes of section 3. Again we give necessary and sufficient conditions for the closedness in L^p. In [DMSSS] the variance optimal martingale measure plays a prominent rôle in the characterization of the closedness of $\mathcal{G}^p_T$. Here the q-optimal measure, i.e. the martingale measure for X, which has minimal L^q norm (q conjugate to p) is an important tool for the main result of section 4.

2 Definitions

Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{0 \leq t \leq T})$ be a stochastic basis satisfying the usual conditions. Let $1 < p < \infty$ be fixed in the whole paper.

Definition 2.1 *For a real valued adapted continuous process H we define*

$$||H||_{\mathcal{R}^p} = ||H^*_t||_{L^p(P)},$$

where $H^*_t = \sup_{0 \leq s \leq t} |H_s|$,
and $\mathcal{R}^p = \{H | ||H||_{\mathcal{R}^p} < \infty\}$ is a Banach space.

Definition 2.2 *For a continuous martingale Y we denote*

$$||Y||_{H^p} = (E[Y]_T^{\frac{p}{2}})^{\frac{1}{p}}.$$

Definition 2.3 *We say that a (not necessarily continuous) martingale N is in bmo_p (see [P]), if there is a constant C, such that for any t*

$$E(|N_T - N_t|^p | \mathcal{F}_t) \leq C,$$

or equivalently, if there is a constant C such that for any t

$$E(([N]_T - [N]_t)^{\frac{p}{2}} | \mathcal{F}_t) \leq C.$$

If N is continuous then above conditions are equivalent for all p. The class of such processes is called BMO (see [K]).

In the sequel $(X_t)_{t \in [0,T]}$ denotes always a fixed R^d-valued continuous semimartingale with decomposition $X = M + A$ and $X_0 = 0$, where M is a continuous local martingale and A is a continuous finite variation process.

Definition 2.4 *For an R^d-valued predictable process θ we define*

$$||\theta||_{L^p(M)} = (E(\int_0^T \theta_t' d[M]_t \theta_t)^{\frac{p}{2}})^{\frac{1}{p}}$$

and $L^p(M) = \{\theta : ||\theta||_{L^p(M)} < +\infty\}$.

Definition 2.5 *For an R^d-valued predictable process θ we define*

$$||\theta||_{L^p(A)} = (E(\int_0^T |\theta_t dA_t|)^p)^{1/p}$$

and $L^p(A) = \{\theta : ||\theta||_{L^p(A)} < +\infty\}$.

Definition 2.6 *We define*

$$\Theta^p = L^p(M) \cap L^p(A),$$

equipped with the seminorm

$$||\theta||_{\Theta^p} = ||\theta||_{L^p(M)} + ||\theta||_{L^p(A)} \cdot$$

This is the space of θ, for which $\theta \cdot X$ is in the space $\mathcal{S}^p$ of semimartingales. We denote by $\tilde{\Theta}^p$ the quotient Banach space obtained from Θ in the canonical way (i.e by identification of zero-seminorm processes with zero).

Definition 2.7 *We define a mapping $i : \Theta^p \mapsto \mathcal{R}^p$ by*

$$i(\theta) = \theta \cdot X .$$

Since by the Burkholder-Davis-Gundy inequality

$$||(\theta \cdot X)^*||_p \leq ||(\theta \cdot M)^*||_p + ||(\theta \cdot A)^*||_p \leq C||\theta||_{L^p(M)} + ||\theta||_{L^p(A)} ,$$

i is continuous. Moreover, the above inequality yields, that i induces a continuous mapping $\tilde{i} : \tilde{\Theta}^p \mapsto \mathcal{R}^p$. This mapping is one-to-one. Indeed, if $\theta \cdot X$ is the zero process, then so are its finite variation part $\theta \cdot A$ and its local martingale part $\theta \cdot M$. Thus $||\theta||_{\Theta^p} = 0$.
The image of the mapping i (or $\tilde{i}$) will be denoted by $\mathcal{G}^p$.
We define $\mathcal{G}_T^p$ as the space of terminal values of processes in $\mathcal{G}^p$, i.e.

$$\mathcal{G}_T^p = \{Y_T | Y \in \mathcal{G}^p\} .$$

We treat it as a subspace of $L^p(P)$.

The notation $\mathcal{G}_S^p = \{Y_S | Y \in \mathcal{G}^p\}$, ${}_S\mathcal{G}^p = \{Y_T - Y_S | Y \in \mathcal{G}^p\} \subset L^p(\Omega, \mathcal{F}, P)$, where S is a stopping time is also used.

Definition 2.8 *$M^s(X)$ denotes the set of signed martingale measures for the process X, i.e. measures $\mu << P$, which fulfill $\mu(\Omega) = 1$, $\frac{d\mu}{dP} \in L^q(P)$ and $E((\theta \cdot X)_T \frac{d\mu}{dP}) = 0$ for all $\theta \in \Theta^p$. q is conjugate to the fixed p, which we use in our paper. $M^e(X) \subset M^s(X)$ denotes the set of equivalent martingale measures for X. We always identify a martingale measure with its density $\frac{d\mu}{dP}$.*

Definition 2.9 *For $1 < q < \infty$, we call the solution of the minimum problem*

$$\int SZdP = \langle S, Z\rangle = 0 \qquad \forall S \in \mathcal{G}_T^p \subset L^p(\Omega, P)$$

$$\int ZdP = \langle 1, Z\rangle = 1$$

$$||Z||_{L^q}^q = \int |Z|^q dP \rightarrow min$$

$Z^{(q)} = \frac{dQ^{(q)}}{dP}$ q-optimal martingale measure for the process X. Noting that $M^s(X)$ is closed in $L^q(P)$ and that L^q for $1 < q < \infty$ is uniformly convex, we can conclude that there is always a unique solution of the minimum problem above, if $M^s(X) \neq \emptyset$.

Definition 2.10 *We say that X satisfies (SC) (structure condition) iff there exists a predictable process λ such that $A = \int_0^T d[M]\lambda$.*

Proposition 2.1 *The following conditions are equivalent*
(i) $\exists C \ \forall \theta \in L^p(M) \ ||\theta||_{L^p(A)} \leq C||\theta||_{L^p(M)}$,
(ii) $\exists C \ \forall \theta \in L^p(M) \ ||\theta||_{\Theta^p} \leq C||\theta||_{L^p(M)}$,
(iii) $\exists C \ \forall \theta \in \Theta^p \ ||\theta||_{\Theta^p} \leq C||\theta||_{L^p(M)}$

Proof. Equivalence of (i) and (ii), and implication (ii) $\Rightarrow$ (iii) are obvious. To prove *(iii)* $\Rightarrow$ *(ii)*, it is enough to consider the case $\theta \in L^p(M), ||\theta||_{L^p(A)} = \infty$. Define stopping times $\tau_n = \inf\{t : \int_0^t |\theta dA| = n\} \wedge T$. Processes $\theta^n = \theta I_{\{t \leq \tau_n\}}$ are in $L^p(A)$, so also in Θ^p. By (iii) $||\theta^n||_{\Theta^p} \leq C||\theta^n||_{L^p(M)} \leq C||\theta||_{L^p(M)}$. Taking the limit $n \rightarrow \infty$ we get a contradiction, that proves (ii).

Definition 2.11 *If one of the conditions (i)-(iii) is fulfilled, then we say that the D_p inequality holds.*

Definition 2.12 *If L is a uniformly integrable martingale such that $L_0 = 1$ and $L_T > 0$ P-a.s, then we say that L satisfies the reverse Hölder inequality under P, denoted by $R_p(P)$, where $1 < p < +\infty$, if and only if there is a constant K such that for every t, we have $E((\frac{L_T}{L_t})^p \mid \mathcal{F}_t) \leq K$.*

Definition 2.13 *Let Z be a positive process. Z satisfies condition (S), if there exists a constant $C > 0$ such that $\frac{1}{C}Z_- \leq Z \leq CZ_-$.*

Definition 2.14 *Let F be a Banach space. Then two vectors $x \in F, x^* \in F^*$ are aligned if $< x, x^* >= ||x||||x^*||$ holds. For L^p-spaces this means equality in the Hölder inequality.*

3 On the closedness of $\mathcal{G}^p$ in $\mathcal{R}^p$.

Throughout this section C denotes a constant, which may vary at each occurrence.
The aim of this section is to give necessary and sufficient conditions for closedness of $\mathcal{G}^p$ in $\mathcal{R}^p$. We shall prove

Theorem 3.1 *Let $1 \le p < \infty$ and let X be a continuous semimartingale. Then the following conditions are equivalent*
(i) D_p holds,
(ii) X satisfies (SC) and $\lambda \cdot M$ is in BMO,
(iii) $\tilde{i}$ is an isomorphism, i.e there exist a constant C such that for any $\theta \in \Theta^p$

$$||\theta||_{\Theta^p} \le C||\theta \cdot X||_{\mathcal{R}^p}$$

(iv) $\mathcal{G}^p$ is closed in $\mathcal{R}^p$.

Remark 1.
Since (ii) does not depend on p, (iv) is valid for all $p > 1$ if it is valid for some $p > 1$.

Remark 2.
Let $K_t = \int_0^t \lambda' d[M]\lambda = [\lambda \cdot M]_t$. This process is called the mean-variance tradeoff process. If (ii) is fulfilled then $(\lambda \cdot M)_T$ possesses all moments and hence so does K_T.

For the proof we will need the following generalization of Fefferman's inequality.

Lemma 3.1 *If $p \ge 1$, $Y \in H^p$, $N \in BMO$ are continuous martingales then*

$$|| \int_0^T |d[Y,N]_T| \, ||_p \le C_p ||N||_{BMO} ||Y||_{H^p} .$$

The proof of the lemma is given in [Y], Corollaire 1.1, p.116.

Proof of Theorem 3.1.

(i)$\Rightarrow$ (ii)
Since by D_p condition $\theta \cdot M = 0$ implies $\theta \cdot A = 0$, the process λ in (SC) exists by the predictable Radon-Nikodym theorem ([DS1]). Let $0 \le u < T$ and $B \in \mathcal{F}_u$. We define a predictable set $D_n = \{|\lambda| \le n, \; ||[M]|| \le n, \; t > u, \; \omega \in B\}$. By (SC) and the D_p inequality we get

$$E(\int I_{D_n} \lambda' d[M]\lambda)^p = E(\int I_{D_n} \lambda' d[A])^p \le C^p E(\int I_{D_n} \lambda' d[M]\lambda)^{\frac{p}{2}} =$$

$$= C^p E I_B (\int I_{D_n} d[\lambda . M])^{\frac{p}{2}} \le C^p P(B)^{\frac{1}{2}} (E(\int I_{D_n} d[\lambda \cdot M])^p)^{\frac{1}{2}}$$

by Schwarz inequality. Since the last expression is finite, the above estimation yields

$$EI_B(\int I_{D_n} d[\lambda \cdot M])^p = E(\int I_{D_n} d[\lambda \cdot M])^p \leq C^{2p} P(B)$$

and taking the limit $n \to \infty$

$$E(\int_u^T I_B(\omega)\lambda' d[M]\lambda)^p \leq C^{2p} P(B).$$

Since $B \in \mathcal{F}_u$ was arbitrary, we get

$$E([\lambda \cdot M]_T - [\lambda \cdot M]_u | \mathcal{F}_u) \leq C^2$$

i.e $\lambda \cdot M$ is in BMO.

(ii)⇒ (i)
By Lemma 3.1 for $Y = \theta \cdot M$, $N = \lambda \cdot M$ we have

$$|| \int |\theta dA| \, ||_p = || \int |\theta \lambda d[M]| \, ||_p = || \int |d[Y, N]| \, ||_p \leq C||\theta \cdot M||_{H^p},$$

i.e the R_p inequality.

(i),(ii)⇒ (iii) First, we will prove that for $\theta \in \Theta^p$

$$||\theta||^2_{L^p(M)} = (E[\theta \cdot X]^{\frac{p}{2}})^{\frac{2}{p}} \leq C||\theta \cdot X||^2_{\mathcal{R}^p} \tag{1}$$

Denote $Y = \theta \cdot M$, $L = \theta \cdot X$. We have $[Y] = [L]$. By Ito's formula $L^2 = 2L \cdot L + [L]$, so for $p \geq 2$ we have the estimation

$$([E[L]^{\frac{p}{2}})^{\frac{2}{p}} = ||L^2 - 2L \cdot L||_{\frac{p}{2}} \leq ||L^{*2}||_{\frac{p}{2}} + 2||L \cdot Y||_{\frac{p}{2}} + 2||L \cdot (\theta \cdot A)||_{\frac{p}{2}}.$$

The first term is equal to $||L^*||^2_p$. The second one can be estimated using Burkholder-Davis-Gundy inequality by

$$C(E[L \cdot Y]^{\frac{p}{4}})^{\frac{2}{p}} = C(E(L^2 \cdot [Y])^{\frac{p}{4}})^{\frac{2}{p}} \leq CE(L^*)^{\frac{p}{2}}[Y]^{\frac{p}{4}}))^{\frac{2}{p}} \leq$$

$$\leq C((EL^{*p})^{\frac{1}{2}}(E[Y]^{\frac{p}{2}})^{\frac{1}{2}})^{\frac{2}{p}} = C(EL^{*p})^{\frac{1}{p}}(E[L]^{\frac{p}{2}})^{\frac{1}{p}}.$$

The third term can be estimated using Lemma 3.1 (with $a = \frac{p}{2}$) by

$$||L \cdot (\theta \cdot A)||_{\frac{p}{2}} \leq ||L \cdot (\theta' \cdot [M] \cdot \lambda)||_{\frac{p}{2}} = ||L \cdot [\theta \cdot M, \lambda \cdot M]||_{\frac{p}{2}} =$$

$$= ||[L \cdot Y, \lambda \cdot M]||_{\frac{p}{2}} \leq C(E[L \cdot Y]^{\frac{p}{4}})^{\frac{2}{p}},$$

which can be estimated as the second term. Putting these estimations together we get a second degree inequality

$$(E[L]^{\frac{p}{2}})^{\frac{2}{p}} \leq ||L^*||^2_p + C||L^*||_p(E[Y]^{\frac{p}{2}})^{\frac{1}{p}},$$

from which we get (1) for $p \geq 2$.
The case $p < 2$ can be derived from the case $p = 2$ using

Lemma 3.2 (Lenglart's domination) *If A and B are positive continuous adapted processes, A is increasing and for any bounded stopping time S*

$$EB_S \leq CEA_S\,,$$

then for any $0 < k < 1$

$$EB_\infty^{*k} \leq C_k E A_\infty^k\,.$$

The proof of this lemma can be found in [RY], Proposition 4.7 in chapter IV.

In order to prove (1) for $p < 2$ it is enough to take $A = (\theta \cdot X)^{*2}$, $B = [\theta \cdot X]$. The assumption of Lemma 3.2 is satisfied by (1) for $p = 2$ and the stopped process X^S.
Combining (1) and D_p inequality we get (iii).

(iii)⇒ (i) We know, that there exists a constant C such that for all $\theta \in \Theta^p$

$$||\theta||_{\Theta^p} \leq C||\theta \cdot X||_{\mathcal{R}^p}.$$

Fix $\delta > 0$ and take an arbitrary $\theta \in \Theta^p$. Since A is a continuous finite variation process, there exist a predictable process ϵ, taking values in $\{1, -1\}$, such that

$$\sup_t |(\epsilon \cdot \theta \cdot A)_t| \leq \delta \tag{2}$$

(Lemma 3.8 in [DMSSS]). Thus

$$||\theta||_{\Theta^p} = ||\epsilon\theta||_{\Theta^p} \leq C||(\epsilon\theta) \cdot X)||_{\mathcal{R}^p} \leq C||\epsilon\theta \cdot M||_{\mathcal{R}^p} + C||\epsilon\theta \cdot A||_{\mathcal{R}^p} \leq$$

$$\leq C||\epsilon\theta||_{L^p(M)} + C\delta \leq C||\theta||_{L^p(M)} + C\delta$$

by Burkholder-Davis-Gundy inequality and (2), where C does not depend on δ. Taking the limit $\delta \to 0$ we get inequality D_p.
(iii)⇔ (iv) $\tilde{i}$ is a continuous linear, one-to-one mapping between two Banach spaces. By Banach's closed graph theorem its image is closed if and only if this mapping is an isomorphism, completing the proof of Theorem 3.1.□

4 The closedness of $\mathcal{G}_T^p$ in $L^p(P)$

In this section we investigate the closedness of $\mathcal{G}_T^p$ in $L^p(P)$ for a fixed continuous semimartingale X. Our main theorem is analogous to Theorem 4.1 of [DMSSS] for the case $p \neq 2$.

Theorem 4.1 *Let X denote a continuous semimartingale, let $1 < p < \infty$ and q conjugate to p. Then the following are equivalent*

(1) There is a martingale measure Q in $M^e(X)$ and $\mathcal{G}_T^p$ is closed in $L^p(P)$.
(2) There is a martingale measure Q in $M^e(X)$ that satisfies the $R_q(P)$ condition.

(3) The q-optimal martingale measure $Q^{(q)}$ *is in* $M^e(X)$ *and satisfies* $R_q(P)$.
(4) $\exists C$ *such that for all* $\theta \in \Theta^p$ *we have*

$$||\theta \cdot X||_{\mathcal{R}^p} = ||(\theta \cdot X)^*_T||_{L^p(P)} \leq C||(\theta \cdot X)_T||_{L^p(P)}.$$

(5) $\exists C$ *such that for all* $\theta \in \Theta^p$ *and all* $\lambda \geq 0$ *we have*

$$\lambda P[(\theta \cdot X)^*_T > \lambda]^{1/p} \leq C||(\theta \cdot X)_T||_{L^p(P)}.$$

(6) $\exists C > 0$ *such that for every stopping time* S, *every* $A \in \mathcal{F}_S$ *and every* $\theta \in \Theta^p$ *with* $\theta = \theta \mathbf{1}_{]S,T]}$ *we have* $||\mathbf{1}_A - (\theta \cdot X)_T||_{L^p(P)} \geq CP[A]^{1/p}$.

In order to prove the theorem we shall need some auxiliary results.

Lemma 4.1 *The density of the q-optimal martingale measure* $Z^{(q)}$ *is aligned to* $(1-f)$, *i.e.*

$$Z^{(q)} = \gamma sgn(1-f)|1-f|^{\frac{p}{q}},$$

where $\gamma = (E(sgn(1-f)|1-f|^{\frac{p}{q}}))^{-1} > 0$ *holds.* f *is the solution of the minimum problem*

$$\min_{g \in \overline{\mathcal{G}^p_T}} ||1-g||_p,$$

p *is conjugate to* q, *and the closure is understood here and in the sequel with respect to the norm of* $L^p(P)$.

Proof. The fact that $Z^{(q)}$ is aligned to $1-f$ for some $f \in \overline{\mathcal{G}^p_T}$ is standard in the theory of minimum norm problems, c.f. [Lb] Theorem 5.8.1. What remains to be proved is that f is the solution of $\min_{g \in \overline{\mathcal{G}^p_T}} ||1-g||_p$. The following inequality holds for all $g \in \overline{\mathcal{G}^p_T}$

$$1 = \langle Z^{(q)}, 1-g \rangle \leq ||Z^{(q)}||_q ||1-g||_p,$$

but equality holds only, if $1-g$ is aligned to $Z^{(q)}$. By the equality

$$1 = \langle Z^{(q)}, 1-f \rangle = \gamma \langle |1-f|, |1-f|^{\frac{p}{q}} \rangle$$

we finally get $\gamma > 0$. □

The next lemma gives a special feature of the q-optimal density.

Lemma 4.2 *If the q-optimal measure* $Q^{(q)} \in M^e(X)$ *exists and the cadlag martingale* L *defined as*

$$L_t = E(\frac{dQ^{(q)}}{dP}|\mathcal{F}_t)$$

satisfies $R_q(P)$, *then* L *satisfies (S).*

Proof. Define for each $f_T \in \overline{\mathcal{G}_T^p}$ the $Q^{(q)}$-martingale $f_t := E_{Q^{(q)}}(f_T | \mathcal{F}_t)$. Let (f_T^n) be a sequence in $\mathcal{G}_T^p$ converging to f_T with respect to the $L^p(P)$-norm, then the sequence (f_t^n) converges uniformly in t with respect to the norm of $L^1(Q^{(q)})$ and hence in probability to (f_t). As each (f_t^n) is a continuous martingale, the $Q^{(q)}$-martingale (f_t) is continuous whenever $f_T \in \overline{\mathcal{G}_T^p}$. From Lemma 4.1 we know that $L_T^{q-1} = \alpha(1-f)$ holds for some $\alpha > 0$ and an $f \in \overline{\mathcal{G}_T^p}$. W_t defined by

$$W_t = E_{Q^{(q)}}(L_T^{q-1}|\mathcal{F}_t) = \frac{E_P(L_T^q|\mathcal{F}_t)}{L_t}$$

is therefore continuous. By assumption L satisfies

$$1 \le \frac{E_P(L_T^q|\mathcal{F}_t)}{L_t^q} \le C,$$

and we conclude that $L_t^{q-1} \le W_t \le CL_t^{q-1}$ holds. Since W is continuous, L satisfies the condition (S). □

The proof of the next lemma can be found in [DS2] (Lemma 3.4) with the only difference that we have to use once Hölder's inequality instead of Cauchy-Schwarz.

Lemma 4.3 *If $U = (U_t)_{0\le t\le T}$ is a non-negative $L^q(P)$-martingale ($1 < q < \infty$), if $U_0 > 0$, if the stopping time $\tau = \inf\{t|U_t = 0\}$ is predictable and announced by a sequence of stopping times $(\tau_n)_{n\ge 1}$, then*

$$E(\frac{U_\tau^q}{U_{\tau_n}^q}|\mathcal{F}_{\tau_n}) \to +\infty$$

on the $\mathcal{F}_{\tau-}$-measurable set $\{U_\tau = 0\}$.

Our next lemma shows roughly that we can give an upper bound for the L^q-norm of a martingale measure for X, if we have a lower bound for the L^p-distance between 1 and $\overline{\mathcal{G}_T^p}$.

Lemma 4.4 *If there is a constant $C > 0$ such that for every stopping time S, every $A \in \mathcal{F}_S$ and every $U \in {}_S\mathcal{G}^p$*

$$||1_A - U||_{L^p} \ge CP[A]^{1/p},$$

then for each stopping time S there is an element $g \in L_+^q(P)$ such that $E(g|\mathcal{F}_S) = 1$, $E(g^q|\mathcal{F}_S) \le C^{-q}$ and $E(gU) = 0$ for each $U \in {}_S\mathcal{G}^p$.

Proof. From Lemma 4.1 applied for the space ${}_S\mathcal{G}^p$ instead of $\mathcal{G}_T^p$ we get a $\hat{Z}^{(q)} = \gamma sgn(1-f)|1-f|^{\frac{p}{q}}$ with $f \in {}_S\mathcal{G}^p$ and $\gamma > 0$. Since $\langle \hat{Z}^{(q)}, 1_A f\rangle = 0$,

$$\langle \hat{Z}^{(q)}, 1_A\rangle = \langle \hat{Z}^{(q)}, 1_A(1-f)\rangle = \langle \gamma sgn(1-f)|1-f|^{\frac{p}{q}}, 1_A(1-f)\rangle =$$
$$\gamma \int_A |1-f|^{\frac{p}{q}+1} dP = \gamma \int_A |1-f|^p dP \ge \gamma C^p P(A)$$

holds, and because A was arbitrary in $\mathcal{F}_S$, we get the estimate

$$E(\hat{Z}^{(q)}|\mathcal{F}_S) = \gamma E(|1-f|^p|\mathcal{F}_S) \geq \gamma C^p.$$

Defining now $g = \frac{\hat{Z}^{(q)}}{E(\hat{Z}^{(q)}|\mathcal{F}_S)}$, yields

$$E(|g|^q|\mathcal{F}_S) = \frac{E(|\hat{Z}^{(q)}|^q|\mathcal{F}_S)}{(E(\hat{Z}^{(q)}|\mathcal{F}_S))^q} = \frac{E(\gamma^q|1-f|^p|\mathcal{F}_S)}{\gamma^q(E(|1-f|^p|\mathcal{F}_S))^q} = \frac{1}{(E(|1-f|^p|\mathcal{F}_S))^{q-1}} \leq \frac{1}{C^q}.$$

By construction $E(gU) = 0$ holds for all $U \in {}_S\mathcal{G}^p$.

The positivity of g is shown exactly as in Theorem 3.1 of [DS2], if we bear in mind Lemma 4.1, which tells us that f in the formula for $Z^{(q)}$ is the element of $\overline{\mathcal{G}_T^p}$ with minimal L^p-distance from 1.□

We also need the following characterization of closedness of $\mathcal{G}_T^p$ in $L^p(P)$.

Lemma 4.5 *If there is an equivalent local martingale measure Q for X with density in $L^q(P)$, then $\mathcal{G}_T^p$ is closed in $L^p(P)$ if and only if there is a constant $C > 0$ such that*

$$\forall \theta \in \Theta^p, \|\theta\|_{\Theta^p} \leq C\|(\theta \cdot X)_T\|_{L^p(P)}.$$

Proof. Using Doob's L^q inequality instead of the L^2-version and exploiting the duality of L^q and L^p, we can prove the lemma in the same way as Proposition 3.6 of [DMSSS]. □

Finally we need the subsequent technical lemma.

Lemma 4.6 *If L is a uniformly integrable positive martingale, that satisfies the R_q inequality, then $L = \mathcal{E}(N)$, where N is in bmo_q.*

Proof.
Since L is a positive martingale, $\frac{1}{L_-}$ is locally bounded, and hence its stochastic logarithm $N = \frac{1}{L_-}.L$ is a well-defined local martingale. Fix $s \geq 0$ and as in [DMSSS] define the sequence of stopping times T_n by

$$T_0 = s, \quad T_n = \inf\{t > T_{n-1} | L_t \leq \frac{1}{2} L_{T_{n-1}}\} \wedge T.$$

We have

$$1 = E(\frac{L_{T_n}}{L_{T_{n-1}}}|\mathcal{F}_{T_{n-1}}) = E(\frac{L_{T_n}}{L_{T_{n-1}}} I_{\{T_n<T\}}|\mathcal{F}_{T_{n-1}}) + E(\frac{L_{T_n}}{L_{T_{n-1}}} I_{\{T_n=T\}}|\mathcal{F}_{T_{n-1}}) \leq$$

$$\leq \frac{1}{2} P(T_n < T|\mathcal{F}_{T_{n-1}}) + C(1 - P(T_n < T|\mathcal{F}_{T_{n-1}}))^{\frac{1}{p}},$$

by Hölder's and R_q inequalities. The obtained inequality implies, that there exists a $\gamma < 1$ such that $P(T_n < T|\mathcal{F}_{T_{n-1}}) \le \gamma$. By induction it implies that $P(T_n < T) \le \gamma^n$. From the conditional Burkholder-Davis-Gundy inequality and the definition of T_n we get

$$E(|N_{T_n} - N_{T_{n-1}}|^q|\mathcal{F}_{T_{n-1}}) \le CE(([N]_{T_n} - [N]_{T_{n-1}})^{\frac{q}{2}}|\mathcal{F}_{T_{n-1}}) =$$

$$= CE([\int_{T_{n-1}}^{T_n} \frac{1}{L_{t-}} dL_t]^{\frac{q}{2}}|\mathcal{F}_{T_{n-1}}) = CE((\int_{T_{n-1}}^{T_n} \frac{1}{L_{t-}^2} \cdot [L])^{\frac{q}{2}}|\mathcal{F}_{T_{n-1}}) \le$$

$$\le 2^q CE((\frac{1}{L_{T_{n-1}}^2}[L]_{T_n})^{\frac{q}{2}}|\mathcal{F}_{T_{n-1}}) \le CE(\frac{L_{T_n}^q}{L_{T_{n-1}}^q}|\mathcal{F}_{T_{n-1}}) \le C$$

by the R_q inequality. Finally we have

$$E(|N_T - N_s|^q|\mathcal{F}_s)^{\frac{1}{q}} \le \sum_n (E|N_{T_n} - N_{T_{n-1}}|^q|\mathcal{F}_s)^{\frac{1}{q}} =$$

$$= \sum E(E(|N_{T_n} - N_{T_{n-1}}|^q|\mathcal{F}_{T_{n-1}})|\mathcal{F}_s)^{\frac{1}{q}} \le$$

$$\le \sum E(I_{\{T_{n-1}<T\}} E(|N_{T_n} - N_{T_{n-1}}|^q|\mathcal{F}_{T_{n-1}})|\mathcal{F}_s) \le$$

$$\le C\sum P(T_{n-1} < T) \le C\sum \gamma^k .$$

Since γ does not depend on s, the above inequality completes the proof of the lemma.

Remark.
If X, Y are strongly orthogonal (i.e [X,Y] is a local martingale) then $[X, Y] = 0$ (indeed, [X,Y] is a continuous local martingale of finite variation).

Corollary.
If $X + Y = N \in bmo_q$, where X continuous and X, Y strongly orthogonal, then $X \in bmo_q$ (so from continuity in BMO).

After these preparatory results we prove now the main result of this section
Proof of Theorem 4.1 First we prove the equivalence of (2)-(6). Obviously (3) implies (2). By Theorem 2.16 of [DMSSS] and Lemma 4.2, (3) implies (4) and (2) implies (5). The strong inequality (4) certainly implies the weak inequality (5). The proof of the equivalence of (5) and (6) works with the same reflection argument as for the case $p = 2$ in Theorem 2.18 of [DMSSS].

We prove now that (6) together with (5) implies (3).
By Lemma 4.1 and the proof of Lemma 4.4 (positivity of the q-optimal measure) we have

$$(\frac{dQ^{(q)}}{dP})^{q-1} = \alpha(1 - f),$$

where $f \in \overline{\mathcal{G}_T^p}$, $f \leq 1$ and $\alpha > 0$ holds. Therefore we can find a sequence $Y^n \in \mathcal{G}^p$ obeying $\|Y_T^n - Y_T^{n+1}\|_{L^p} \leq 3^{-n}$ and $Y_T^n \to f$ in $L^p(P)$. The weak inequality (5) yields

$$\sum_{n\geq 1} P(\sup_{0\leq t\leq T} |Y_t^n - Y_t^{n+1}| > 2^{-n}) < +\infty,$$

and we can conclude that Y_t^n converges uniformly in t a.s to a continuous process denoted by f_t ($f_T = f$). We define now $|\widetilde{Z}_t|^{q-1} sgn(\widetilde{Z}_t) = \alpha(1 - f_t)$ and write L_t for the density process of the q-optimal measure. Because

$$L_t Y_t = E_P(L_T Y_T | \mathcal{F}_t)$$

holds for all $Y \in \mathcal{G}^p$, $L_T, L_t \in L^q(P)$ and Y_T^n tends to f with respect to the norm of $L^p(P)$, we infer that

$$L_t |\widetilde{Z}_t|^{q-1} sgn(\widetilde{Z}_t) = E_P(L_T |\widetilde{Z}_T|^{q-1} sgn(\widetilde{Z}_T) | \mathcal{F}_t) = E_P[|L_T|^q | \mathcal{F}_t)$$

holds. Defining the stopping time $\tau = \inf\{t \mid L_t \widetilde{Z}_t = 0\}$, we have

$$0 = \int_{\tau<T} |L_T|^q$$

yielding $L_T = 0$ on $\{\tau < T\}$ and hence $L_\tau = 0$ on $\{\tau < T\}$. Using the continuity of $\widetilde{Z}$, Lemma 4.3 and Lemma 4.4, we can finish the proof in completely the same way as the proof of Theorem 2.18 in [DMSSS], if we replace 2 by q at the appropriate places.
Since we have now proved the equivalence of (2)-(6), it is still to be shown that (1) is equivalent to (2)-(6). Assuming (1) yields by the continuity of the map i (see Definition 2.7) and Lemma 4.5 the validity of (4).
Conversely assuming (2)-(6) there is an equivalent martingale measure Q satisfying $R_q(P)$. The density process of Q denoted by L_t is necessarily of the form $L = \mathcal{E}(-\lambda \cdot M + U)$, where U is a local martingale strongly orthogonal to M (see [AS]).). Lemma 4.6 and its corollary show that $-\lambda \cdot M + U$ as well as $-\lambda \cdot M$ are in bmo_q. By Theorem 3.1, our hypothesis (4) and the Lemma 4.5 we conclude that $\mathcal{G}_T^p$ is closed. □

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