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INTRODUCTION

We follow the notations of [6], although that paper is not required for the present work. $B'(t)$ denotes $B(t \wedge T(-1))$ where $B(t)$ is a Brownian motion on R, $B(0) = 0$, and $T(-1) = \inf\{t : B(t) = -1\}$ (we assume the paths of B are unbounded above and below, so that $T(-1) < \infty$).

We set $\alpha_n = 2^{-n}$, and define a random walk R_n by $R_n(0) = 0$, $R_n(k\alpha_n^2) = B'(T_k)$ where $T_0 = 0$ and inductively $T_{k+1} = \inf\{t > T_k : B'(t) - B'(T_k) = \pm\alpha_n\}; \inf \emptyset = \infty, B'(\infty) = -1$.

Our main objects of concern are the upcrossing chains $N_n(k) = \#\{j < M_n : R_n(j\alpha_n^2) = k\alpha_n,\ R_n((j+1)\alpha_n^2) = (k+1)\alpha_n\}, -2^n \leq k$, where $M_n = \inf\{k : R_n(k\alpha_n^2) = -1\}$. Clearly, $N_n(k) = 0$ for $k \leq -2^n$, or for $k \geq K(n) := \inf\{j \geq 0 : N_n(j) = 0\}$, and it is known [4] that, for each $n \geq 0$, $N_n(k)$ is a Markov chain in k with negative binomial one-step transition function

$$p(i,j) = \begin{cases} \binom{i+j-1}{j} \alpha_{i+j};\, k \geq 0; i,j \geq 0 \\ \binom{i+j}{j} \alpha_{i+j+1};\, -2^n \leq k < 0; i,j \geq 0. \end{cases}$$

Thus, in the parameter range $k \geq 0$ we have a Galton-Watson branching chain with geometric offspring ($p = \frac{1}{2}$) while for $-2^n < k < 0$ there is a superimposed geometric immigration ($p = \frac{1}{2}$). Only the parameter range depends on n (since our definition of N_n does not include the scaling used for R_n). These Markov chains are both elementary and much-studied, and they are not the subject here. What is not as well understood, and will be our principle concern, is the dependence of $N_n(\cdot)$ on $n(\geq 0)$. It turns out that N_n is also a Markov chain with parameter n. Its "upward" ($n \downarrow$) and "downward" ($n \uparrow$) one-step transition functions will be investigated (Section 2), and it turns out that they are "almost" homogeneous in n.

The original motivation for this work was a question of J. Pitman and M. Yor (unpublished) which we understand as follows. Let $L(x)$, $-1 \leq x$, denote the (continuous) local time of $B' : L(x) := \frac{d}{dx} \int_0^{T(-1)} I_{(-\infty,x)}(B'(s))ds$. The problem is to construct the law of $B'(\cdot)$ conditional on $\sigma(L(\cdot))$. Now our approach is to introduce N_n into the problem, so that it has two stages: first, one constructs the law of $\{N_n(\cdot), 0 \le n\}$ given $L(\cdot)$; second, one constructs the law of $B'(\cdot)$ given $\{N_n(\cdot), 0 \le n\}$. Since we have $L(x) = \lim_{n\to\infty} 2\alpha_n N_n [2^n x]$ uniformly x, P-a.s. (see [1], [4]), it is not necessary to include $L(\cdot)$ explicitly in the given data for the second stage. This convergence was recently studied in [6], where other references are given. There, we obtained the law of $L(\cdot)$ given $N_n(\cdot)$ for fixed n. The principle obstacle in stage one is to reverse this to find the law of N_n given $L(\cdot)$. We emphasize that a simple Bayes rule application does not succeed in the function space setting. Nor does it seem possible to find the higher order transition functions of $N_n(\cdot)$ and pass to a limit as $n \to \infty$. A very plausible conjecture, for example, is that $\sigma(L(\cdot)) = \lim_{N\to\infty} \sigma(N_n, n \ge N)$ up to P-nullsets, but we could not prove it.

Nevertheless, we are able by means of a comparison theorem (Theorem 2.5 below) to get solid information about the law of $N_n(\cdot)$ given $L(\cdot)$. This has emboldened us to write down, in the following Sections 2 and 3, what we have found along these lines, in the hope that it may be useful for some more skillful subsequent treatment elsewhere. Indeed, the subject seems to us attractive, both because of its relevance to Brownian motion and because of its combinatorial overtones.

Before entering into the dependence of N_n on n, however, we give in Section 1 a construction of the law of $B'(\cdot)$ given N_n for a single n. This is easy and no doubt known, but it gives the first step in the solution of stage 2 of the Pitman-Yor problem, and we imagine it may take the place of Section 4 for all but the more diligent readers. Thus the outline of the paper is as follows. In Section 1 we construct the law of B' given N_n, n fixed. In Section 2 we discuss the dependence of N_n on n, and study the explicit transition functions. In Section 3 we obtain estimates for the law of N_n given L, and carry out stage 1 of the Pitman-Yor problem as far as we are able. Then, in Section 4, we carry out stage 2 of the Pitman-Yor problem, which does not depend on stage 1.

After completion of this paper, we received a first draft of a paper [9] by J. Warren and M. Yor which gives a "solution" to the problem when B' is replaced by a reflected Brownian motion. This paper has virtually nothing in common with ours, which we regard as a paper on the random walk approximation as much as on the Pitman-Yor problem per se. Nevertheless, the solution of [9] is remarkable, both as to completeness and conciseness. From the standpoint of the present paper, its main implication is that it suffices only to treat the case $L(\cdot) \equiv 1$ — the general case follows from this by changes of scale and time. It remains to be seen whether the conditional law of $N_n(k)$ given $L \equiv 1$ can be given explicitly.

Section 1. Construction of the law of B' given N_n.

For the rest of this section, $n (\ge 0)$ is fixed. For $i > -2^n$, a "random walk path" starting at i is a sequence $(i_0, i_1, \dots, i_{M_n})$ where $0 < M_n < \infty$, $i_0 = i$, $i_{M_n} = -2^n < i_k$ and $i_{k+1} = i_k \pm 1$ for $0 \le k < M_n$. We define $N_n^i(k)$ as the number of upcrossings $k \uparrow (k+1)$, as a function of paths starting at i, $-2^n \le k$, where $N_n^i(-2^n) = 0$ (thus $N_{n\cdot}^0 \equiv N_n$ in an obvious sense). For $-2^n \le k$, let

$X_k^i(j)$, $0 \le j \le N_n^i(k)$, denote the number of upcrossings $(k+1) \uparrow (k+2)$ occurring after the jth upcrossing $k \uparrow k+1$ but before the $(j+1)$-st, where if $j = 0$ the first requirement is absent, and if $j = N_n^i(k)$ the second requirement is absent (for example, if $j = 0 = N_n^i(k)$ then $X_k^i(0) = 0$ unless $k \le i-1$, but if $k \le i-1$ then $X_k^i(0)$ is the total number of upcrossings $k \uparrow k+1$).

Lemma 1.1. *The function $X_k^i(j)$ determines the path starting at i uniquely. Indeed, there is an algorithm for its determination.*

Remark We assume that at least one path for $X_k^i(j)$ exists.

Proof. The algorithm is as follows. If $X_{i-1}^i(0) > 0$, the first step is to $i+1$, and the X-function for the subsequent path (starting at $i+1$) is obtained from $X_k^i(j)$ as follows:

(a) $X_{i-1}^{i+1}(0) = X_{i-1}^i(0) - 1$
(b) $X_i^{i+1}(j) = X_i^i(j+1)$ for $0 \le j \le N_n^{i+1}(i) = N_n^i(i) - 1$
(c) $X_k^{i+1}(j) = X_k^i(j)$ for all other (k, j).

On the other hand, if $X_{i-1}^i(0) = 0$, the first step is to $i-1$, and the X-function of the subsequent path is obtained simply by changing superscript i to $i-1$ (if $i = -2^n + 1$, we leave X^{i-1} undefined).

It is an elementary task to see that the first step is the only possibility consistent with the given $X_k^i(j)$, and that the modified X-function is actually the X-function uniquely determined by the subsequent path. This being true, our proof of uniqueness is by induction on the total number of steps M_n. If $M_n = 2^n + i$, then there is only one possible path, namely all steps are down, and $X_k^i(0) = 0$, $k < i$. Clearly this is the path determined by repetition of the algorithm. So we make the induction assumption that, for a certain $k \ge 1$ and all $i > -2^n$, the uniqueness has been proved for $M_n < 2^n + i + k$. Then if $M_n = 2^n + i + k$ for a certain path, either $X_{i-1}^i(0) > 0$, the first step is up and M_n is reduced by 1, so that the induction assumption applies to the subsequent path (which is thus uniquely determined) or $X_{i-1}^i(0) = 0$, the first step is down and M_n is reduced by 1, so that the equality is maintained for the subsequent path starting at $i-1$. Since this step is uniquely determined, we can repeat the procedure on the subsequent path, leading eventually to a path for which the first step is up (since there were more than $2^n + i$ steps). At that point the induction assumption applies to the subsequent path, and the whole path is thus uniquely determined.

We now determine the law of R_n given N_n. In view of Lemma 1.1, it is enough to determine the joint law of the random vectors $(X_k(j),\ 0 \le j \le N_n^0(k))$, $-2^n \le k$; where we omit the superscript 0 in $X_k^0(j)$. We observe that $\Sigma_j X_k(j) = N_n(k+1)$. It turns out that this is the only restriction imposed on an X-function $X_k(j)$ when N_n is given. Following the terminology of W. Feller [2] we introduce

Definition 1.2. Non-negative, integer-valued random variables $X_1, \ldots, X_n$ are said to be determined by Bose-Einstein sampling of size (n, N), $N \ge 0$, if all distinct $(x_1, \ldots, x_n)$ with $\Sigma_1^n x_j = N$ are equally likely. Then $P\{(X_1, \ldots, X_n) = (x_1, \ldots, x_n)\} = \binom{N+n-1}{N}^{-1}$.

Now we have

Lemma 1.3. *Given N_n, the $(X_k(j),\ 0 \le j \le N_n(k))$ are independent over $k \ge -2^n$. For $-2^n \le k < 0$ they have the law of Bose-Einstein sampling of size $(N_n(k)+1,\ N_n(k+1))$, whereas, for $0 \le k$, $X_k(0) = 0$ and $(X_k(j),\ 1 \le j \le N_n(k))$ is either vacuous (if $N_n(k) = 0$) or Bose-Einstein of size $(N_n(k),\ N_n(k+1))$ (if $N_n(k) > 0$).*

Proof. For fixed $k < 0$, it follows from the transition function of N_n in case $i = 0$, namely $p(0,j) = \alpha_{j+1}$, that $X_k(0)$ is geometric with $p = \frac{1}{2}$ (apply the Markov property of R_n at its passage time to $(k+1)\alpha_n$). Similarly, by the Markov property of R_n at its subsequent returns to $(k+1)\alpha_n$ from $k\alpha_n$, given that they occur, each $X_k(j)$, $j \le N_n(k)$, is geometric ($p = \frac{1}{2}$) and they are independent conditionally on $N_n(k)$. Thus, given $N_n(k)$, $P\{\bigcap_{0 \le j \le N_n(k)} X_k(j) = x_j\} = \alpha_{N_n(k)+1+\Sigma x_j}$, and when $N_n(k+1) = \Sigma_j x_j$ is also given, the $X_k(j)$, $0 \le j \le N_n(k)$, are Bose-Einstein of size $(N_n(k)+1,\ N_n(k+1))$, as asserted. For fixed $k \ge 0$ an analogous reasoning applies to $(X_k(j),\ 1 \le j \le N_n(k))$ based on the passage times of R_n to $(k+1)\alpha_n$ from $k\alpha_n$.

It remains to see that the vectors $(X_k(j), 0 \le j \le N_n(k))$ are mutually independent given $N_n(\cdot)$, and that each is conditionally dependent only on $(N_n(k),\ N_n(k+1))$. This follows by Proposition 1.1, p. 92, of J. B. Walsh [8]. In brief, for each k we introduce the upcrossing field $\mathcal{U}_{k,k+1}$ generated during the successive upcrossings $k\alpha_n \uparrow (k+1)\alpha_n$. We also introduce downcrossing field $\mathcal{V}_{k+1,k}$ generated during the downcrossings of $(k+1)\alpha_n \downarrow k\alpha_n$, say $Z_1(t), \ldots, Z_N(t)$, where $N = N_n(k)+1$ for $k < 0$, $N = N_n(k)$ for $k \ge 0$. Then $\sigma\{N_n(j),\ j \le k\} \subset \mathcal{U}_{k,k+1}$, $\sigma\{N_n(j),\ j > k\} \subset \mathcal{V}_{k+1,k}$, and $\mathcal{U}_{k,k+1}$ is conditionally independent of $\mathcal{V}_{k+1,k}$ given $N_n(k)$. Hence $N_n(k+1)$ is independent of $\sigma\{N_n(j),\ j \le k\}$ given $N_n(k)$, and we have therefore obtained, as above, the conditional law of $(X_k(j),\ 0 \le j \le N_n(k))$ given $\sigma\{N_n(j),\ j \le k+1\}$ (note that for every $i \ge 0$, $(\{N_n(k) = i\} \bigcap \sigma\{X_k(j),\ 0 \le j \le i\}) \subset (\{N_n(k) = i\} \bigcap \mathcal{V}_{k+1,k})$, so that when $N_n(k)$ is given, $\sigma(X_k(j),\ 0 \le j \le N_n(k))$ is independent of $\sigma\{N_n(j),\ j \le k\}$). On the other hand, $\sigma\{N_n(j),\ j > k+1\} \subset \mathcal{V}_{k+2,k+1}$, and given $N_n(k+1)$, $\mathcal{V}_{k+2,k+1}$ is independent of $\mathcal{U}_{k+1,k+2}$, which contains not only $\sigma\{N_n(j),\ j \le k+1\}$ but also $\sigma(X_k(j),\ 0 \le j \le N_n(k))$ (as a picture will show). Thus the law of $(X_k(j),\ 0 \le j \le N_n(k))$ given $\sigma\{N_n(k),\ N_{n+1}(k)\}$ is not only the same as given $\sigma\{N_n(j),\ j \le k+1\}$ but also the same as given $\sigma\{N_n(j),\ -2^n < j\}$, as asserted.

Corollary 1.4. *Given N_n, every family $x_k(j)$ with $0 \le x_k(j)$, $x_k(0) = 0$ for $k \ge 0$, and $\sum_{j \le N_n(k)} x_k(j) = N_n(k+1)$, $-2^n \le k$, corresponds to a unique random walk path starting at 0, and all such families are equally likely.*

Proof. Obvious.

Remark. It is also easy to see that any sequence $0 \leq N_n(k)$, $-2^n < k$, with $N_n(k) = 0$ for all $k \geq K(n) := \inf\{j \geq 0 : N_n(j) = 0\} < \infty$, has positive probability for N_n. Indeed, the probability follows by iteration of the transition function $p(i,j)$. The total probability of all such sequences is one. The number of such paths is included below in Theorem 4.5.

To complete the construction of B' conditional on N_n, it now remains only to fill in B' given R_n (the conditional law of R_n given N_n being identified by Lemmas 1.1 and 1.3, where $\sigma(N_n) \subset \sigma(R_n) \subset \sigma(B')$). For this we need one more lemma, which appears as Lemma 1.1 of [6] and will be used repeatedly in the sequel. For the reader's convenience we repeat it here with a different (and simpler) proof.

Lemma 1.5. *For $0 \leq k < M_n$, set*

$$Y_k(t) = sgn_k(B((T_k + t) \wedge T_{k+1}) - B(T_k)); \; 0 \leq t,$$

where sgn_k is the choice of sign in the definition of T_{k+1}. Then conditional on $\sigma(R_n)$ (which includes $\sigma(M_n)$), $Y_0, \ldots, Y_{M_n-1}$ are independent and identically distributed with the law of a $BES^3_{\alpha_n}(t \wedge T(2\alpha_n)) - \alpha_n$.

Terminology. We call $Y_k(t)$ an "n-insert", $k < M_n$, and the result "(conditional) independence of n- inserts".

Proof. Instead of stopping at M_n, we continue the sequence T_k by using $B(t)$ instead of $B'(t)$, and in this way define $R_n(k\alpha_n^2)$ for all k so that it becomes an unstopped symmetric random walk. Then we obtain a sequence of processes Y_k of which the first M_n are those of the lemma. The strong Markov property of B at T_k, together with the symmetry $B \longleftrightarrow -B$, shows that for each k, given $\{R_n(j\alpha_n^2),\ j \leq k\}$, Y_k is conditionally independent of $Y_0, \ldots, Y_{k-1}$ and has the same law as Y_0. It is plain that the law of Y_0 is that of B starting at 0, stopped at α_n, and conditioned to reach α_n before $-\alpha_n$. Then it is a familiar fact that this is the asserted law of a BES^3 starting at α_n and translated by $-\alpha_n$ (indeed, this is the law of the h-path transform of B killed at $\pm\alpha_n$ with $h(x) = x + \alpha_n$, which has the BES^3 generator by a simple calculation). Now the same law holds if $R_n((k+1)\alpha_n^2)$ is also given (considering the two possibilities separately), and then the strong Markov property at T_{k+1} shows that we may as well be given $R_n(j\alpha_n^2)$ for all j. Finally, since $T(-1)(= T_{M_n})$ is a stopping time, the same reasoning shows that $\{R_n((j + M_n)\alpha_n^2),\ 1 \leq j\}$ is independent of $\sigma(B'(t),\ 0 \leq t)$, so it suffices to be given only $\{R_n((k \wedge M_n)\alpha_n^2),\ 1 \leq k\}$, proving the Lemma.

We summarize these findings as

Theorem 1.6. *To construct the law of B' given N_n, since $M_n = (2^n + 2\Sigma_k N_n(k)) \in \sigma(N_n)$, we may begin with a random sequence $Y_0, \ldots, Y_{M_n-1}$ of independent n-inserts, with absorption times $\zeta_0, \ldots, \zeta_{M_N-1}$, respectively. Then we determine R_n from N_n by Bose-Einstein sampling (Lemma 1.3), and, setting $T_k = \zeta_0 + \cdots + \zeta_{k-1} (0 < k)$, we define $B'((T_k + t) \wedge T_{k+1}) = (sgn_k)Y_k(t) + R_n(k\alpha_n^2)$,*

$0 \le t,\ 0 \le k$, where $sgn_k = sgn(R_n((k+1)\alpha_n^2) - R_n(k\alpha_n^2))$. This defines $B'(t),\ T_k \le t \le T_{k+1}$, for all k, as required.

Section 2. The N_n-chain, with n as parameter.

The random walks R_n are nested, in such a way that $\sigma(R_n) \subset \sigma(R_{n+1})$, but that is not true for $N_n : \sigma(N_n)$ is not comparable to $\sigma(N_{n+1})$ although, of course, $\sigma(N_n) \subset \sigma(R_n)$. We do have a Markov property, as consequence of

Theorem 2.1. *Given N_n, for n fixed, $R_n(\cdot)$ is conditionally independent of $\{N_{n+k},\ 1 \le k\}$. In particular, since $\sigma\{N_0, \ldots, N_n\} \subset \sigma(R_n)$, N_n is a Markov chain in n.*

Proof. Intuitively, the assertion is reasonably obvious from independence of inserts (Lemma 1.5). Indeed, to go from N_n to R_n involves only an ordering of the n-inserts at each level, while to go to N_{n+k} involves only interpolation of $(n+k)$-inserts into the n-inserts without changing level and counting those at each sub-level, a result which is not affected by the reordering. As to a proof, it is enough to show conditional independence, for all $n \ge 0$, of R_n and N_{n+1} given N_n, because then, for every k, $\sigma\{R_n, N_n, \ldots, N_{n+k}\}$ is conditionally independent of $\sigma(N_{n+k+1})$ given $\sigma(N_{n+k})$. Then if the law of $N_{n+1}, \ldots, N_{n+k}$ given $\sigma(R_n)$ depends only on $\sigma(N_n)$ (by induction assumption), the same is true for $N_{n+1}, \ldots, N_{n+k+1}$. To write this out, let $f_j,\ 1 \le j \le k+1$, be bounded Borel functions. Then we have

$$
\begin{aligned}
&E\left(\prod_1^{k+1} f_j(N_{n+j}) \middle| R_n\right) \\
&= E\left[E(f_{k+1}(N_{n+k+1}) | R_n, N_{n+1}, \ldots, N_{n+k}) \prod_1^k f_j(N_{n+j}) \middle| R_n\right] \\
&= E\left[E(f_{k+1}(N_{n+k+1}) \middle| N_{n+k}) \prod_1^k f_j(N_{n+j}) \middle| R_n\right] \\
&= E\left[E(f_{k+1}(N_{n+k+1}) | N_{n+k}) \prod_1^k f_j(N_{n+j}) \middle| N_n\right] \\
&\in \sigma(N_n).
\end{aligned}
$$

The argument is completed by appeal to the monotone class theorem applied to the linear algebra generated by such products, and then by letting $k \to \infty$.

Now for $n \ge 0$, we observe that if N_n is given, so are the numbers of downcrossings at each level $(k+1) \downarrow k$ (namely, $N_n(k)$ for $k \ge 0$, and $N_n(k) + 1$ for $-2^n \le k < 0$). The law of N_{n+1} given $\sigma(R_n)$ may be constructed by independently interpolating R_{n+1} into each of the independent n-inserts, and then adding the upcrossings over those n-inserts which can contribute to a given level for N_{n+1} (for example, the level $2k$ is only contributed to from upcrossings $k \uparrow k+1$ and downcrossings $k \downarrow k-1$ of R_n). All that matters is the number of summands of each type, which in turn depends only on $\sigma(N_n)$ when R_n is given. This suffices for the proof (the enumeration procedure will be explained in full detail below when we obtain the transition mechanism).

We take the point of view that a "step" $N_n \to N_{n+1}$ is "down" (toward $L(\cdot)$), and discuss first the downward transition mechanism. There are two approaches, leading to different (but of course equivalent) descriptions, and (since we cannot iterate either one to obtain explicitly the $N_n \to N_{n+k}$ transition function) we shall present them both quite briefly.

The first approach is based on symmetry. Let $Y(t)$ be an n-insert (Lemma 1.5), and let us interpolate a random walk of step size α_{n+1} into Y, just as we did R_{n+1} into B'. Clearly the total number of steps $0 \to \pm\alpha_{n+1}$ has the same law as if we were interpolating into $B'(t \wedge T(1))$ (since the condition $R_n(\alpha_n^2) = \alpha_n$ is analogous to $R_n(\alpha_n^2) = -\alpha_n$). Thus the law is that of $1 + \text{geo}(\frac{1}{2})$, where for brevity we shall write geo(p) for (a) random variable X with $P\{X = k\} = p(1-p)^k$, $0 \le k$ with an analogous interpretation for bin(n,p) and neg. bin.(n,p). Moreover, given this variable geo$(\frac{1}{2})$, the number of passages $0 \to \alpha_{n+1}$ has the law of $1 + \text{bin}(\text{geo}(\frac{1}{2}), \frac{1}{2})$, namely 1 plus a binomial variable with $p = \frac{1}{2}$, which determines the number of passages $0 \to -\alpha_{n+1}$ as $\text{geo}(\frac{1}{2}) - \text{bin}(\text{geo}(\frac{1}{2}), \frac{1}{2})$. Again, this is obvious enough when we interpolate into $B'(t \wedge T(1))$ instead of into Y, and condition on $B'(T(1)) = \alpha_n$. In more detail, we can use the strong Markov property of $B'(t \wedge T(1))$ to write for $k > 0$

$$\begin{aligned}
&P^0\{B' \text{ reaches } -\alpha_{n+1} \text{ before } \alpha_{n+1} \big| k \text{ returns to } 0, \text{ then to } \alpha_n\} \\
&= 2^{+(k+1)} P^0\{-\alpha_{n+1} \text{ before } \alpha_{n+1} \text{ then } k \text{ returns to } 0, \text{ then to } \alpha_n\} \\
&= 2^{+(k+1)} 2^{-2} P^0\{(k-1) \text{ returns to } 0, \text{ then to } \alpha_n\} \\
&= \frac{1}{2}.
\end{aligned}$$

Thus, conditional on $k > 0$ returns to 0, the first step of R_{n+1} interpolated into the n-insert Y goes to $\pm\alpha_{n+1}$ each with probability $p = \frac{1}{2}$. Then, given this return to 0, by the strong Markov property given $k - 1 > 0$, the second exit is to α_{n+1} with $p = \frac{1}{2}$, independently of the first, and so forth to the kth exit from 0, giving the law of bin$(k, \frac{1}{2})$.

We observe that the same law applies to an interpolation into $-Y$ since $(\text{geo}\,\frac{1}{2}) - \text{bin}(\text{geo}\,\frac{1}{2}, \frac{1}{2}) \stackrel{d}{=} \text{bin}(\text{geo}\,\frac{1}{2}, \frac{1}{2})$, except that the extra 1 adds to $\#(0 \to -\alpha_{n+1})$. Secondly, the set of all n-inserts which are in a position to contribute interpolated steps from $j\alpha_n \to j\alpha_n \pm \alpha_{n+1}$ for given j are precisely those with $R_n(T_k) = j\alpha_n$. By independence of inserts it follows that we have

Theorem 2.2. *For $j > 0$ (resp. $-2^n < j \le 0$) the law of $N_{n+1}(2j-1) + N_{n+1}(2j)$ conditional on N_n is that of $(N_n(j-1) + N_n(j) + \text{neg.bin.}(N_n(j-1) + N_n(j), \frac{1}{2}))$ (resp. replace $N_n(j-1)$ by $N_n(j-1)+1$), and conditional on this* neg. bin.*(= k, say)*

$$(N_{n+1}(2j-1), N_{n+1}(2j)) \stackrel{d}{=} \tag{2.1}$$

$$= (N_n(j-1) + \text{bin}(k, \frac{1}{2}), N_n(j) + k - \text{bin}(k, \frac{1}{2}))$$

(where the two variables $\text{bin}(k, \frac{1}{2})$ *are identical). Given* N_n*, these conditional pairs of random variables are mutually independent in* j*, and thus determine the conditional law of* N_{n+1} *given* N_n*.*

Proof. For $j > 0$, we have only to sum the interpolated steps over all contributing inserts and appeal to their mutual independence. The sum of independent $\text{geo}(\frac{1}{2})$ terms gives the $\text{neg. bin.}(N_n(j-1) + N_n(j), \frac{1}{2})$ terms, $j > 0$, and the sum over the conditionally independent $\text{bin}(\text{geo}(\frac{1}{2}), \frac{1}{2})$ terms then gives the $\text{bin}(k, \frac{1}{2})$. For $j \leq 0$ there are $1 + N_n(j-1)$ steps $j\alpha_n \to (j-1)\alpha_n$, and the result is the joint law of $(1 + N_{n+1}(2j-1),\ N_{n+1}(2j))$. But, for given $k (= \text{neg. bin.}(1 + N_n(j-1) + N_n(j), \frac{1}{2}))$ the two ones drop out, and no change in (2.1) is needed.

Remark. The transition mechanism described by Theorem 2.2 does not depend on n *except* for the fact that the state of N_n is a sequence $(N_n(k); -2^n \leq k)$. This dependence cannot be avoided by defining $N_n(k) = 0$ for $k < -2^n$ since then 0 is preserved only for $k < -2^{n+1}$, which depends on n.

Another way to present the result of Theorem 2.2 is to exhibit $N_{n+1}(k)$ given N_n as a (conditional) Markov chain in k. Clearly there is only a 2-step dependence on N_n, and it is equivalent to use either increasing or decreasing k. In terms of increasing k, the result is

Theorem 2.3. *For* $k > 0$*, given* N_n*, we have* $N_{n+1}(2k-1) \stackrel{d}{=} N_n(k-1) + \text{neg. bin.}(N_n(k-1) + N_n(k), \frac{2}{3})$*, and given both* N_n *and* $N_{n+1}(2k-1)$*, we have* $N_{n+1}(2k) \stackrel{d}{=} N_n(k) + \text{neg. bin.}(N_{n+1}(2k-1) + N_n(k), \frac{3}{4})$*. For* $k \leq 0$*, analogous facts hold after replacing* $N_n(k-1)$ *by* $1 + N_n(k-1)$ *and* $N_{n+1}(2k-1)$ *by* $1 + N_n(2k-1)$*.*

Proof. It is possible to prove this by deriving the corresponding result for an n-insert, and then adding over contributing inserts as we did for Theorem 2.2, but it is somewhat long. For brevity, we derive the result directly from Theorem 2.2. For $j > 0$, we had $N_{n+1}(2j-1) \stackrel{d}{=} N_n(j-1) + \text{bin}(k, \frac{1}{2})$ where $k \stackrel{d}{=} \text{neg. bin.}(N_n(j-1)+N_n(j), \frac{1}{2})$, while for $j \leq 0$ this last becomes $k \stackrel{d}{=} \text{neg. bin.}(1 + N_n(j-1) + N_n(j), \frac{1}{2})$. Writing "bin" for $\text{bin}(k, \frac{1}{2})$ with the random k, we have for $j > 0$ $P\{\text{bin} = i\} =$

$$\sum_{k=i}^{\infty} \binom{k}{i} \binom{N_n(j-1) + N_n(j) + k - 1}{k} 2^{-(2k + N_n(j-1) + N_n(j))}$$

$$= \frac{2^{-(2i)}}{i!(N_n(j-1) + N_n(j) - 1)!} \left[\sum_{\ell=0}^{\infty} \binom{N_n(j-1) + N_n(j) + i + \ell - 1}{\ell} 4^{-\ell} (\frac{3}{4})^{N_n(j-1)+N_n(j)+i} \right] \cdot (\frac{2}{3})^{(N_n(j-1)+N_n(j))} (\frac{4}{3})^i (etc.)$$

$$= P\{\text{neg. bin.}(N_n(j-1) + N_n(j); \frac{2}{3}) = i\},$$

as asserted for the marginal distribution, where for $j \le 0$ we replace $N_n(j-1)$ by $(1+N_n(j-1))$ throughout. [Of course, the marginal law of $N_{n+1}(2j)$ follows by replacing $N_n(j-1)$ by $N_n(j)$, using (2.1)].

Clearly $N_{n+1}(j)$ is a conditional Markov chain given N_n. It remains to derive the asserted transition function for $N_{n+1}(2k)$ given $N_{n+1}(2k-1)$ and N_n. But with i and k as before, for $j > 0$ we have

$$P(N_{n+1}(2j) = N_n(j) + k - i \big| N_{n+1}(2j-1))$$

$$= P\left(\text{neg. bin.}\,(N_n(j-1) + N_n(j))\,, \frac{1}{2}\right) = k \big| \,\text{bin}\,(\text{neg. bin.}(etc.) = i)$$

$$= P(\text{bin} = i \big| \,\text{neg. bin.} = k) \cdot P(\text{neg. bin.} = k) P^{-1}(\text{bin} = i)$$

$$= \binom{k}{i} 2^{-k} \frac{\binom{N_n(j-1)+N_n(j)+k-1}{k} 2^{-(N_n(j-1)+N_n(j)+k)}}{\binom{N_n(j-1)+N_n(j)+i-1}{i} (\frac{2}{3})^{N_n(j-1)+N_n(j)} (\frac{1}{3})^i}$$

$$= (\frac{3}{4})^{N_n(j-1)+N_n(j)+i} (\frac{1}{4})^{k-i} \frac{(N_n(j-1) + N_n(j) + k - 1)!}{(N_n(j-1) + N_n(j) + i - 1)!(k-i)!}$$

$$= (\frac{3}{4})^{N_n(2j-1)+N_n(j)} (\frac{1}{4})^{k-i} \binom{N_n(j-1)+N_n(j)+k-1}{k-i}$$

$$= P\{\text{neg. bin.}\left(N_n(2j-1) + N_n(j), \frac{3}{4}\right) = k - i\},$$

as required for the second assertion. For $k \le 0$, as before, we need to add 1 to $N_n(k-1)$ and $N_n(2k-1)$ in the argument of the neg. bin.

In view of Theorems 2.2 and 2.3, let us note that while the downward mechanism is perhaps more tractable than we had a right to expect, it does not seem tractable enough to iterate explicitly to 2 or more steps. As indicated in the Introduction, what we really need is information about iteration to k steps as k becomes large, and this seems to be out of reach for the downward transitions. It turns out, thanks to some fortuitous comparisons, that it is not entirely out of reach for the "upward" transitions $N_n \to N_{n-1}$. Accordingly, we turn our attention now to these.

Since $\sigma(N_0, \ldots, N_n) \subset \sigma(R_n)$, the conditional law of N_{n-1} given N_n is implicit in the construction of R_n from N_n in Section 1. Indeed, referring to Lemma 1.1, given N_n we see that $N_{n-1}(k)$ equals the number of $X_{2k}(j)$, $1 \le j \le N_n(2k)$, which are non-zero. We note that, even for $k < 0$, we do not count $X_{2k}(0)$ since it only gives the steps of N_n from $(2k+1)$ to $2(k+1)$ before reaching $2k$, which do not yield steps of N_{n-1} from k to $k+1$. On the other hand, for $1 \le j$, even if $X_{2k}(j) > 1$ there is at most one step of N_{n-1} from k to $k+1$ starting with the j^{th} of N_{n+1} from $2k$ to $2k+1$ and before the $(j+1)^{st}$ (void if $j = N_n(2k)$). Now according to Lemma 1.3, for $k \ge 0$ when N_n is given, $\{X_{2k}(j),\ 1 \le j \le N_n(2k)\}$ are determined by Bose-Einstein statistics of size $(N_n(2k),\ N_n(2k+1))$, and for $k < 0$, $\{X_{2k}(j),\ 0 \le j \le N_n(2k)\}$ are determined by Bose-Einstein statistics of size $(1+N_n(2k),\ N_n(2k+1))$. Thus, for $k \geq 0$, the number of $X_{2k}(j) > 0$ is equal to the *number of non-empty boxes* in Bose-Einstein statistics with $N_n(2k)$ boxes and $N_n(2k+1)$ balls. This is a familiar combinatorial problem, and the answer is easily derivable from Exercise 17 of W. Feller [2, II.11] (see below). On the other hand, for $k < 0$ we need the law of the number of non-empty boxes *excluding the first box*, in Bose-Einstein statistics with $1 + N_n(2k)$ boxes and $N_n(2k+1)$ balls. This is a mixture of the former. The probability of i balls in box 1 is

$$\frac{\binom{N_n(2k)+(N_n(2k+1)-i)-1}{N_n(2k+1)-i}}{\binom{N_n(2k)+N_n(2k+1)}{N_n(2k+1)}}, 0 \leq i \leq N_n(2k+1),$$

and given i the problem reduces to the former with $N_n(2k+1)$ replaced by $N_n(2k+1) - i$. Thus we can prove

Theorem 2.4. *For $n > 0$, given $\sigma(N_n)$ the variables $N_{n-1}(k), -2^{n-1} < k$, are conditionally independent. Moreover*

(a) *For* $k \geq 0$, $P\left(N_{n-1}(k) = j \middle| N_n\right) = \frac{\binom{N_n(2k)}{j}\binom{N_n(2k+1)-1}{j-1}}{\binom{N_n(2k)+N_n(2k+1)-1}{N_n(2k+1)}}$; $1 \leq j \leq N_n(2k) \wedge N_n(2k+1)$, $(= 1$ *if* $j = N_n(2k) \wedge N_n(2k+1) = 0$ *or* $1)$.

(b) *For* $-2^{n-1} < k < 0$, $P\left(N_{n-1}(k) = j \middle| N_n\right) = \frac{\binom{N_n(2k)}{j}\binom{N_n(2k+1)}{j}}{\binom{N_n(2k)+N_n(2k+1)}{N_n(2k+1)}}$; $0 \leq j \leq N_n(2k) \wedge N_n(2k+1)$ $(= 1$ *if* $j = N_n(2k) \wedge N_n(2k+1) = 0)$.

Proof. We first show (a) $\Rightarrow$ (b). Indeed, by the remarks before the theorem (b) is a mixture of (a) in which $\binom{N_n(2k)+N_n(2k+1)-i-1}{N_n(2k+1)-i}$ cancels out leaving (for $k < 0$)

$$P\left(N_{n-1}(k) = j \middle| N_n\right) = \\ = \binom{N_n(2k)}{j}\left(\sum_{i=0}^{N_n(2k+1)-j}\binom{N_n(2k+1)-i-1}{j-1}\right) \\ \cdot \binom{N_n(2k)+N_n(2k+1)}{N_n(2k+1)}^{-1}$$

and the sum reduces to $\binom{N_n(2k+1)}{j}$ by identity (12.8a) of [2, II. 12, p. 64]. Now to prove (a), we write for Bose-Einstein statistics with $N_n(2k)$ boxes and $N_n(2k+1)$ balls

$$\begin{aligned} &P\ \{\text{exactly } j \text{ boxes are not empty}\} \\ &= P\ \{\text{exactly } N_n(2k) - j \text{ boxes remain empty}\} \\ &= \binom{N_n(2k)}{j} P\{N_n(2k) - j \text{ given boxes are empty, the other } j \text{ nonempty}\} \\ &= \binom{N_n(2k)}{j}\binom{N_n(2k+1)-1}{j-1}\binom{N_n(2k)+N_n(2k+1)-1}{N_n(2k+1)}^{-1}, \end{aligned}$$

as required, where we used [2, II, (5.2), p. 38] at the last step.

Remark. As with the downward transitions, the upward transition mechanism is practically free of n, but this time there is no exception. We need only extend it to $-\infty < k < \infty$ by the convention $P\left(N_{n-1}(k) = 0 \middle| N_n\right) = 1$ if $N_n(2k) \wedge N_n(2k+1) = 0$, and also define $P\{N_n(k) = 0,\ k \leq -2^n\} = 1$. This preserves the necessary zeros of $N_{n-1}(k)$ for
$k \leq -2^{n-1}$ and for $k > -2^{n-1}$ we note that neither (a) nor (b) depends on n explicitly.

In view of Theorem 2.4 (a), we are led to study the (hypergeometric) transition kernels

$$P\{Y = k | X_1 = j_1,\ X_2 = j_2\} = \binom{j_1}{k}\binom{j_2 - 1}{k - 1} \Big/ \binom{j_1 + j_2 - 1}{j_2}, \qquad 1 \leq k \leq j_1 \wedge j_2, \tag{2.2}$$

for $1 \leq j_1 \wedge j_2$, while $P\{Y = 0 | X_1 = j_1,\ X_2 = j_2\} = 1$ if $0 = j_1 \wedge j_2$. It turns out that when L is given the joint law of X_1 and X_2, when $X_1 = N_n(2k)$ and $X_2 = N_n(2k+1)$, is that of (conditionally) independent random variables. Consequently, we only examine the case when X_1 and X_2 are independent. In fact, we shall mainly be interested in the additional assumption that X_1 and X_2 also are identically distributed. Then we can prescribe a distribution $F = F_{X_1} = F_{X_2}$ for X_1 and X_2, and study the iteration of the transition mechanism with $F_Y = F_{Y_1} = F_{Y_2}$ in place of F (Y_1 and Y_2 being taken independent) and so on to the higher iterates (this applies also for $k < 0$, where there is an analogous iteration which one derives from Theorem 2.4 (b)). To be sure, $N_n(2k)$ and $N_n(2k+1)$ are not identically distributed given L except in special cases, but we aim for a comparison theorem with the identically distributed case. Anyway, this extra assumption is not needed for the basic comparison (Theorem 2.5). We introduce the familiar ordering of distribution functions:

Notation 2.5. For integer-valued random variables $X_1 \geq 0$ and $X_2 \geq 0$, we write $X_1 << X_2$ if, for all $k \geq 0$, $F_{X_1}(k) \geq F_{X_2}(k)$.

Now we will derive

(Comparison) Theorem 2.5. If F_{Y_1} is determined from (independent) $X_{1,1}$ and $X_{1,2}$ by (2.2), and F_{Y_2} similarly from (independent) $X_{2,1}$ and $X_{2,2}$, then if $X_{1,1} << X_{2,1}$ and $X_{1,2} << X_{2,2}$, it follows that $Y_1 << Y_2$.

Note. The probability space and joint distribution of (Y_1, Y_2) is irrelevant and unspecified.

Proof. The proof is rather long, but simple in outline. Viewing (2.2) as a transformation of pairs of point-probability distributions $(\delta_{j_1}, \delta_{j_2}) \to F_Y$, we first observe that it suffices that this be monotone increasing in both variables (j_1, j_2) in the sense of the order of Notation 2.5. Indeed, the relation $X_1 << X_2$ means that F_{X_1} may be obtained from F_{X_2} by transfer of probability mass from larger to smaller values. In brief, we can first obtain $F_{X_1}(0)$ by successive transfer from $\{k > 0 : F_{X_2}(k+1) - F_{X_2}(k) > F_{X_1}(k+1) - F_{X_1}(k)\}$ to 0, adding the surplus mass to $F_{X_2}(0)$ starting with the smallest possible k. Then with $F_{X_1}(0) = F_{X_2}(0)$ having been obtained for the resultant distribution F_{X_2}, we proceed analogously to obtain $F_{X_1}(1) - F_{X_1}(0)$ by transfers from $\{k > 1\}$; and so forth to $F_{X_1}(k+1) - F_{X_1}(k)$ for all $k \geq 0$. Obviously such transfers of mass reduce the distribution F_{X_2} in the sense of Notation 2.5 without destroying the relation $X_1 << X_2$ for the new distributions F_{X_2}. To see that such an operation also reduces F_Y (whether performed for $F_{X_{1,1}} << F_{X_{1,2}}$ or for $F_{X_{2,1}} << F_{X_{2,2}}$) it obviously suffices to show that (2.2) is monotone increasing in each variable when $Y = k$ is replaced by $Y \geq k$. Evidently (see Theorem 2.4 (a)) there is no difficulty if $j_1 \wedge j_2 = 0$ or if $j_1 \wedge j_2 = 1$. To examine the other cases, it is useful intuitively to regard (2.2) as a special case of the hypergeometric distribution, in which Y is the number of objects of "type one" chosen at random from $j_1 + j_2 - 1$ objects in j_2 choices when there are initially j_1 of type 1 and $j_2 - 1$ of type 2 (to see this, note that $\binom{j_2-1}{k-1} = \binom{j_2-1}{j_2-k}$; this interpretation also makes clear why $Y \equiv 1$ if $j_1 \wedge j_2 = 1$).

We now examine the dependence on the variable j_2 (with j_1 fixed) by looking at the difference of (2.2) at j_2 and at $j_2 + 1$, for $1 \leq k \leq j_1 \wedge j_2$. We obtain by routine cancellations

$$\binom{j_1}{k}\binom{j_2-1}{k-1}\Big/\binom{j_1+j_2-1}{j_2} - \binom{j_1}{k}\binom{j_2}{k-1}\Big/\binom{j_1+j_2}{j_2+1}$$
$$:= D_1(j_1, j_2, k) = C_1(j_1, j_2, k)(j_1 j_2 - (j_1+j_2)k + j_1),$$

where $C_1(j_1, j_2, k)$ is a *non-negative* common factor whose complicated exact expression need not concern us further. The lesson derived from this is that it suffices, in order to prove monotonicity in j_2, to observe that $\sum_{k=1}^{j_1 \wedge j_2} D_1(j_1, j_2, k) \geq 0$. Indeed, since the sums $\sum_{k=1}^{j} D_1(j_1, j_2, k)$ are manifestly unimodal in j (i.e., increasing to a positive maximum and decreasing thereafter), we need only show that the last is non-negative, which follows since for $j = j_1 \wedge j_2$ the sum of the first terms is 1 and that of the second is ≤ 1.

A similar argument applies to the dependence on j_1. We obtain, for $1 \leq k \leq j_1 \wedge j_2$,

$$\binom{j_1}{k}\binom{j_2-1}{k-1}\Big/\binom{j_1+j_2-1}{j_2} - \binom{j_1+1}{k}\binom{j_2-1}{k-1}\Big/\binom{j_1+j_2}{j_2}$$
$$:= D_2(j_1, j_2, k) = C_2(j_1, j_2, k)(j_1 j_2 - (j_1+j_2)k)$$

where C_2 is non-negative. Again the partial sums in k are unimodal, and to show that they are all non-negative it suffices to observe that the sum from 1 to $j_1 \wedge j_2$ is obviously non-negative.

Corollary 2.5. *The comparison of Theorem 2.5 remains valid if, in place of (2.2), we replace $N_n(2k)$ by X_1 and $N_n(2k+1)$ by X_2 in the conditional distribution of Theorem 2.4 (b).*

Proof. As observed before Theorem 2.4, part (b) is a mixture, over i, of part (a) with $N_n(2k+1)$ replaced by $N_n(2k+1) - i$, where i is the number of balls in box 1 for Bose-Einstein statistics of size $(1 + N_n(2k),\ N_n(2k+1))$. We need only realize that if the distribution of $N_n(2k)$ is increased in the sense of Notation 2.5, that of i is decreased, hence that of $N_n(2k+1)-i$ is again increased. Obviously it also increases along with the distribution of $N_n(2k+1)$, so our assertion follows from that of Theorem 2.5.

Theorem 2.5 of course carries over to iterations of the transition mechanism (2.2): in particular if $X_{1,1}$ and $X_{1,2}$ (resp. $X_{2,1}$ and $X_{2,2}$) are i.i.d. with $X_{1,1} << X_{2,1}$, then $Y_1 << Y_2$ generate i.i.d. pairs to which the same operation applies, preserving the order.

A key to applying this is to identify distributions for which the operation and its iterates can be more or less explicitly calculated to serve as a basis for comparisons. If we take the obvious $F_X = \delta_k$ it turns out that F_Y is too complicated to iterate again (even once?). However, as sometimes happens in such situations, the Poisson distributions provide a better candidate for iteration.

Lemma 2.6. *If X_1 and X_2 in (2.2) are i.i.d. with the Poisson distribution, parameter $\lambda > 0$, then*

$$P\{Y=k\} = \begin{cases} e^{-\lambda}(2-e^{-\lambda});\ k=0 \\ 2e^{-\lambda}\lambda^{2k}/(2k)!;\ k \geq 1. \end{cases}$$

In particular, if X_1 and X_2 are i.i.d. with law

$$P\{X_1=k\} = \begin{cases} 0; k=0 \\ \frac{e^{-\lambda}}{1-e^{-\lambda}}\lambda^k/k!;\ k \geq 1, \ \textit{then} \end{cases}$$

$$P\{Y=k\} = \begin{cases} 0;\ k=0 \\ (cosh\lambda - 1)^{-1}\lambda^{2k}/(2k)!;\ 1 \leq k. \end{cases}$$

In other words, given $X_1 > 0$ and $X_2 > 0$, $2Y$ has a cosh(λ)-distribution conditioned to be non-zero.

Proof. The second assertion follows from the first by observing that the condition $X_1 > 0$ and $X_2 > 0$ is equivalent to $Y > 0$. As to the first, we need only calculate for $k > 0$

$$\begin{aligned} P\{Y=k\} &= e^{-2\lambda}\sum\sum_{i,j\geq k} \frac{\lambda^{i+j}}{i!j!}\binom{i}{k}\binom{j-1}{k-1}\Big/\binom{i+j-1}{j} \\ &= e^{-2\lambda}\sum_{n=2k}^{\infty}\frac{\lambda^n}{(n-1)!}\sum_{i=k}^{n-k}\binom{i-1}{k-1}\binom{n-i-1}{k-1} \\ &= e^{-2\lambda}k^{-1}\lambda^{2k}\sum_{n=2k}^{\infty}\frac{\lambda^{n-2k}}{(n-1)!}\binom{2k+(n-2k)-1}{n-2k} \\ &= e^{-2\lambda}k^{-1}\frac{\lambda^{2k}}{(2k-1)!}e^{\lambda} = 2e^{-\lambda}\lambda^{2k}/(2k)!. \end{aligned}$$

We now turn to a comparison of the law of $2Y$ from Lemma 2.6 with a Poisson law. Let us set

Notation 2.7. For any integer-valued random variable $X \geq 0$, let $\mathcal{O}X$ denote (any) random variable with the law derived from (2.2) when X_1 and X_2 are independent with the law of X. Further, let $\mathcal{P}_\lambda$ denote any random variable with the Poisson law, parameter $\lambda > 0$.

Noting that from Lemma 2.6 we have

$$P\{2\mathcal{O}\mathcal{P}_\lambda = k\} = \begin{cases} 0;\ 1 \leq k \text{ odd} \\ e^{-\lambda}(2 - e^{-\lambda});\ k = 0\,, \\ 2e^{-\lambda}\frac{\lambda^k}{k!};\ 0 < k \text{ even} \end{cases} \qquad \text{we prove}$$

Theorem 2.8. *For $\lambda > 6$, set $c = (\lambda - 1)^{-1}$. Then for $n \geq 1$, $(2\mathcal{O})^n\mathcal{P}_\lambda << \mathcal{P}_{\mu_n}$, where*
$\mu_n = \lambda \prod_{k=o}^{n-1}(1 + c(1+c)^k \ell n\lambda)$.

Remark. This result can probably be improved, however it represents a compromise. Due to the nonlinearity of (2.2) it does not seem possible to extract the factors 2 from Theorem 2.8 to get an estimate of $\mathcal{O}^n\mathcal{P}_\lambda$. Consequently, Theorem 2.8 is not used in the sequel, and an uninterested reader can skip to Theorem 2.9.

Proof. We first note that $2\mathcal{O}\mathcal{P}_\lambda << X$, where $P\{X = 0\} = e^{-\lambda} - e^{-2\lambda}$, $P\{X = 1\} = e^{-\lambda}$, and for $1 \leq k$, $P\{X = 2k\} = P\{X = 2k+1\} = e^{-\lambda}\frac{\lambda^{2k}}{(2k)!}$. We wish to minimize μ such that $X << \mathcal{P}_\mu$. We set $T_k = P\{X = k\} - e^{-\mu}\frac{\mu^k}{k!}$, so we want $\sum_{j=0}^k T_j \geq 0$ for $0 \leq k$. $T_0 \geq 0$ means $e^{-\lambda} - e^{-2\lambda} - e^{-\mu} \geq 0$, and $T_1 \geq 0$ means $e^{-\lambda} - \mu e^{-\mu} \geq 0$. Now if $T_1 \geq 0$ holds, then $T_0 \geq \mu e^{-\mu} - \mu^2 e^{-2\mu} - e^{-\mu}$, so it suffices that $\mu - \mu^2 e^{-\mu} \geq 1$. Here the left side is increasing for $\mu > 1$, and exceeds 1 for $\mu = \lambda(1 + \frac{\ell n\lambda}{\lambda - 1})$. Indeed, logarithmic differentiation shows this last is increasing for $\lambda > 1$ with limit 2 at $\lambda = 1$, and indeed $2 - 4e^{-2} \geq 1$. Hence to ensure that $T_0 \geq 0$ it suffices to require $T_1 \geq 0$. We need to show that this will also imply that $\sum_{j=o}^n T_j \geq 0$ for every n. Treating the even and odd terms separately, we first show that both $\sum_{j=1}^n T_{2j}$ and $\sum_{j=0}^n T_{2j+1}$ are unimodal, in the sense that the signs $sgnT_{2j}$ are decreasing in j (once -1, then remaining -1), and likewise $sgnT_{2j+1}$. Indeed, for $T_{2k} > 0$ it is equivalent that $\ell n\left(e^{-\lambda}\frac{\lambda^{2k}}{(2k)!}\right) > \ell n\left(e^{-\mu}\frac{\mu^{2k}}{(2k)!}\right)$, or $-\lambda + (2k)\ell n\lambda > -\mu + 2k\ell n\mu$, and for $1 < \lambda < \mu$ this holds if and only if $2k < (\mu - \lambda)(\ell n\frac{\mu}{\lambda})^{-1}$, $1 \leq k$. Similarly, for $0 \leq k$, $T_{2k+1} > 0$ is equivalent to $\ell n\left(e^{-\lambda}\frac{\lambda^{2k}}{(2k)!}\right) > \ell n\left(e^{-\mu}\frac{\mu^{2k+1}}{(2k+1)!}\right)$, which reduces to $-\lambda + (2k)\ell n\lambda > -\mu + (2k+1)\ell n\mu - \ell n(2k+1)$. Replacing $2k+1$ by x, we find

$$\frac{d}{dx}[(\mu - \lambda) - x(\ell n\mu - \ell n\lambda) - \ell n\lambda + \ell nx] = -\ell n(\mu/\lambda) + \frac{1}{x},$$

decreasing in x, so the inequality holds only in an initial interval. Setting $0 = (\mu - \lambda) - x(\ell n\mu - \ell n\lambda) - \ell n\lambda + \ell nx$, we get a unique root of $x = ((\mu - \lambda) +$

$\ell n\frac{x}{\lambda})(\ell n\frac{\mu}{\lambda})^{-1}$. Now if $\mu-\lambda=\frac{\lambda}{\lambda-1}\ell n\lambda$ from the Theorem at $n=1$, then at $x=\lambda$ the right side becomes $\frac{\lambda}{\lambda-1}\ell nx/\ell n(1+\frac{1}{\lambda-1}\ell n\lambda)$ which exceeds λ. Thus the root x exceeds λ, and since $\ell n\frac{x}{\lambda}>0$ we see that it exceeds $(\mu-\lambda)(\ell n\frac{\mu}{\lambda})^{-1}$ as well. On the other hand, at $x=\mu=\lambda+\frac{\lambda}{\lambda-1}\ell n\lambda$ the right side is $(\mu-\lambda)(\ell n\frac{\mu}{x})^{-1}+1$, which equals $(\frac{\lambda}{\lambda-1}\ell n\lambda)/\ell n(1+\frac{\ell n\lambda}{\lambda-1})+1$. Setting $\frac{\ell n\lambda}{\lambda-1}=\epsilon$, this is $\lambda(\frac{\epsilon}{\ell n(1+\epsilon)})+1<(\frac{\lambda}{1-\frac{\epsilon}{2}})+1=\lambda\left(1+\frac{\epsilon/2}{1-\epsilon/2}\right)+1$. Now for $\epsilon\leq\frac{1}{2}$ this is at most $\lambda(1+\frac{\epsilon}{3})+1$, and if $\frac{\lambda}{6}\geq 1$ this is less than μ. A check shows that for $\lambda>6$ the condition on ϵ is met, so that the root x is less than μ. Since the range of the right side in $\lambda\leq x\leq\mu$ is 1, we see that the root exceeds $(\mu-\lambda)(\ell n\frac{\mu}{\lambda})^{-1}$ by less than 1.

Now we have shown that the first $2k>0$ for which $T_{2k}<0$ is the first even integer exceeding $(\mu-\lambda)(\ell n\frac{\mu}{\lambda})^{-1}$, while the first $2k+1$ for which $T_{2k+1}<0$ exceeds this by less than 1. Thus, whether the first integer exceeding $(\mu-\lambda)(\ell n\frac{\mu}{\lambda})^{-1}$ is even or odd, if $2k_0$ is the first even integer exceeding it, then $T_{2k_0+1}<0$. In all cases, the index of the first even negative term differs from the index of the first odd negative term by 1. This implies, since $T_0\geq 0$ and $T_1\geq 0$, and $\sum_{j=0}^{\infty}T_j=0$, that $\sum_{j=0}^{n}T_j\geq 0$ for all n, which will complete the proof for $n=1$.

It remains to examine the requirement $T_1\geq 0$, i.e. $e^{-\lambda}-\mu e^{-\mu}\geq 0$, and to iterate it over n. With λ fixed we estimate the root of $-\lambda=\ell n\mu-\mu$ by using Newton's method starting with $\mu_0=\lambda>1$. Then $\mu_1=\lambda+(\frac{\lambda}{1-\lambda})\ell n\lambda$ over-estimates the root, since $\frac{d}{d\mu}(\ell n\mu-\mu)=\frac{1-\mu}{\mu}$ is decreasing, so Theorem 2.8 is proved for $n=1$. For $n=2$ we replace λ by μ_1 and repeat the procedure to get, setting $c=(\lambda-1)^{-1}$,

$$\begin{aligned}\mu_2&=\lambda(1+c\ \ell n\lambda)(1+c\ell n[\lambda(1+c\ell n\lambda])\\&<\lambda(1+c\ell n\lambda)(1+c(\ell n\lambda+c\ell n\lambda))=\lambda(1+c\ell n\lambda)(1+c+c^2)\ell n\lambda,\end{aligned}$$

using the estimate $\ell n(1+x)<x$ for $x>0$. In the same way, we get for induction step

$$\begin{aligned}&\mu_n(1+c\ \ell n\mu_n)=\\&=\lambda\prod_0^{n-1}(1+c(1+c)^k\ell n\lambda)(1+c(\ell n\lambda+\sum_0^{n-1}\ell n(1+c(1+c)^k\ell n\lambda))\\&<\lambda\prod_0^{n-1}(1+c(1+c)^k\ell n\lambda)(1+c\ \ell n\lambda+c^2\sum_0^{n-1}(1+c)^k\ell n\lambda)\\&=\lambda\prod_0^{n}(1+c(1+c)^k\ell n\lambda),\quad\text{as required.}\end{aligned}$$

While we do not have a reasonable bound on the law of $\mathcal{O}^nX$, even if $X=\mathcal{P}_\lambda$, we do obtain good upper bounds on the first two moments $E\mathcal{O}^nX$ and $E(\mathcal{O}^nX)^2$ for general X. These will lead in Section 3 to bounds on the two moments of the law of $(N_n\mid L)$. With more work, the following for $n>1$ may also be sharpened to some extent, but it seems quite satisfactory as stated.

Theorem 2.9. *For any random variable $X \geq 0$ (integer-valued), we have*
(a) $E(\mathcal{O}X) \leq \frac{1}{2}(EX+1)$; $E(\mathcal{O}^n X) \leq 2^{-n}EX + 1$ *for* $1 \leq n$.
(b) $0 \leq 2E(\mathcal{O}X) - E(\mathcal{O}(2X)) \leq \frac{2}{3}$, *and*
(c) $E(\mathcal{O}X)^2 \leq \frac{1}{4}EX^2 + \frac{5}{8}(EX+1)$; $E(\mathcal{O}^n X)^2 \leq 4^{-n}EX^2 + 5 \cdot 2^{-1(n+2)}EX + \frac{5}{6}$; $1 \leq n$.

Proof.

(a) As noted in the proof of Theorem 2.5, given $X_1 = i \geq 1$ and $X_2 = j \geq 1$, $\mathcal{O}X$ has a hypergeometric distribution with mean $j(\frac{i}{i+j-1})$. Thus if X has distribution $p_i = P(X = i)$, $0 \leq i$, we have

$$
\begin{aligned}
E\mathcal{O}X &= \frac{1}{2}\sum\sum_{i,j\geq 1} p_i p_j 2ij/i+j-1 \\
&\leq \frac{1}{4}\sum\sum_{i,j\geq 1} p_i p_j \left(\frac{(i+j)^2}{i+j}\right)\left(\frac{i+j}{i+j-1}\right) \\
&= \frac{1}{4}\sum\sum p_i p_j (i+j)(1+\frac{1}{i+j-1}) \\
&= \frac{1}{4}\sum_i p_i(i+EX) + \frac{1}{4}\sum\sum p_i p_j \left(1+\frac{1}{i+j-1}\right) \\
&\leq \frac{1}{2}EX + \frac{1}{4} + \frac{1}{4}\sum\sum_{i,j\geq 1} p_i p_j/i+j-1 \\
&\leq \frac{1}{2}(EX+1),
\end{aligned}
$$

proving the first assertion. Now by iteration, setting $e_0 = EX$, $e_{n+1} = \frac{1}{2}(e_n+1)$, we obtain easily $e_n = 2^{-n}e_0 + (\frac{1}{2}+\frac{1}{4}+\cdots+2^{-n}) \leq 2^{-n}e_0 + 1$, as required.

(b) Evidently we can assume again, without loss of generality, that $1 \leq X$, so that, given (i,j), $\mathcal{O}X$ has a hypergeometric distribution with mean $\frac{ij}{i+j-1}$. Then we need only note for $1 \leq i \wedge j$

$$
\begin{aligned}
0 &\leq 2(ij/i+j-1) - \frac{(2i)2j)}{2(i+j)-1} \\
&= 2ij\left(\frac{1}{i+j-1} - \frac{1}{i+j-\frac{1}{2}}\right) \\
&= ij\frac{1}{(i+j-1)(i+j-\frac{1}{2})} \\
&= \frac{ij}{(i+j)^2 - \frac{3}{2}(i+j) + \frac{1}{2}} \\
&< \frac{(i+j)^2}{4((i+j)^2 - \frac{3}{2}(i+j)) + \frac{1}{2})} \\
&< \frac{2}{3}
\end{aligned}
$$

(the last expression is monotone decreasing in $i+j > 1$ by routine differentiation, so obviously stronger inequalities hold for $i+j > K$, depending on K, but $\frac{1}{4}$ is a lower bound obtained by letting $i+j \to \infty$).

(c) As to the square, it is well-known that the variance of a hypergeometric distribution is less than that of the corresponding binomial $b(n,p)$ where, in our case (given $X_1 = i$, $X_2 = j$), we have $n = j$, $p = \frac{i}{i+j-1}$. Thus (for $1 \leq i, j$) $(Var\mathcal{O}X \mid i,j) \leq j\frac{i}{i+j-1}\left(1 - \frac{i}{i+j-1}\right)$. Thus $E((\mathcal{O}X)^2 \mid i,j) \leq (E(\mathcal{O}X \mid i,j))^2 + E(\mathcal{O}X \mid i,j) - \frac{ji^2}{(i+j-1)^2}$, and taking expectations gives

$$
\begin{aligned}
E(\mathcal{O}X)^2 &\leq E(\mathcal{O}X) + \sum\sum p_i p_j \left(\frac{ij}{i+j-1}\right)\frac{(i-1)j}{(i-1)+j} \\
&\leq E(\mathcal{O}X) + \sum\sum p_i p_j \frac{(ij)^2}{(i+j-1)(i+j)},
\end{aligned}
$$

where we used the observation that $\frac{d}{dx}\left(\frac{xy}{x+y}\right) = \frac{y^2}{(x+y)^2} > 0$ for $0 < x, y$. Now

$$
\begin{aligned}
\sum\sum p_i p_j \frac{(ij)^2}{(i+j-1)(i+j)} &= \frac{1}{4}\sum\sum p_i p_j \frac{(2ij)^2}{(i+j-1)(i+j)} \\
&\leq \frac{1}{16}\sum\sum p_i p_j (i+j)^2 \left(\frac{i+j}{i+j-1}\right) \\
&\leq \frac{1}{4}EX^2 + \frac{1}{16}\sum\sum p_i p_j \frac{(i+j)^2}{(i+j-1} \\
&\leq \frac{1}{4}EX^2 + \frac{1}{8}(EX+1),
\end{aligned}
$$

where we used the inequality from the proof of (a) at the last step. Combining with $E\mathcal{O}X$ now gives the first assertion of (c).

As to the second, we will prove by induction that $E(\mathcal{O}^n X)^2 \leq 4^{-n}EX^2 + 5(\sum_{k=n+2}^{2n+3} 2^{-k})EX + \frac{5}{2}\sum_{k=1}^{n} 4^{-k}$. For $n = 1$ it follows by the first assertion, so we assume it for n. Then

$$
\begin{aligned}
E(\mathcal{O}^{n+1}X)^2 &\leq \frac{1}{4}E(\mathcal{O}^n X)^2 + \frac{5}{8}E(\mathcal{O}^n X + 1) \\
&\leq 4^{-(n+1)}EX^2 + \left[\frac{5}{4}\left(\sum_{k=n+2}^{2n+3} 2^{-k}\right) + \frac{5}{8}2^{-n}\right]EX \\
&\quad + \frac{5}{8}\sum_{k=1}^{n} 4^{-k} + \frac{5}{8} \\
&= 4^{-(n+1)}EX^2 + \frac{5}{4}\left(\sum_{k=n+1}^{2n+3} 2^{-k}\right)EX + \frac{5}{2}\left(\sum_{k=1}^{n+1} 4^{-k}\right),
\end{aligned}
$$

and the proof is completed by summing the series.

Remark. For brevity, and because it is not needed for Theorem 3.6 below, we do not attempt any analog of Theorem 2.9 using the operation of Theorem 2.4 (b) in place of (2.2).

Section 3. The law of $(N_n \mid L), n$ fixed.

We start by admitting that we do not have any closed expression for the above law. One approach would be to examine $\lim_{m\to\infty}(N_n \mid N_m)$ in law; however, as stated in the Introduction, we could not prove that $\sigma(L) \equiv \lim_{m\to\infty} \sigma(N_k, k \geq m)$, and besides, an explicit expression for the $(m-n)^{th}$ iterate of the transition mechanism is lacking. The approach via Bayes Formula also seems doomed to failure, although the ingredients are known from [6]. A more rewarding method is to find, first of all, the law of N_n given $\{L(k\alpha_n),\ -2^n < k\}$. This is done explicitly below. Then we can use the fact that $(N_n \mid L) = \lim_{m\to\infty}(N_n \mid L(k\alpha_m),\ -2^m < k)$ (in law) together with the results of Section 2, to obtain information about $(N_n \mid L)$.

Theorem 3.1. *Choose $0 < x_k;\ -2^n < k \leq \kappa$, and set $0 = x_k$ for $k > \kappa$, for some $\kappa \geq 0$. Then for $-2^n < j < \kappa$*

$$P(N_n(j) = n_j \mid L(k\alpha_n) = x_k,\ -2^n < k) = \tag{3.1}$$

$$\begin{cases} \left(\frac{x_j x_{j+1}}{4\alpha_n^2}\right)^{n_j} (n_j!)^{-2} I_0^{-1}\left(\sqrt{\frac{x_j x_{j+1}}{\alpha_n^2}}\right); j < 0,\ 0 \leq n_j \\ \left(\frac{x_j x_{j+1}}{4\alpha_n^2}\right)^{n_j - \frac{1}{2}} (n_j!(n_j-1)!)^{-1} I_1^{-1}\left(\sqrt{\frac{x_j x_{j+1}}{\alpha_n^2}}\right); j \geq 0,\ 0 < n_j \end{cases}$$

where I_0 and I_1 are the modified Bessel functions (so that either sum over n_j equals 1).

Remark 1. For $j \geq \kappa$ the conditional probability is 1 for $n_j = 0$. The exact meaning of the conditioning is that, for $\kappa := \inf\{k \geq 0 : L((k+1)\alpha_n) = 0\}$ (note that $\{N_n(k) = 0\} \equiv \{N_n(j) = 0 \text{ for } j \geq k\} \equiv \{L((k+1)\alpha_n) = 0\}$) the expression gives a regular conditional distribution of $N_n(j)$ given $\{L(k\alpha_n),\ -2^n < k\}$. Moreover, as seen easily from the proof below, the variables $N_n(j)$ are conditionally independent given $\{L(k\alpha_n),\ -2^n < k\}$, hence by multiplication this also gives the regular conditional joint distributions.

Remark 2. Following J. Pitman and M. Yor [7] we call the first case the (discrete) Bessel I_0-distribution with parameter $z = \sqrt{\frac{x_j x_{j+1}}{\alpha_n^2}}$ and the second case the I_1-distribution with the same z. This appearance of these Bessel distributions has a long history, the case of I_1 going back at least to F. Knight [5, p. 180] and the case of I_0 to Pitman and Yor [loc sit., p. 449] where the other Bessel distributions are also discussed. But the present derivation, which includes also the joint distributions, seems to be new.

Proof. By Lemma 1.5, if $N_n(\cdot)$ is given, $L(k\alpha_n)$ equals a sum over inserts which "start" at $k\alpha_n$ (i.e. start at some T_j with $B(T_j) = k\alpha_n$), and these are all independent. On the other hand, a well-known fact about the local time of B at 0 and time $T(\pm\alpha_n)$ is that it has the exponential law with $\lambda = \alpha_n^{-1}$ (it suffices to check the case $n = 0$ and apply Brownian scaling). This is clearly the same for the local time at 0 of an n-interest. Thus we see that, given $N_n(\cdot)$, the $L(k\alpha_n)$ are mutually independent with marginal law having the gamma densities $\Gamma(N_n(k-1) + N_n(k);\ \alpha_n^{-1})$ for $k > 0$ and $\Gamma(N_n(k-1) + N_n(k) + 1;\ \alpha_n^{-1})$ for $-2^n < k \le 0$ (note that for $k \le 0$ there is a first trip from $k\alpha_n$ to $(k-1)\alpha_n$ not counted in $N_n(k-1)$).

We now work out the joint distribution

$$P(N_n(k) = n_k,\ 1 \le k < \kappa \mid N_n(0) = n_0,\ L(k\alpha_n) = x_k,\ 1 \le k \tag{3.2}$$

for $n_k > 0,\ 0 \le k < \kappa$ fixed. Applying Bayes' rule, this is proportional (for n_0 fixed, $n_\kappa = 0$) to $P(L(k\alpha_n) = x_k,\ 1 \le k \le \kappa \mid N_n(k) = n_k,\ 0 \le k \le \kappa) \cdot P(N_n(k) = n_k,\ 1 \le k < \kappa \mid N_n(0) = n_0,\ \kappa)$ with a factor of proportionality such that the sum over $n_k > 0$, $1 \le k < \kappa$, equals 1, and by the Markov property of $N_n(k)$ this would not be changed if $\{N_n(k),\ k \le 0\}$ were given instead of only $N_n(0)$. Referring to the transition function of $N_n(k)$ from the Introduction, and setting $n_\kappa = 0$, the last product equals

$$\begin{aligned}
&\prod_{k=1}^{\kappa} \frac{x_k^{n_{k-1}+n_k-1} e^{-\alpha_n^{-1} x_k}}{\alpha_n^{n_{k-1}+n_k-1}\Gamma(n_{k-1}+n_k)} \prod_{k=1}^{\kappa} \binom{n_{k-1}+n_k-1}{n_k} \alpha_{n_{k-1}+n_k} \\
&= \prod_{k=1}^{\kappa} \left(\frac{x_k}{2\alpha_n}\right)^{n_{k-1}+n_k-1} \left((n_{k-1}-1)!\,n_k!\right)^{-1} e^{-\alpha_n^{-1} x_k} \\
&= \frac{\left(\frac{x_1}{2\alpha_n}\right)^{n_0-\frac{1}{2}}}{(n_0-1)!} \left(\prod_{k=1}^{\kappa-1} \sqrt{\frac{x_k x_{k+1}}{4\alpha_n^2}}^{\,2n_k-1} \left((n_k-1)!\,n_k!\right)^{-1} e^{-\alpha_n^{-1} x_k}\right).
\end{aligned} \tag{3.3}$$

Thus, in so far as the dependence on $(n_1, \ldots, n_{\kappa-1})$ is concerned, we recognize for each k the $(n_k - 1)^{th}$ term in the series expansion of $I_1\left(\sqrt{\frac{x_k x_{k+1}}{\alpha_n^2}}\right)$. Hence by normalization the whole expression (3.2) must reduce to

$$\frac{\prod_{k=1}^{\kappa-1} \sqrt{\frac{x_k x_{k+1}}{4\alpha_n^2}}^{\,2n_k-1} \left((n_k-1)!\,n_k!\right)^{-1}}{\prod_{k=1}^{\kappa-1} I_1\left(\sqrt{\frac{x_k x_{k+1}}{\alpha_n^2}}\right)}.$$

We note the curious fact that there is no dependence on the given n_0. Moreover, since $N_n(k) = n_k > 0$ implies $k < \kappa$, we can fix $K \ge 1$ and sum over all κ and $n_k : K < k < \kappa$ to obtain, for $1 \le n_k,\ 1 \le k \le K$,

$$\begin{aligned}
&P(N_n(k) = n_k,\ 1 \le k \le K \mid N_n(0),\ L(k\alpha_n) = x_k,\ 1 \le k \le K+1) \\
&= \prod_{k=1}^{K} \sqrt{\frac{x_k x_{k+1}}{4\alpha_n^2}}^{\,2n_k-1} \left[(n_k-1)!\,n_k!\,I_1\left(\sqrt{\frac{x_k x_{k+1}}{\alpha_n^2}}\right)\right]^{-1},
\end{aligned} \tag{3.4}$$

where $\{\kappa > K\}$ is also given since $x_{K+1} > 0$.

With this as model, let us now work out

$$(3.5) \qquad P(N_n(k) = n_k,\ -2^n < k < \kappa \mid L(k\alpha_n) = x_k,\ -2^n < k \le \kappa)$$

where $n_k \ge 0$ for $k \le 0$; $n_k > 0$ for $0 < k < \kappa$, and $x_k > 0$ for $2^n < k \le \kappa$ (and we set $x_{-2^n} = 0 = n_{-2^n}$, $x_{\kappa+1} = 0 = n_\kappa$). Applying Bayes' Rule we get a result proportional to (as the n_k vary with κ fixed) the product of (3.4) with

$$\begin{aligned}
&P(L(k\alpha_n) = x_k,\ -2^n < k \le 0 \mid N_n(\cdot))P(N_n(k) = n_k,\ -2^n < k \le 0) \\
&= \prod_{-2^n+1}^{0} \frac{x_k^{n_{k-1}+n_k-1} e^{-\alpha_n^{-1} x_k}}{\alpha_n^{n_{k-1}+n_k}\Gamma(n_{k-1}+n_k)} \prod_{-2^n+1}^{0} \binom{n_{k-1}+n_k}{n_k} \alpha_{n_{k-1}+n_k-1} \\
&= \prod_{-2^n+1}^{0} \left(\frac{x_k}{2\alpha_n}\right)^{n_{k-1}+n_k} (n_{k-1}!n_k!)^{-1} e^{-\alpha_n^{-1} x} \\
&= \frac{\left(\frac{x_0^{n_0}}{2\alpha_n}\right)}{n_0!} \prod_{-2^n+1}^{-1} \sqrt{\frac{x_k x_{k+1}}{4\alpha_n^2}}^{\,2n_k} (n_k!)^{-2}.
\end{aligned}$$

Combining the first factor on the right with the factor $\frac{\left(\frac{x_1}{2\alpha_n}\right)^{n_0-\frac{1}{2}}}{(n_0-1)!}$ from the right side of (3.3) we recognize the general term in the expansion of $I_1\left(\sqrt{\frac{x_0 x_1}{\alpha_n^2}}\right)$, for $n_0 > 0$, while the terms of the subsequent product are those of $I_0\left(\sqrt{\frac{x_k x_{k+1}}{\alpha_n^2}}\right)$, where we permit $n_k = 0$. On the other hand, if $n_0 = 0$ then $\kappa = 0$ and (3.3) is vacuous. In that case we have only the terms from $I_0\left(\sqrt{\frac{x_k x_{k+1}}{\alpha_n^2}}\right)$; $-2^n < k < 0$. This completes the proof of Theorem 3.1. Moreover, since our conditional probability given $\{L(k\alpha_n) = x_k,\ -2^n < k\}$ factors into a product of terms in n_j, we see that the events $\{N_n(j) = n_j\}$ are all conditionally independent given $\{L(k\alpha_n);\ -2^n < k\}$. Indeed, since the conditional law of $N_n(k)$ depends only on $L(k\alpha_n)$ and $L((k+1)\alpha_n)$, we can state

Corollary 3.1. *For $-2^n < k_1 < \cdots < k_m$, $\{N_n(k_j);\ 1 \le j \le m\}$ are conditionally independent given $\{L(k_j\alpha_n),\ L(k_j+1)\alpha_n;\ 1 \le j \le m\}$ and have the marginal distributions of Theorem 3.1.*

Remark 1. The distribution of κ is not contained in Theorem 3.1. However, it is easy to find:

$$\begin{aligned}
P\{\kappa \le K\} &= P\{\max B'(t) < (K+1)\alpha_n\} \\
&= (K+1)\alpha_n\,(1 + (K+1)\alpha_n)^{-1},\ 0 \le K.
\end{aligned}$$

Remark 2. The independence assertion of Corollary 3.1 can be seen as a consequence of the conditional independence, given $\{L(k_j\alpha_n),\ L((k_j+1)\alpha_n)\}$, of the processes in $[k_j\alpha_n, (k_j+1)\alpha_n]$ obtained by excising the excursions of B' outside the interval (not proved here).

For application of Theorem 2.9 to these distributions, we need to work out the first two moments of the second marginal in (3.1) (namely, of the I_1-distribution).

Lemma 3.2. *Let X have the I_1-distribution with parameter $\beta := \left(\frac{x_j x_{j+1}}{4\alpha_n^2}\right)$. Then*

$$EX = \sqrt{\beta} I_0(2\sqrt{\beta})/I_1(2\sqrt{\beta}), \quad \text{and}$$
$$EX^2 = \beta + EX; \ \beta > 0.$$

Remark. we pass over the case when X has an I_0-distribution.

Proof. We have

$$\begin{aligned} EX &= \sum_{n=1}^{\infty} \beta^{n-\frac{1}{2}}((n-1)!)^{-2}(I_1(2\sqrt{\beta}))^{-1} \\ &= \sqrt{\beta}\sum_{n=0}^{\infty} \beta^n (n!)^{-2}(I_1(2\sqrt{\beta}))^{-1} \\ &= \sqrt{\beta} I_0(2\sqrt{\beta})/I_1(2\sqrt{\beta}), \quad \text{and} \\ EX^2 &= \left[\sum_{n=2}^{\infty} \beta^{n-\frac{1}{2}}((n-1)!(n-2)!)^{-1} + \sqrt{\beta} I_0(2\sqrt{\beta})\right] \left(I_1(2\sqrt{\beta})\right)^{-1} \\ &= \beta + \sqrt{\beta} I_0(2\sqrt{\beta})(I_1(2\sqrt{\beta}))^{-1}, \quad \text{as required} \end{aligned}$$

To apply the Comparison Theorem 2.5, we need

Lemma 3.3. *Both the I_1 distribution and the I_0 distribution are increasing functions of the parameter $\beta := \left(\frac{x_j x_{j+1}}{4\alpha_n^2}\right)$, in the sense of Notation 2.5.*

Proof. For the I_1-distribution, we must show that $(I_1(2\sqrt{\beta}))^{-1}\sum_{k+1}^{\infty} \beta^{n-\frac{1}{2}}(n!(n-1)!)^{-1}$ is increasing in $\beta > 0$ for $k \geq 1$. Denoting the sum by $B(\beta) := I_1(2\sqrt{\beta}) - A(\beta)$, and differentiating the whole expression with respect to β, we have $\frac{d}{d\beta}\left(\frac{B(\beta)}{A(\beta)+B(\beta)}\right) = (A(\beta)B'(\beta) - B(\beta)A'(\beta))/(A+B)^2$, which has the same sign as

$$B'/B - A'/A =$$

$$\beta^{-1}\left(\frac{\sum_{k+1}^{\infty}(n-\frac{1}{2})\beta^{n-\frac{1}{2}}(n!(n-1)!)^{-1}}{B} - \frac{\sum_1^k (n-\frac{1}{2})\beta^{n-\frac{1}{2}}(n!(n-1)!)^{-1}}{A}\right)$$

$$> \beta^{-1}(k + \frac{1}{2} - (k - \frac{1}{2})) = \beta^{-1} > 0,$$

as required. The proof for the I_0 distribution is precisely the same.

We may now state

Theorem 3.5. *There exists a choice of conditional law $\mathcal{L}(N_n(\cdot),\ 0 \leq n \mid L)$ with the following properties:*

(a) *Apart from a P-null set in $\sigma(L)$, $(N_n(\cdot) \mid L)$ is an $\mathcal{L}$-Markov chain in n with the same transition mechanism as $N_n(\cdot)$, (in other words, $\mathcal{L}(\cdot \mid L)$ is, P-a.s., an entrance law for N_n).*

(b) *(Local Independence). Apart from a P-null set, for every* $-1 < x_1 < x_2$*, and* $\forall(k,n)$ *with* $(k\alpha_n,\ (k+1)\alpha_n) \subset (x_1, x_2)$*,* $N_n(k)$ *under* $\mathcal{L}$ *is jointly independent of all* $N_m(j)$ *with* $(j\alpha_m, (j+1)\alpha_m)$ *outside* $[x_1, x_2]$*, and its conditional law depends only on* $\{L(x);\ x_1 \le x \le x_2\}$*. In words: the* $N_n(k)$ *which count upcrossings of disjoint intervals are mutually independent given L, and their laws depends only on L inside the intervals counted.*

(c) *(Local monotonicity). Apart from* $w_i, i = 1$ *or 2, in a P-null set, for every* $-1 < x_1 < x_2$*,* $L(x, w_1) < L(x, w_2)$ *for* $x_1 \le x \le x_2$ *implies that* $(N_n(k) \mid L(\cdot, w_1)) << (N_n(k) \mid L(\cdot, w_2))$ *for* $\forall(k,n)$ *with* $x_1 \le k\alpha_n$ *and* $(k+1)\alpha_n \le x_2$*, where* $(N_n(k) \mid L(\cdot, w))$ *is any choice of* $N_n(\cdot)$ *having the conditional law* $\mathcal{L}$ *at w.*

Proof. Since $L(x)$ is a known diffusion process (we may assume continuous paths and generated filtrations continuous in x, so that "fringe effects" do not play a role: $\sigma(L(y),\ y \le x) \equiv \bigcap_{\epsilon>0} \sigma(L(y),\ y < x+\epsilon))$ we have $\sigma(L) = \bigvee_n \sigma(L(k\alpha_n),\ -2^n < k)$, the right term being monotone in n. For each n, let $\mathcal{L}_n(w)$ be a regular conditional probability for $(N_n,\ N_{n-1}, \ldots, N_0)$ given $(L(k\alpha_n),\ -2^n < k)$. Since $\sigma(L) \subset \sigma(N_{n+m},\ 0 \le m)$, it follows easily by Theorem 2.1 that, with probability one, $\mathcal{L}_n(w)$ makes $(N_n, N_{n-1}, \ldots, N_0)$ a Markov chain with the same transition mechanism as (N_n, P) (Theorem 2.4) and initial distribution of N_n that of Corollary 3.1. Indeed, we may and do take this as *definition* of $\mathcal{L}_n$.

Now we define the limit in distribution

$$\mathcal{L}(N_n(\cdot),\ 0 \le n \mid L(w)) = \lim_{N\to\infty} \mathcal{L}_N(w) \tag{3.6}$$

if, for every finite subset $(N_{n_1}(k_1), \ldots, N_{n_j}(k_j))$, the joint law for $\mathcal{L}_N$ converges (hence the limit is uniquely extendible to a probability on a product of discrete spaces by Kolmogorov's Extention Theorem), and $L(N_n(\cdot),\ 0 \le n \mid L(w)) = \delta_{N_n(\cdot,w')}$ elsewhere, where $w' \in \Omega$ is fixed but arbitrary. By a simple consideration of martingale convergence the first case has probability 1 in $\sigma(L(x), -1 < x)$, and defines a regular conditional probability of $\sigma(N_n, 0 \le n)$ given $\sigma(L(x), -1 < x)$. To see that (a) is true, we need only observe that the transition mechanism is obviously continuous under convergence in law of the marginal distributions. As $N \to \infty$ with n fixed, the Markov chain law $\mathcal{L}_N$ applied to $(N_n, N_{n-1}, \ldots, N_0)$ yields a Markov chain limit law when we condition on $\sigma(N_k;\ n \le k \le N)$ (which converges to $\sigma(N_k; n \le k)$). Since n is arbitrary, (a) follows.

Turning to (b), we have first to observe that it suffices to prove that, for every n, $\{N_n(k),\ -2^n < k\}$ are mutually independent for $\mathcal{L}$ (with probability 1) and that their conditional laws (as w varies) depend only on L inside the intervals covered (we refer to $(k\alpha_n, (k+1)\alpha_n)$ as the interval "covered by" $N_n(k)$). Indeed, any finite collections $\{N_{n_j,i}(k^i_j),\ 1 \le j \le \ell_i\},\ 1 \le i \le I$, such that the covered intervals are disjoint over i has a law built up by applying the transition mechanism to disjoint subsets of $\{N_{\hat{n}}(k),\ -2^{\hat{n}} < k\}$ where $\hat{n} = \max_{i,j} n_{j,i}$. Given that these disjoint subsets are mutually independent for $\mathcal{L}$, it follows that so are the $\{N_{n_j}(k_j)\}$ as i varies. This will prove the independence assertion, and the dependence assertion also follows (indeed, it is clear by independence that, if the law of each $N_{\hat{n}}(k)$ depends only on L in the covered interval, then any finite subset covering a subset of $[x_1, x_2]$ has joint law depending only on L in $[x_1, x_2]$).

Now to prove the conditional independence of the $N_n(k)$ for fixed n we start with $\mathcal{L}_n$, for which the independence is part of Corollary 3.1. More generally, for $M \geq n$, since $\mathcal{L}_M$ makes $\{N_M(k),\ -2^{+M} < k\}$ independent, and given these the $\mathcal{L}_M$-law of $\{N_n(k),\ -2^n < k\}$ is built up by applying the transition mechanism $M - N$ times to the disjoint subsets of $\{N_M(k)\}$ which cover subintervals of the intervals covered by $\{N_n(k)\}$, we see that $\mathcal{L}_M$ makes these last independent. Similarly, since the $\mathcal{L}_M$-law of $N_M(k)$ depends only on L at the endpoints of the covered interval (by Corollary 3.1 again), the $\mathcal{L}_M$-law of $N_n(k)$ depends only on L at the $N_M(j)$-endpoints contained in the interval covered by $N_n(k)$, hence on L in the covered interval. Then as $M \to \infty$, the $\mathcal{L}_M$-law of the $\{N_n(k),\ -2^n < k\}$ converges to the $\mathcal{L}$-law (with probability 1) preserving both the independence and the individual limits of dependence on L to the covered intervals. This finishes (b).

As to (c), it follows by Lemma 3.3 that if $L(x, w_1) < L(x, w_2), x_1 \leq x \leq x_2$, and if $N_n(k)$ covers a subinterval of (x_1, x_2), then the $\mathcal{L}_n$-laws of $N_n(k)$ (given by Corollary 3.1) are ordered in the same direction (in the sense of $<<$). Therefore since by Theorem 2.5 and Corollary 2.5 the transition mechanism preserves this ordering, and the $\mathcal{L}_M$-law of $N_n(k)$ is developed from that of $\{N_M(j); -2^M < j\}$ by applying the transition mechanism to those $N_M(j)$ covering subintervals of (x_1, x_2), the ordering is also preserved by $\mathcal{L}_M$, $M \geq n$. Letting $M \to \infty$, and noting again that the order $<<$ is conserved under convergence in law, we see that (apart from w_1 or w_2 in the set where (3.6) fails) the ordering is also preserved by $\mathcal{L}$. This finishes (c).

We come now to our upper bounds for the first two moments of $N_n(k)$ under $\mathcal{L}$ from Theorem 3.5.

Theorem 3.6. *Apart from the P-null set where (3.6) of Theorem 3.5 (Proof) fails, for all $n \geq 0$ and $-2^n < k$, if $L^* := \max_{k\alpha_n \leq x \leq (k+1)\alpha_n} L(x)$, then under $\mathcal{L}$ we have*

(a) $E(N_n(k) \mid L) \leq 2^{n-1}L^* + 1$, *and*
(b) $E(N_n^2(k) \mid L) \leq 4^{n-1}(L^*)^2 + 5 \cdot 2^{n-3}L^* + \frac{5}{6}$.

Proof. We may and do define both conditional expectations using $\mathcal{L}$. Now for $M \geq n$, since $L(j\alpha_M) < L^*$ for $k\alpha_n \leq j\alpha_M \leq (k+1)\alpha_m$, by Lemma 3.3 the two conditional expectations computed under $\mathcal{L}_M$ instead of $\mathcal{L}$ (see proof of Theorem 3.5) will not be decreased if we replace L by L^* in the interval. Suppose, first, that $0 \leq k$. Then the $L(j\alpha_M)$ under $\mathcal{L}_M$ have the I_1-distributions with parameter $\beta = \frac{L(j\alpha_M)L((j+1)\alpha_m)}{4\alpha_M^2}$. If we replace L by L^*, β becomes $\frac{(L^*)^2}{4\alpha_M^2}$ and the distribution is increased in the sense of the order <<. By Lemma 3.2, the corresponding moments become respectively $\frac{L^*}{2\alpha_M} I_0(\frac{L^*}{\alpha_M})/I_1(\frac{L^*}{\alpha_M})$, and this plus $\frac{(L^*)^2}{4\alpha_M^2}$. Now applying (a) and (c) of Theorem 2.9, respectively, to the $(M-m)^{th}$ iterate of the transition mechanism $\mathcal{O}$ (which preserves <<) it follows that under $\mathcal{L}_M$,

$$E(N_n(k) \mid L(j\alpha_M), \forall_j) \le 2^{+n-1} L^* I_0\left(\frac{L^*}{\alpha_M}\right) / I_1\left(\frac{L^*}{\alpha_M}\right) + 1 \quad \text{and}$$

$$E(N_n^2(k) \mid L(j\alpha_M), \forall_j) \le 4^{+n-1}(L^*)^2 + 4^{-(M-n)}\frac{L^*}{2\alpha_M}\frac{I_0}{I_1}\left(\frac{L^*}{\alpha_M}\right)$$

$$+ 5 \cdot 2^{-(M-n)-2}\frac{L^*}{2\alpha_M}\frac{I_0}{I_1}\left(\frac{L^*}{\alpha_M}\right) + \frac{5}{6}$$

$$= 4^{+n-1}(L^*)^2 + \left[2^{-M-1}(4^n) + 5 \cdot 2^{n-3}\right] L^* \frac{I_0}{I_1}\left(\frac{L^*}{\alpha_M}\right) + \frac{5}{6}$$

Then as $M \to \infty$, except on the P-null set we have $\mathcal{L}_M \to \mathcal{L}$, and convergence in law implies that the moments for $\mathcal{L}$ are bounded by the liminf of those for $\mathcal{L}_M$ (approximate x or x^2 from below by bounded continuous functions). Meanwhile, we have $\lim_{x \to \infty} I_0(x)/I_1(x) = 1$ [10, p. 343]. Combining these observations gives the asserted bounds when $0 \le k$.

It remains to discuss the case $-2^n < k < 0$. This reduces to the former case. We have only to decompose $L(x) = L_1(x) + L_2(x)$, $k\alpha_n \le x$, where $L_1(x) := L(x, T(k\alpha_n))$ is the local time before reaching $k\alpha_n$, and $L_2(x)$ is the local time in $T(k\alpha_n) \le t \le T(-1)$. By the strong Markov property, L_1 and L_2 are independent processes (we need not use their characterization as diffusions) and $L(k\alpha_n) = L_2(k\alpha_n)$. We claim that the case $k < 0$ reduces to $k \ge 0$ with L_2 in place of L. Indeed, just as we can treat $k \ge 0$ in terms of the process after reaching $k\alpha_n$ (which does not change L in $(k\alpha_n, (k+1)\alpha_n)$), we can do the same for $k < 0$, but then we have to reduce L by L_1. The salient facts here are

(a) the process before reaching $k\alpha_n$ has no effect on $N_n(k)$ (apart from the case $T(k\alpha_n) = \infty$ when $0 < k$) and
(b) $B'(T(k\alpha_n)+t) - B'(T(k\alpha_n))$ looks like B' with $T(-1)$ replaced by $T(-1+k\alpha_n)$, and this reduces to B' by Brownian scaling. Using these facts, for $k < 0$ the inequalities of Theorem 3.6 are seen to hold with L_2 in place of L. But then they also hold given $\sigma(L_1, L_2)$, and with L^* in place of $\max_{k\alpha_n \le x \le (k+1)\alpha_n} L_2(x)$ (since L_1 is independent of $N_n(k)$, and L^* is larger than $\max L_2$). Since L^* depends only on L, we may as well only assume L, and the case $k < 0$ follows as stated.

Remark 1. In order to accurately evaluate the sharpness of these inequalities, one could develop reverse inequalities in Theorem 2.9 (a), (c), but it looks complicated. Meanwhile, at least for n reasonably large (where the proof of these inequalities is tightest) there is reason to believe that they are quite sharp. Indeed, as n becomes large, $2\alpha_n N_n(k) \to L$ and $L^* \to L$, in such a way that after multiplying (a) by $2\alpha_n$ and replacing $E(N_n(k) \mid L)$ by $N_n(k)$, it converges to the tautology $L \le L$ and similarly for (b), using $4\alpha_n^2 N_n^2(k) \to L^2$. Moreover, without multiplying by $2\alpha_n$, if we replace L^* by L on the right and take expectations, using the identities $EN_n(k) = 2^n$, $EN_n^2(k) = 2^{2n+1} + k2^{n+1} + 2^n$, and $EL(x) = 2$, $EL^2(x) = 8(x+1)$ for $0 \le x = k\alpha_n$ (by standard calculations), then for $0 \le x = k\alpha_n$ (a) becomes $2^n \le 2^n + 1$, while (b) becomes $2^{2n+1} + k2^{n+1} + 2^n \le 2^{2n+1} + k2^{n+1} + 5 \cdot 2^{n-2} + \frac{5}{6}$. This verifies that (a) and (b) hold "on the average", and at the same time shows that they are quite sharp on the average, particularly if n and k are large with $x = k\alpha_n$ fixed.

Remark 2. It may be of interest to calculate the law of $\mathcal{O}X$ explicitly when X has the I_1-distribution (for $\mathcal{O}X$ see Notation 2.8 — when $X = \mathcal{P}_\lambda$ this was done in Lemma 2.6). The result, after simplification, is

$$P\{\mathcal{O}X = k\} = \left(\frac{\beta}{2}\right)^{k-\frac{1}{2}} (k!(k-1)!)^{-1} I_{2k-1}(2\sqrt{2\beta})/I_1^2(2\sqrt{\beta}).$$

Thus modified Bessel functions of all odd orders appear, and it looks to us hopeless to simplify the law of $\mathcal{O}^2X$, much less that of $\mathcal{O}^nX$ for general n.

Remark 3. In terms of $\mathcal{L}(N_n \mid L)$, the unverified conjecture of the Introduction that $\sigma(L(\cdot)) \equiv \lim_{N\to\infty} \sigma(N_n,\ n \ge N)$ is equivalent to asserting that, for every m, $\lim_{n\to\infty} \mathcal{L}(N_m \mid N_n) = \mathcal{L}(N_m \mid L)$, P-a.s., where the convergence is to be in law. It seems that a proof would require additional results along the general lines of Theorem 3.6.

4. Construction of the law of B' given $(N_n;\ 0 \le n)$.

Here we assume that $(N_n(\cdot),\ 0 \le n)$ has one of the entrance laws $\mathcal{L}(N \mid L)$ with properties (a)—(c) of Theorem 3.5 and also (a)—(b) of Theorem 3.6 (i.e. we exclude the P-nullset where (3.6) fails), and we seek to define the law of B' consistent with this $\mathcal{L}(N \mid L)$, to obtain $\mathcal{L}(B' \mid L)$. Let us then also impose one further condition, namely that $\lim_{n\to\infty} 2\alpha_n N_n[2^n x] = L(x)$ uniformly in x, $P^{\mathcal{L}(N|L)}$-a.s. Since this holds P-a.s., it also holds $P^{\mathcal{L}(N|L)}$-a.s. for P-almost all $w \in \Omega$, so the condition holds except on a P-nullset. Then if we define the law $\mathcal{L}(B' \mid N_.)$ for P-almost all laws $\mathcal{L}(N \mid L)$, we can write

$$\text{(4.1)} \qquad \mathcal{L}(B' \mid L) = E[\mathcal{L}(B' \mid \sigma(L, N_.))|\sigma(L)] = E(\mathcal{L}(B' \mid N)|\sigma(L)),$$

where E is to be calculated using $\mathcal{L}(N \mid L)$. This gives the solution to our problem.

In fact, we define $\mathcal{L}(B' \mid N_.)$ for all paths of $N_.$ having the *consistency properties* that for all $0 \le n$,

(a) $N_n(k) > 0$ for $0 \le k < K(n) := \inf\{j \ge 0 : N_n(j) = 0\}$,
(b) $N_{n+1}(2k) \wedge N_{n+1}(2k+1) \ge N_n(k)$ for $-2^n < k$.

This conditional law is entirely free of the given L. We do this by defining $\mathcal{L}(R_n, 0 \leq n \mid N_.)$, which solves our problem since $P\{\lim_{n\to\infty} R_n(t) = B'(t)$ uniformly in $0 \leq t \leq T(-1) \mid N(\cdot)\} = 1$, P-a.s. The first step is to define $\mathcal{L}(R_0 \mid N_.)$ which, in accordance with Theorem 2.1 at $n = 0$ is simply $\mathcal{L}(R_0 \mid N_0)$. But this has already been done in Section 1 (taking again $n = 0$). For induction, suppose we have defined $\mathcal{L}(R_n \mid N_.)$ (since we have $\sigma(R_0, \ldots, R_n) = \sigma(R_n)$, this gives $\mathcal{L}(R_k,\ 0 \leq k \leq n \mid N_.)$, and by Theorem 2.1 it depends only on $(N_0, \ldots, N_n)$). The induction step reduces to defining the law $\mathcal{L}(R_{n+1} \mid R_n, N_{n+1})$ (since $\sigma(N_0, \ldots, N_n) \subset \sigma(R_n)$). Once this conditional law is defined for general (and hence by induction for every) n our problem is reduced to a last appeal to the Kohmogorov-Bochner extension theorem for the consistent families of laws $\mathcal{L}(R_n \mid N_.)$. Here the canonical space of each R_n is countable, and there is no serious problem in topologizing the projective limit space so as to apply the extension theorem. This was done in [3] for the unconditional case, and deserves no further mention here. So it remains only to define $\mathcal{L}(R_{n+1} \mid R_n, N_{n+1})$, and here we may take $n = 1$ for convenience.

The picture to keep in mind is the reduction of B' to its 1-inserts (Lemma 1.5), which are independent when R_1 is given. The effect of being given also N_2 is to enumerate how many 2-inserts there are with given starting values $k\alpha_2$ and given sign ± 1 (i.e. up or down). The problem is simply to interpolate into the 1-inserts (whose starting and ending values are dictated by R_1) the internal R_2-steps, subject to the total multiplicities N_2.

To derive the law of this interpolation (which in non-explicit terms is just the law that all possible R_2-paths are equally likely) we need to examine the law of the $(n+1)$-upcrossings embedded into an n-insert. This is the same for every n, so taking $n = 1$, we introduce

Notation 4.1. Let V denote the number of upcrossings of $(-\alpha_2, 0)$ of a 1-insert, and let $U + 1$ be the number of upcrossings of $(0, \alpha_2)$ for the same 1-insert.

Lemma 4.2. *The pair (V, U) has a symmetric joint distribution. In terms of the notations* geo(p) *and* neg. bin.(n, p) *of Section 2, the marginal law of V and U is* geo$(\frac{2}{3})$ *and given $\{V = k\}$, the conditional law of U is* neg. bin.$(k+1, \frac{3}{4})$, *as is that of V given $\{U = k\}$.*

Proof. This is mostly a consequence of Theorem 2.3 when $k = 0$, $n = 1$, and $N_1(0) = 0$, $N_1(-1) = 0$. This gives $N_2(-1) \stackrel{d}{=} \text{neg. bin.}(1, \frac{2}{3}) = \text{geo}(\frac{2}{3})$, and, given $N_2(-1)$, $N_2(0) \stackrel{d}{=} \text{neg. bin.}(1 + N_2(-1), \frac{3}{4})$. Since in this case $N_2(-1)$ and $N_2(0)$ are entirely contained in a single insert starting at 0 and ending at -1, we have the identification $(V, U) \stackrel{d}{=} (N_2(0), N_2(-1))$, where we noted that since this insert has sign -1 the roles of V and U are interchanged. It remains only to note that the sign of the first 1-insert described by B' is determined at the last exit of B' from 0 before reaching $\{\pm 1\}$, independently of the crossings from 0 to $\{\pm\frac{1}{2}\}$ before that time. If we discount the single crossing to $\{\pm\frac{1}{2}\}$ after this last exit (as we did in defining U), the remaining crossings are determined by symmetric Beromoulli trials, hence the law of (V, U) is also symmetric (from an analytic viewpoint, this was already treated before Theorem 2.2).

Remark. It follows easily, although not needed below, that $P\{V = i,\ U = j\} = \frac{1}{2}\binom{i+j}{j}\frac{1}{4}^{i+j};\ 0 \le i, j$.

We return to our problem of the interpolation of the 2- inserts into the 1-inserts when (R_1, N_2) is given. Actually we need only determine the joint law of the total numbers of 2-inserts of sign $+1$ in each 1-insert. Indeed, for a 1-insert of sign $+1$ starting at $k/2$ there are no 2-inserts $\frac{(k+1)}{2} \downarrow \frac{2k+1}{4}$; the number of 2-inserts $\frac{2k+1}{4} \downarrow \frac{k}{2}$ is 1 less than the number $\frac{k}{2} \uparrow \frac{2k+1}{4}$, the number $\frac{k}{2} \downarrow \frac{2k-1}{4}$ equals the number $\frac{2k-1}{4} \uparrow \frac{k}{2}$, and there are none $\frac{2k-1}{4} \downarrow \frac{k-1}{2}$. Similar reasoning applies to a 1-insert of sign -1. Now if R_1 is given but not N_2, the numbers of 2-inserts of sign $+1$ at each level in each 1-insert are determined independently (by independence of 1-inserts; Lemma 1.5) with the law of (V, U) from Lemma 4.2 (if the 1-insert has sign $+1$, we must add 1 to U). When N_2 is also given, for each k we are given the total number of 2-inserts from $\frac{k}{2} - \frac{1}{4} \uparrow \frac{k}{2}$, and the total number from $\frac{k}{2} \uparrow \frac{k}{2} + \frac{1}{4}$. Now 2-inserts $\frac{k}{2} - \frac{1}{4} \uparrow \frac{k}{2}$ can only occur during 1-inserts $\frac{k}{2} \uparrow \frac{k+1}{2}$, or during 1-inserts $\frac{k}{2} \downarrow \frac{k-1}{2}$, *except that* exactly one occurs during each 1-insert $\frac{k-1}{2} \uparrow \frac{k}{2}$, and the same holds for 2-inserts $\frac{k}{2} \uparrow \frac{k}{2} + \frac{1}{4}$ without exception. Conversely, these two types of 1-insert have all their embedded 2-inserts of sign 1 partitioned among these two types, *except for* one extra 2-insert from $\frac{k}{2} + \frac{1}{4} \uparrow \frac{k+1}{2}$ in each 1-insert $\frac{k}{2} \uparrow \frac{k+1}{2}$. It follows that the conditional law of the numbers of embedded 2-inserts of sign 1 in the 1-inserts, given (R_1, N_2), is determined independently for each k, by the conditional law of the embedded 2-inserts of sign 1 in the one inserts starting at $\frac{k}{2}$. This is slightly different for $k > 0$ and for $-2 < k \le 0$. We introduce a notation for the random frequencies whose conditional law is to be determined. Let $1 + U_i^+(k)$, $1 \le i \le N_1(k)$, denote the numbers of upcrossings $\frac{k}{2} \uparrow \frac{k}{2} + \frac{1}{4}$ in the successive 1-inserts of sign 1 starting at $\frac{k}{2}$ (the order being that determined by R_1), and let $V_i^+(k)$, $1 \le i \le N_1(k)$, denote the analogous numbers of upcrossings $\frac{k}{2} - \frac{1}{4} \uparrow \frac{k}{2}$. Similarly, for $k > 0$, let $V_i^-(k)$, $1 \le i \le N_n(k-1)$, denote the numbers of upcrossings of $\frac{k}{2} \uparrow \frac{k}{2} + \frac{1}{4}$ in the successive 1-inserts of sign -1 starting at $\frac{k}{2}$ (and for $-2 < k \le 0$, the same notation with $1 \le i \le N_n(k-1) + 1$), and similarly let $U_i^-(k)$ denote the numbers of upcrossings $\frac{k}{2} - \frac{1}{4} \uparrow \frac{k}{2}$ (since the 1-inserts have sign -1, U^- counts those farthest in the direction of advance). Then for $0 < k \le K(1)$, we have easily

$$\sum_{i=1}^{N_1(k)} U_i^+(k) + \sum_{i=1}^{N_1(k-1)} V_i^-(k) = N_2(2k) - N_1(k); \tag{4.2}$$

$$\sum_{i=1}^{N_1(k)} V_i^+(k) + \sum_{i=1}^{N_1(k-1)} U_i^-(k) = N_2(2k-1) - N_1(k-1), \tag{4.3}$$

where we subtracted $N_1(k)$ on both sides of the first equation for reasons of symmetry. For $-2 < k \le 0$, the analogous equations hold except that $N_1(k-1)$ in the two upper limits is replaced by $N_1(k-1) + 1$.

We observe that if R_1 is given but N_2 is not given, then by independence of inserts and Lemma 4.2, the left sides of (4.2) and (4.3) have the same law, namely a neg. bin.$(N_1(k)+N_1(k-1),\frac{2}{3})$ if $k>0$, or a neg. bin.$(N_1(k)+N_1(k-1)+1,\frac{2}{3})$ if $-2<k\le 0$. But suppose the collection $S:=\{U_i^+(k),\ 1\le i\le N_1(k);\ V_i^-(k),\ 1\le i\le N_1(k-1)\}$ for $k>0$ is also given (replace $N_1(k-1)$ by $N_1(k-1)+1$ if $k\le 0$). Then by Lemma 4.2 the left side of (4.3) has the law neg. bin.$(N_2(2k)-N_1(k)+N_1(k)+N_1(k-1),\frac{3}{4})=$ neg. bin.$(N_2(2k)+N_1(k-1),\frac{3}{4})$, depending only on the single parameter $N_2(2k)$ from N_2. Therefore, given $N_2(2k)$ (as well as R_1), the collection S is independent of $N_2(2k-1)$. So if (R_1,N_2) is given, the law of S is the same as if only $(R_1,N_2(2k))$ were given, i.e. the sum $N_2(2k)-N_1(k)$ is the only restraint on the otherwise independent terms (4.2). It follows that the joint law of S is determined by Bose-Einstein sampling of size $(N_1(k)+N_1(k-1),\ N_2(2k)-N_1(k))$ if $0<k\le K(1)$, or of size $(N_1(k)+N_1(k-1)+1,\ N_2(2k)-N_1(k))$ if $-2<k\le 0$ (see Definition 1.2 for the probabilities). Of course, by symmetry an analogous fact holds for the collection on the left of (4.3). It remains only to discuss the joint law of the two collections. Suppose, to this effect, that the collection S is given. Since the pairs $(V_i^+(k),U_i^+(k))$ and $(V_j^-(k),U_j^-(k))$ are jointly independent over i and j when S is not given, it follows from Lemma 4.2 that when S is given (but not $N_2(2k-1)$) the variables $V_i^+(k)$ and $U_j^-(k)$ are mutually independent over i and j (they all pertain to different 1-inserts), and the conditional law of $V_i^+(k)$ is neg. bin.$(U_i^+(k)+1,\frac{3}{4})$ while that of $U_j^-(k)$ is neg. bin.$(V_j^-(k)+1,\frac{3}{4})$. Let us introduce a corresponding

Definition 4.3. Integer-valued random variables $X_i\ge 0,\ 1\le i\le n$, have the law of a *Bose-Einstein partition* of size $(x_1,\dots,x_n;N)$ if the joint law is obtained by defining $(Y_1,\dots,Y_K)$ by Bose-Einstein sampling of size (K,N), where $K=\sum_{i=1}^n x_i$, and then combining to get $X_1=\sum_{i=1}^{x_1}Y_i;\ X_2=\sum_{i=x_1+1}^{x_1+x_2}Y_i,\dots,X_n=\sum_{i=x_1+\cdots+x_{n-1}+1}^{x_1+\cdots+x_n}Y_i$.

Lemma 4.4. *In a Bose-Einstein partition of size $(x_1,\dots,x_n;N)$ we have (setting again $K=\sum_{i=1}^n x_i$)*

$$P\{X_1=k_1,\dots,X_n=k_n\}=\binom{K+N-1}{N}^{-1}\prod_{j=1}^n\binom{x_j+k_j-1}{k_j};$$
$$\sum_1^n k_j=N.$$

Proof. The first factor on the right is the probability of a sample point in Bose-Einstein sampling of size (K,N). For the same reason, the factors in the product are the numbers of sample points in Bose-Einstein sampling of size (x_j,k_j), which is the same as the number of occupancy x_j-tuples of x_j cells by k_j balls. The event $\bigcap_1^n\{X_j=k_j\}$ occurs if and only if each consecutive set of x_j cells receives k_j balls. Clearly the total number of such occupancies is just the product, as asserted.

Now the gist of the discussion preceding Definition 4.3 for $k > 0$ is

(a) the conditional joint law of S given (R_1, N_2) is Bose-Einstein of size $(N_1(k) + N_1(k-1),\ N_2(2k) - N_1(k))$, and

(b) given S as well as (R_1, N_2), the summands on the left of (4.3) have law determined as that of independent, neg. bin.$(U_i^+(k)+1, \frac{3}{4})$ and neg. bin.$(V_j^-(k)+1, \frac{3}{4})$ random variables whose total sum is given by $N_2(2k-1) - N_1(k-1)$. It is easy to see by writing neg. bin.(n,p) as a sum of n independent geo(p) terms in each of these negative binomials that the effect is to determine the conditional law of $\{V_i^+(k), U_j^-(k)\}$ by Bose-Einstein partitioning of size $(U_i^+(k) + 1$ $(1 \leq i \leq N_1(k))$, $V_j^-(k) + 1$ $(1 \leq j \leq N_1(k-1))$; $N = N_2(2k-1) - N_1(k-1)$ if $0 < k \leq K(1)$ and for $-2 < k \leq 0$ one need only replace $N_1(k-1)$ by $N_1(k-1)+1$ in the range of j. Of course, a parallel argument applies for every $n \geq 0$ to the conditional law of the numbers of $(n+1)$-inserts interpolated into the n-inserts when (R_n, N_{n+1}) is given. To determine, now, the conditional law of R_{n+1}, it remains only to condition also on these numbers of $(n+1)$-inserts (as we noted before, the numbers of sign -1 are determined by those of sign 1). Then by the independence of n-inserts given R_n, the *ordering* of the two types of random $(n+1)$-inserts (namely, those from $k\alpha_n$ to $k\alpha_n + \alpha_{n+1}$, and those from $k\alpha_n - \alpha_{n+1}$ to $k\alpha_n$) is made independently in each n-insert, given the totals of each type, and moreover this ordering determines the R_{n+1}-path uniquely. Indeed for an n-insert of sign 1, each step $k\alpha_n$ to $k\alpha_n + \alpha_{n+1}$ except the last (hence, a total of $U_i^+(k)$ if $n = 1$) is followed by a step from $k\alpha_n + \alpha_{n+1}$ back to $k\alpha_n$, while each from $k\alpha_n - \alpha_{n-1}$ to $k\alpha_n$ is preceded by one from $k\alpha_n$ to $k\alpha_n - \alpha_{n+1}$ (a total of $V_i^+(k)$ if $n = 1$), and an analogous argument applies to n-inserts of sign -1. This obviously determines the R_{n+1}-path uniquely. Finally, it is clear from the symmetry of R_{n+1} that these orderings are all equally likely, given the totals (for example, given $(U_i^+(k),\ V_i^-(k))$ if $n = 1$ and the 1-insert in question is the i^{th} of sign $+1$ starting from $\frac{k}{2}$). For given totals (U, V), the number of such orderings is $\binom{U+V}{V}$.

To write the conditional probability of a choice of $\{U_i^+(k),\ V_i^-(k)\}$ and $\{V_i^+(k),\ U_j^-(k)\}$, $1 \leq i \leq N_1(k)$ and $1 \leq j \leq N_1(k-1)$ for $0 < k \leq K(1)$ (resp.$1 \leq j \leq N_1(k-1)+1$ if $-2 < k \leq 0$), given (R_1, N_2), we need only multiply the Bose-Einstein probabilities of the former by the Bose-Einstein partition probabilities of the later, and then divide by the corresponding product of the order counts $\binom{U+V}{V}$. When we carry this out, it is seen that the order counts precisely cancel the products in the numerator of the Bose-Einstein partition probabilities (i.e. the $\prod_{j=1}^{n} \binom{x_j + k_j - 1}{k_j}$ in Lemma 4.4) leaving only, for $0 < k \leq K(1)$, the expression

$$(4.4) \qquad \left[\binom{N_2(2k) + N_1(k-1) - 1}{N_2(2k) - N_1(k)} \binom{N_2(2k) + N_2(2k-1) - 1}{N_2(2k-1) - N_1(k-1)}\right]^{-1}$$

$$= \left[\binom{N_2(2k) + N_1(k-1) - 1}{N_2(2k) - N_1(k)} \binom{N_2(2k) + N_2(2k-1) - 1}{N_2(2k) + N_1(k-1) - 1}\right]^{-1}$$

$$= \left[(N_2(2k) + N_2(2k-1) - 1)!\right]^{-1} \cdot$$

$$(N_1(k) + N_1(k-1) - 1)!(N_2(2k) - N_1(k))!(N_2(2k-1) - N_1(k-1))! \ .$$

An analogous calculation $-2 < k \leq 0$ gives the result

$$\left[(N_2(2k) + N_2(2k-1))!\right]^{-1} \cdot \tag{4.5}$$

$$(N_1(k) + N_1(k-1))!(N_2(2k) - N_1(k))!(N_2(2k-1) - N_1(k-1))! \ .$$

Thus, the probability of a sample point of R_2 given (R_1, N_2) is the product of (4.4) over $k > 0$ times (4.5) for $-2 < k \leq 0$. To be sure, precisely the same result holds for any R_{n+1} given (R_n, N_{n+1}), replacing the subscripts and letting the product for $k \leq 0$ range from $-2^n < k \leq 0$. This may not appear especially simple, until one takes into account the amount of cancellation which has already occurred. Nevertheless, we shall state

Theorem 4.5. *For every $n \geq 0$, the conditional probability of a sample path of R_{n+1} given (R_n, N_{n+1}) is (either 0, if the point is inconsistent with (R_n, N_{n+1}), or) the product of (4.4) (with subscripts $(1,2)$ replaced by $(n, n+1)$) over $0 < k \leq K(n)$, and of (4.5) (with the same replacement) over $-2^n < k \leq 0$.*

Remark. One fact which does not seem quite obvious, but follows from this expression, is that for given (R_n, N_{n+1}), all possible paths of R_{n+1} are equally likely, their total number being the reciprocal of this product. Of course, if we change the given R_n, we get an entirely disjoint but equinumerous set of possible R_{n+1}-paths. Indeed, since the conditional result depends only on (N_n, N_{n+1}), when we change R_n keeping N_n fixed the cardinality of the set of possible R_{n+1}-paths does not change.

We want to consider, finally, the law of R_{n+1} given $(N_0, \ldots, N_{n+1})$. We obtain these probabilities by multiplying the conditional laws, viz $\mathcal{L}(R_0 \mid N_0) \cdot \mathcal{L}(R_1 \mid R_0, N_1) \cdot \mathcal{L}(R_2 \mid R_1, N_2) \ldots \mathcal{L}(R_{n+1} \mid R_n, N_{n+1})$.

Now by Lemma 1.3 $\mathcal{L}(R_0 \mid N_0)$ is determined by equally likely cases with the probability of a case being

$$p_0 = \left[\prod_{k=0}^{K(0)-1} \binom{N_0(k) + N_0(k+1) - 1}{N_0(k+1)}\right]^{-1} . \tag{4.6}$$

Next, by Theorem 4.5 $\mathcal{L}(R_1 \mid R_0, N_1)$ has equally likely cases with probability

$$\begin{aligned} &\left[(N_1(0) + N_1(-1)!\right]^{-1} N_0(0)!(N_1(0) - N_0(0))!(N_1(-1) - N_0(-1))! \\ &\cdot \prod_{k=1}^{K(0)} \left[(N_1(2k) + N_1(2k-1) - 1)!\right]^{-1} (N_0(k) + N_0(k-1) - 1)!(N_1(2k) \\ &- N_0(k))!(N_1(2k-1) - N_0(k-1))! \end{aligned} \tag{4.7}$$

The product of (4.6) and (4.7) gives the probability of a case for $\mathcal{L}(R_1 \mid N_0, N_1)$ as

$$
\begin{aligned}
p_1 &= \left[(N_1(0)+N_1(-1))!\right]^{-1} N_0(0)! \left[\prod_{k=0}^{K(0)} (N_1(2k)-N_0(k))!(N_1(2k-1) \right. \\
&\quad \left. - N_0(k-1))!(N_0(k)-1)!N_0(k+1)! \right] \cdot \\
&\quad \cdot \prod_{k=1}^{K(0)} \left[(N_1(2k)+N_1(2k-1)-1)!\right]^{-1} \quad \text{(where (-1)! := 1)} \\
&= \left[(N_1(0)+N_1(-1))!\right]^{-1} \left(\prod_{k=0}^{K(0)} (N_0(k)-1)!N_0(k)! \right) \\
&\quad \left(\prod_{k=0}^{K(0)} (N_1(2k)-N_0(k))!(N_1(2k-1)-N_0(k-1))! \right) \\
&\quad \cdot \left[\prod_{k=1}^{K(0)} (N_1(2k)+N_1(2k-1)-1)! \right]^{-1}
\end{aligned}
\tag{4.8}
$$

Now to write $\mathcal{L}(R_2 \mid N_0, N_1, N_2)$ we have to multiply (4.8) by the product of (4.4) over $0 < k \leq K(1)$ and of (4.5) over $-2 < k \leq 0$. We get for the probabilities

(4.9)

$$
\begin{aligned}
p_2 &= \left(\prod_{k=0}^{K(0)} (N_0(k)-1)!N_0(k)! \right) \left[N_1(-1) \prod_{k=1}^{K(0)} (N_1(2k-1)+N_1(2k-1)-1)! \right] \cdot \\
&\quad \cdot \left[\prod_{k=0}^{K(0)} (N_1(2k)-N_0(k))!(N_1(2k-1)-N_0(k-1))! \prod_{k=-1}^{K(0)} (N_2(2k) \right. \\
&\quad \left. - N_1(k))!(N_2(2k-1)-N_1(k-1))! \right] \\
&\quad \cdot \left[\prod_{k=-1}^{0} (N_2(2k)+N_2(2k-1))! \prod_{k=1}^{K(1)} (N_2(2k)+N_2(2k-1)-1)! \right]^{-1} .
\end{aligned}
$$

To detect the general rule, it is necessary only to write the next case $\mathcal{L}(R_3 \mid N_0, N_1, N_2, N_3)$ by the same method. Omitting the details, we obtain

Theorem 4.6. *For every $n \geq 0$, $\mathcal{L}(R_n \mid N_{(\cdot)}, L) = \mathcal{L}(R_n \mid N_0, \ldots, N_n)$ is determined by equally likely cases over the set of consistent R_n-paths. The probabilities are given by (4.6) for $n = 0$, by (4.8) for $n = 1$, by (4.9) for $n = 2$, and* *for $n \geq 2$ they have the form*

$$p_n = T_0 \left(\prod_{j=1}^{n-1} T_1(j) \right) \left(\prod_{j=0}^{n-1} T_2(j) \right) T_n^{-1}, \quad \textit{where}$$

$$T_0 := \prod_{k=1}^{K(0)} (N_0(k)-1)!(N_0(k))!; T_1(j) := \prod_{-2^{j-1}}^{-1} (N_j(2k+1)+N_j(2k))!.$$

$$\prod_{k=1}^{K(j-1)} (N_j(2k-1)+N_j(2(k-1))-1)!;$$

$$T_2(j) := \prod_{-2^j+1}^{K(j)} (N_{j+1}(2k)-N_j(k))!(N_{j+1}(2k-1)-N_j(k-1))!, T_n :=$$

$$\prod_{-2^{n-1}+1}^{0} (N_n(2k)+N_n(2k-1))! \prod_{k=1}^{K(n-1)} (N_n(2k)+N_n(2k-1)-1)!\ .$$

We observe here that, except for $T_2(j)$, each factor depends only on N_j for a simple j, while $T_2(j)$ depends only on (N_j, N_{j+1}). The terms $T_1(j)$ and T_n are quite similar, but $T_1(j)$ combines pairs $(2k, 2k+1)$ while T_n combines pairs $(2k-1, 2k)$. The terms $T_2(j)$ represent the product of the extra $(j+1)$-upcrossings beyond those necessitated by the j-upcrossings of the α_j-intervals containing them. Note, finally, that it is the reciprocal of T_n that figures in p_n.

Final Remark. We have considered B stopped at $T(-1)$, but of course B stopped at any $c \neq 0$ can be covered by scale changes if we replace α_n by $|c|\alpha_n$. It takes only a little more thought to see that B stopped at $\inf\{t : L(t,0) \geq c\}$, $c > 0$, is also covered. Actually, for this we need only adapt the arguments for $k \geq 0$ to construct B' on $[0,\infty)$ reflected at 0 (the excursions into $(-\infty, 0)$ are spliced out). To see this note that, by excursion theory from 0, B' on $[0,\infty)$ may be constructed by replacing $T(-1)$ by $\inf\{t : L(t,0) = e\}$ where e is exponential, independent of B (since $L(T(-1),0) \overset{(d)}{=} e$). Now given $e = c$, the law of $\{L(x), x \geq 0\}$ is the same as its law for c in the other case, and the same is true of the reflected B'.

Furthermore, the dependence of N_n on $L(x)$ is local, in such a way that $\mathcal{L}(N_n(k), 0 \leq k \mid L\}$ depends only on $\{L(x), 0 \leq x\}$. Thus stage one of our construction carries over. Stage 2, to construct $\mathcal{L}(R_n, 0 \leq n \mid N)$, also carries over if R_n is replaced by its analog on $[0,\infty)$ (splicing out the negative excursions). For $n = 0$, we need only determine the variables $X_k(j)$ for $0 \leq k$ and $1 \leq j \leq N_0(k)$. The same argument gives a Bose-Einstein law when N_0 is given, and these determine R_0 in each positive excursion by the (classical) branching process argument. The induction step, to define the law $\mathcal{L}(R_{n+1} \mid R_n, N_{n+1})$, relied on the independence of inserts, which is equally valid in the reflected case. The only change needed beyond ignoring $k < 0$ is that, for $k = 0$, we again have a simple Bose-Einstein distribution of the $N_{n+1}(0)$ upcrossings among the $N_n(0)$

n-inserts, since only the crossings $0 \uparrow \alpha_n$ can contribute. For $k > 0$, however, the argument of Section 4 remains applicable. Thus we again have equally likely cases, and the precise expressions are analogous to those of Theorem 4.6 but somewhat simpler.

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