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SOME CALCULATIONS FOR PERTURBED BROWNIAN MOTION

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1. INTRODUCTION

If B is a standard Brownian motion starting from zero, and $\bar{B}_t = \sup_{0 \leq s \leq t} B_s$, then the process X defined by

$$(1) \quad X_t = B_t + \frac{\alpha}{1-\alpha} \bar{B}_t,$$

where $\alpha < 1$ is called an α -perturbed Brownian motion. It is immediate from (1) that if $\bar{X}_t = \sup_{0 \leq s \leq t} X_s$ then

$$(2) \quad \bar{X}_t = \frac{1}{1-\alpha} \bar{B}_t,$$

so that (1) shows that X is the unique pathwise solution of the functional equation

$$(3) \quad X_t = B_t + \alpha \bar{X}_t.$$

This is a special case of the equation

$$X_t = B_t + \alpha \bar{X}_t + \beta \underline{X}_t,$$

where $\underline{X}_t = \inf_{0 \leq s \leq t} X_s$, which has been studied by a number of authors; see ([3], [5], [4], and [8]). It should also be mentioned that, by the Lévy equivalence, (1) can be written as

$$-X_t = W_t - (1-\alpha)^{-1} L_t,$$

where W is a reflected Brownian motion whose local time at zero is L , so X is often referred to as “reflected Brownian motion perturbed by its local time”. (See e.g. [11].)

From (1) it is clear that X is a non-Markov process which moves like Brownian motion except when it is at its maximum, and, moreover, X has the Brownian scaling property. Many other results known for Brownian motion have analogues for perturbed Brownian motion, including Lévy’s Arc-sine law, the Ray-Knight theorems, and the solution to the two-sided exit problem. (See [7], [2], [10], and [11].)

In this note we give an excursion theory approach, based on the excursions of X away from its maximum, which leads to simple proofs of some of these results, and to new ones. In particular, we give new proofs of the Ray-Knight theorems and extend the results known about the two-sided exit problem by computing the transition density of the bivariate Markov process $(X, \bar{X})$, killed when X exits the interval, at an exponential time. From these results we are able to deduce some information about “ X conditioned to stay positive”.

The basis for our calculations is the following observation; write $P^{(\alpha)}$ for the measure of X and $n^{(\alpha)}$ for the characteristic measure, under $P^{(\alpha)}$, of the excursions away from zero of $\bar{X} - X$. Note that $n = n^{(0)}$ coincides with the characteristic measure of excursions away from zero of reflected Brownian motion.

Proposition 1.1. *The measures $n^{(\alpha)}$ and n are related by*

$$(4) \quad n^{(\alpha)} = (1 - \alpha)n.$$

Proof. From (1) and (3) we have

$$\bar{X}_t - X_t = (1 - \alpha)^{-1} \bar{B}_t - \{B_t + \alpha(1 - \alpha)^{-1} \bar{B}_t\} = \bar{B}_t - B_t,$$

which tells us that $n^{(\alpha)}$ is a multiple of n . But (2) tells us that the local times at zero of $\bar{X} - X$ and $\bar{B} - B$ are related by $l^{(\bar{X}-X)} = (1 - \alpha)^{-1} l^{(\bar{B}-B)}$, and this identifies the constant. ■

2. RAY-KNIGHT THEOREMS

Let L_t^x denote a jointly continuous version of the local time at level x and time t of X , and write Q_x^δ for the law of the square of a Bessel process of dimension δ starting from x .

Theorem 2.1. For fixed $b > 0$ let $Z = \{Z(x), 0 \leq x \leq b\}$, where $Z_x = L_{T_b}^{b-x}$. Then the law of Z is the restriction to $[0, b]$ of $Q_0^{2\bar{\alpha}}$, where $\bar{\alpha} = 1 - \alpha$.

Proof. Since the result is classical for $\alpha = 0$, it follows from the Lévy-Khintchine representation of Q_0^δ (see Theorem 3.2, p30 of [11]) that it suffices to show that for any Borel function $f \geq 0$

$$P^{(\alpha)}\left\{\exp - \int_0^b f(x)Z(x)dx\right\} = [P^{(0)}\{\exp - \int_0^b f(x)Z(x)dx\}]^{\bar{\alpha}}.$$

However, if we write $g(\cdot) = f(b - \cdot)$, the occupation density theorem gives

$$\int_0^b f(x)Z(x)dx = \int_0^{T_b} g(X_s)ds = \int_0^{\tau_b} g(l_s - Y_s)ds,$$

where $Y = \bar{X} - X$ and τ is the inverse of $l = l^{(Y)}$. Applying the master formula of excursion theory gives, with $\zeta = \zeta(\varepsilon)$ standing for the lifetime of a generic excursion ε ,

$$\begin{aligned} P^{(\alpha)}\left\{\exp - \int_0^b f(x)Z(x)dx\right\} &= \exp - \left\{\int_0^b dt \int_{\Omega} n^{(\alpha)}(d\varepsilon)[1 - \exp - \int_0^\zeta g(t - \varepsilon(u))du]\right\} \\ &= [P^{(0)}\{\exp - \int_0^b f(x)Z(x)dx\}]^{\bar{\alpha}} \end{aligned}$$

by virtue of (4), and the result follows. ■

Next, we deduce the second Ray-Knight theorem. We write σ for the inverse of L^0 and $\tilde{Q}_x^\delta$ for the measure of the square of a Bessel process of dimension δ , starting from x and killed on hitting zero.

Theorem 2.2. For fixed $t > 0$ let $U^{(t)} = \{U_x^{(t)}, x \geq 0\}$, where $U_x^{(t)} = L^x(\sigma_t)$. Then under $P^{(\alpha)}$ the law of $U^{(t)}$ is $\tilde{Q}_t^{2\alpha}$.

Proof. For $x_0 > 0$ it is clear that, given $L^{x_0}(\sigma_t) = t_0$, $\{L^{x_0+x}(\sigma_t), x \geq 0\}$ is independent of $\{L^y(\sigma_t), 0 \leq y < x_0\}$, and is distributed as $U^{(t_0)}$. Thus $\{U_x^{(t)}, x \geq 0\}$ is Markov, and the result will follow if we can show that, for all Borel subsets A of $[0, \infty)$ and any $t > 0, x > 0$,

$$(5) \quad P^{(\alpha)}\{U_x^{(t)} \in A\} = \tilde{Q}_t^{2\alpha}\{X_x \in A\}.$$

Now by Theorem 2,

$$P^{(\alpha)}\{U_x^{(t)} = 0\} = P^{(\alpha)}\{L^0(T_x) > t\} = Q_0^{2\bar{\alpha}}\{X_x > t\},$$

whereas, writing $\lambda_t = \sup\{s : X_s = t\}$, it follows by time reversal (see e.g. Ex.1.23, p420 of [9]) that

$$\tilde{Q}_t^{2\alpha}\{X_x = 0\} = Q_t^{2\alpha}\{T_0 \leq x\} = Q_0^{2+2\bar{\alpha}}\{\lambda_t \leq x\}.$$

Finally, using the scaling property and the fact that the $Q_0^{2+2\bar{\alpha}}$ distribution of λ_1 coincides with the $Q_0^{2\bar{\alpha}}$ distribution of $\{X_1\}^{-2}$ (see Ex 1.18, p418 of [9]), we see that (5) holds for $A = \{0\}$. Next, on $\{U_x^{(t)} > 0\}$, we set $\tilde{T} = \inf\{s > T_x : X(s) = 0\}$, and write

$$(6) \quad U_x^{(t)} = L^x(\tilde{T}) + \{L^x(\sigma_t) - L^x(\tilde{T})\}.$$

Since the excursions of X below x after time T_x have the same structure as the excursions below zero of a Brownian motion, it is clear that, given $L^0(T_x) = s$, the terms on the RHS of (6) are independent and, by the Ray-Knight theorems for Brownian motion, have the distribution of X_x under Q_0^2 and Q_{t-s}^0 respectively. Using the composition law for squares of Bessel processes (Theorem 1.2, p410 of [9]) and appealing again to Theorem 2 gives

$$\begin{aligned} P^{(\alpha)}\{U_x^{(t)} \in dy\} &= \int_0^t Q_0^{2\bar{\alpha}}\{X_x \in ds\} Q_{t-s}^2\{X_x \in dy\} \\ &= \frac{dy}{dt} \int_0^t Q_0^{2\bar{\alpha}}\{X_x \in ds\} Q_y\{X_x \in dt - s\} \\ &= \frac{dy}{dt} Q_y^{2+2\bar{\alpha}}\{X_x \in dt\}. \end{aligned}$$

Finally, time reversal gives

$$\frac{1}{dt} Q_y^{2+2\bar{\alpha}}\{X_x \in dt\} = \frac{1}{dy} Q_t^{2\alpha}\{X_x \in dy; T_0 > x\} = \frac{1}{dy} \tilde{Q}_t^{2\alpha}\{X_x \in dy\},$$

which completes the proof of (5). ■

3. THE PROCESS KILLED ON LEAVING $[-a, b]$.

We will write $S = S(a, b) = T_{-a} \wedge T_b$ for the first exit time of $[-a, b]$, and V_{θ^*} for an independent, exponentially distributed random variable with parameter $\theta^* = \theta^2/2$.

Theorem 3.1. *It holds that, for $a > 0, b > 0$,*

$$(7) \quad P^{(\alpha)}\{S \leq V_{\theta^*}; X(S) = b\} = \left(\frac{\sinh a\theta}{\sinh(a+b)\theta} \right)^{\bar{\alpha}},$$

for $0 < y < b$,

$$(8) \quad P^{(\alpha)}\{S \leq V_{\theta^*}; X(S) = -a, \bar{X}(S) \in dy\} = \frac{\bar{\alpha}\theta(\sinh a\theta)^{\bar{\alpha}}}{\{\sinh(a+y)\theta\}^{\bar{\alpha}+1}} dy,$$

and for $-a < z < y, 0 < y < \infty$,

$$(9) \quad P^{(\alpha)}\{T_{-a} > V_{\theta^*}; X(V_{\theta^*}) \in dz, \bar{X}(V_{\theta^*}) \in dy\} = \frac{\bar{\alpha}\theta^2(\sinh a\theta)^{\bar{\alpha}} \sinh(a+z)\theta}{\{\sinh(a+y)\theta\}^{\bar{\alpha}+1}} dy dz.$$

Proof. Write $A(\theta^*, c)$ for $\{\varepsilon : \zeta(\varepsilon) > V_{\theta^*}\} \cup \{\varepsilon : \zeta(\varepsilon) \leq V_{\theta^*}, \bar{\varepsilon}(\zeta) > c\}$, and recall that

$$(10) \quad n(A(\theta^*, c)) = \theta \coth c\theta.$$

Then $P^{(\alpha)}\{S \leq V_{\theta^*}; X(S) = b\} = P^{(\alpha)}\{\phi > b\}$, where $\phi = \inf\{s : \varepsilon_s \in A(\theta^*, a + s)\}$. Thus

$$\begin{aligned} P^{(\alpha)}\{S \leq V_{\theta^*}; X(S) = b\} &= \exp\left\{-\int_0^b n^{(\alpha)}(A(\theta^*, a + s))ds\right\} \\ &= \exp\left\{-\bar{\alpha} \int_0^b \theta \coth(a + s)\theta ds\right\}, \end{aligned}$$

from (4) and (10), and (7) follows. Also

$$\begin{aligned} P^{(\alpha)}\{S \leq V_{\theta^*}; X(S) = -a\} &= P^{(\alpha)}\{T_{-a} < T_b \wedge V_{\theta^*}\} \\ &= \int_0^b P^{(\alpha)}\{\phi > y\} n^{(\alpha)}\{\varepsilon : T_{a+y} < \zeta(\varepsilon) \wedge V_{\theta^*}\} dy \\ &= \int_0^b \left\{ \frac{\sinh a\theta}{\sinh(a + y)\theta} \right\}^{\bar{\alpha}} \frac{\bar{\alpha}\theta}{\sinh(a + y)\theta} dy, \end{aligned}$$

where we have used another standard result for Brownian motion, and this is equivalent to (8). Similarly

$$\begin{aligned} P^{(\alpha)}\{S > V_{\theta^*}; X(V_{\theta^*}) \in dz\} \\ = \int_{z^+}^b P^{(\alpha)}\{\phi > y\} n^{(\alpha)}\{\varepsilon : \zeta(\varepsilon) > V_{\theta^*}, \bar{\varepsilon}(V_{\theta^*}) \leq a + y, \varepsilon(V_{\theta^*}) \in y - dz\} dy \end{aligned}$$

and since, for $0 < u < v$

$$\begin{aligned} n\{\varepsilon : \zeta(\varepsilon) > V_{\theta^*}, \bar{\varepsilon}(V_{\theta^*}) \leq v, \varepsilon(V_{\theta^*}) \in du\} \\ = n\{\varepsilon : T_u < \zeta(\varepsilon) \wedge V_{\theta^*}\} P_u^{(0)}\{X(V_{\theta^*}) \in du, T_0 \wedge T_v > V_{\theta^*}\} \\ = \left\{ \frac{\theta}{\sinh u\theta} \right\} \cdot \left\{ \frac{\theta \sinh u\theta \sinh(v - u)\theta du}{\sinh v\theta} \right\} = \frac{\theta^2 \sinh(v - u)\theta}{\sinh v\theta} du, \end{aligned}$$

(9) is also immediate. ■

From this some known results in [2] and [8] follow immediately.

Corollary 3.2. *For α -perturbed Brownian motion we have*

$$(11) \quad P^{(\alpha)}\{X \text{ exits } [-a, b] \text{ at } b\} = \left(\frac{a}{a+b}\right)^{\bar{\alpha}},$$

$$E^{(\alpha)}\{e^{-\theta^* T_b}\} = e^{-\bar{\alpha} b \theta},$$

and

$$E^{(\alpha)}\{e^{-\theta^* T_{-a}}\} = \int_0^\infty \frac{\bar{\alpha}\theta(\sinh a\theta)^{\bar{\alpha}}}{\{\sinh(a + y)\theta\}^{\bar{\alpha}+1}} dy.$$

We can also deduce some facts about X conditioned “to stay positive”;

Corollary 3.3. *It holds that*

$$(12) \quad \lim_{a \downarrow 0} \lim_{k \uparrow \infty} P^{(\alpha)} \{ e^{-\theta^* T_b} \mid X \text{ exits } [-a, k] \text{ at } k \} = \left(\frac{b\theta}{\sinh b\theta} \right)^{\bar{\alpha}},$$

and

$$(13) \quad \begin{aligned} \lim_{a \downarrow 0} \lim_{k \uparrow \infty} P^{(\alpha)} \{ X(V_{\theta^*}) \in dz \mid X \text{ exits } [-a, k] \text{ at } k \} \\ = \theta^{1+\bar{\alpha}} z \sinh z\theta \cdot \int_z^\infty \frac{\bar{\alpha}\theta}{y^\alpha \{\sinh y\theta\}^{\bar{\alpha}+1}} dy dz. \end{aligned}$$

Proof. Note that for $k > b$,

$$\begin{aligned} P^{(\alpha)} \{ T_b \leq V_{\theta^*} \mid X \text{ exits } [-a, k] \text{ at } k \} \\ = \frac{P^{(\alpha)} \{ S \leq V_{\theta^*}, X(S) = b \} P^{(\alpha)} \{ X \text{ exits } [-(a+b), k-b] \text{ at } k-b \}}{P^{(\alpha)} \{ X \text{ exits } [-a, b] \text{ at } b \}} \\ = \left\{ \frac{\sinh a\theta}{\sinh(a+b)\theta} \right\}^{\bar{\alpha}} \left\{ \frac{a+b}{a+k} \right\}^{\bar{\alpha}} \left\{ \frac{a}{a+k} \right\}^{\bar{\alpha}}, \end{aligned}$$

which does not depend on k . So (12) follows by letting $a \downarrow 0$.

Similarly, we see that for $z < y < k$

$$\begin{aligned} & P^{(\alpha)} \{ X(V_{\theta^*}) \in dz, \bar{X}(V_{\theta^*}) \in dy \mid \text{exits } [-a, k] \text{ at } k \} \\ = & P^{(\alpha)} \{ T_{-a} > V_{\theta^*}, X(V_{\theta^*}) \in dz, \bar{X}(V_{\theta^*}) \in dy \} P^{(0)} \{ X \text{ exits } [-(a+z), y-z] \text{ at } y-z \} \\ & \times \left\{ \frac{P^{(\alpha)} \{ X \text{ exits } [-(a+y), k-y] \text{ at } k-y \}}{P^{(\alpha)} \{ X \text{ exits } [-a, k] \text{ at } k \}} \right\} \\ = & \frac{\bar{\alpha}\theta^2 (\sinh a\theta)^{\bar{\alpha}} \sinh(a+z)\theta}{\{\sinh(a+y)\theta\}^{\bar{\alpha}+1}} dy dz \cdot \frac{z+a}{y+a} \cdot \left(\frac{a+y}{a+k} \right)^{\bar{\alpha}} \cdot \left(\frac{a+k}{a} \right)^{\bar{\alpha}}, \end{aligned}$$

and this leads to (13). ■

REMARK Using (13), it is not difficult to show that there is a probability measure $R^{(\alpha)}$ say, which is the weak limit of $P^{(\alpha)}(\cdot \mid X \text{ exits } [-a, k])$ as $k \uparrow \infty$ and $a \downarrow 0$, and it would be interesting to describe X under $R^{(\alpha)}$. Of course $R^{(0)}$ corresponds to the BES(3) process, and one way to realize that is as $|B_t| + L_t$, where L is the local time at zero of $|B|$. This suggests the process $\Sigma^{(\delta)} = |B| + \frac{2}{\delta} L$, which has been studied in [11], chapter 4, as a candidate to have the $R^{(\alpha)}$ measure, for some suitable δ . Furthermore, when $\delta = 2(1-\alpha)$, one can check that, under $P^{(\alpha)}$, the time-reversed process $\{1 - X_{T_1-t}, 0 \leq t \leq T_1\}$ has the same measure as $\{\Sigma_t^{(\delta)}, 0 \leq t \leq \lambda_1^{(\delta)}\}$, where $\lambda_1^{(\delta)} = \sup\{s : \Sigma_s^{(\delta)} = 1\}$. (I owe this observation, which extends a well-known connection between Brownian motion and BES(3), to Loic Chaumont.) However it follows from results in [1] that if $T^{(\delta)}$ is the hitting time process of Σ^δ , then

$$E\{e^{-\theta^* T_b^{(\delta)}}\} = \frac{\bar{\alpha}\theta}{(\sinh b\theta)^{\bar{\alpha}}} \int_0^b \frac{dy}{(\sinh y\theta)^\alpha}.$$

Since this disagrees with (12), we conclude that $\Sigma^{(\delta)}$ does not have $R^{(\alpha)}$ as its measure. This question is discussed further in [6].

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