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Criteria of regularity at the end of a tree

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Abstract

For a random walk on a tree, we give analogues of Wiener's test relatively to Dirichlet's problem for the endpoints of the tree.

Résumé

Étant donnée une marche aléatoire sur un arbre, nous établissons pour les points de la frontière des critères de régularité analogues à des critères classiques relatifs au problème de Dirichlet pour le mouvement brownien dans $\mathbb{R}^n$, dont celui de Wiener pour $n = 2$.

Keywords Dirichlet problem, resistance, regularity criteria.

1 Introduction

Let $\mathcal{A} = (A, \mathcal{U}, 0)$ be a non oriented infinite tree with a root : A is the set of vertices x, y, α etc., $\mathcal{U}$ the set of edges (x, y) or $[x, y]$, and 0 a fixed point in A. We denote by $x \sim y$ the symmetric relation $(x, y) \in \mathcal{U}$ and $d(x)$ the cardinality of $\{y \in A : x \sim y\}$. We suppose

$$2 \leq \inf_{x \in A} d(x) \leq \sup_{x \in A} d(x) < \infty$$

verified; in particular A is countably infinite. A *geodesic ray* (starting at 0) of $\mathcal{A}$ is any one to one sequence $\eta = (x_n)$ of vertices such that $x_0 = 0$ and $x_n \sim x_{n+1}$ for all $n \in \mathbb{N}$, and the *end* of $\mathcal{A}$ is the set of all geodesic rays.

We consider a *resistance* R on $\mathcal{A}$, i.e. a function from $\mathcal{U}$ to $\mathbb{R}_+$ such that $R[x, y] = R[y, x]$ for all $[x, y] \in \mathcal{U}$ and we associate to R a random walk $X = (X_n)_{n \geq 0}$ with transition $P(X_{n+1} = x / X_n = y) = p_{xy} = \dfrac{R[x, y]^{-1}}{\sum_{\{z : y \sim z\}} R[y, z]^{-1}}$ if $x \sim y$ and $= 0$ otherwise, where $p_{xy} = 1/d(y)$ if $x \sim y$ in the simple random walk ($R \equiv 1$). We denote by P_x the law of $X_0 = x$, $T_y = \inf\{n \geq 0 : X_n = y\}$ the first hitting time of $y \in A$, and $S_B = \inf\{n > 0 : X_n \in B\}$ the first return time to the subset B of A. *We assume in all this article* that X is *transient*, i.e. $P_x[S_{\{x\}} = \infty] > 0$ for all $x \in A$.

Following [1] we say that a geodesic ray $\eta = (x_n)_{n \in \mathbb{N}}$ is *regular* for the Dirichlet problem if $\lim_{n \to \infty} P_{x_n}[T_0 < \infty] = 0$; this is analogous to classical definition of a regular point of a Dirichlet problem. In [4] and [11], Wiener's test in the continuous case is presented. In [12], [6] the description of the Dirichlet problem on graph and conditions to obtain a regular problem are given. In [2] another description is given.

In §2 we establish a criterion of regularity, for geodesic ray for random walk on a tree, analogue in the simple case to Wiener's test [11], [4] for the brownian motion in $\mathbb{R}^2$, and we give the analogue of Frostman criterion.

In §3 we give a characterization of the regularity of a geodesic ray, analogous in the simple case to Wiener's test which we find in [5] for brownian motion in $\mathbb{R}^n$, $n \geq 2$. This characterization is based on the behaviour of the potential kernel in the neighbourhood of geodesic ray.

2 Electrical network and Wiener's test

To each $\alpha \in A$ we associate a partial order (orientation) $<_\alpha$ on A as : for $x \neq y$ we have $x <_\alpha y$ if and only if x belongs to a geodesic ray between y and α. We call a *flow started at* α any function I^α from $\mathcal{U}$ to $\mathbb{R}$ such that

1. $\sum_{y:\alpha\sim y} I^\alpha([\alpha, y]) = 1$ and $\sum_{y:\beta\sim y} I^\alpha([\beta, y]) = 0$ for all $\beta \neq \alpha$;
2. $I^\alpha([x,y]) = -I^\alpha([y,x])$ for all $[x,y] \in \mathcal{U}$ and $I^\alpha([x,y]) \geq 0$ if $x <_\alpha y$.

The *energy* of the flow I^α is the number $E(I^\alpha) = \frac{1}{2}\sum_{x\sim y} R[x,y] I^\alpha([x,y])^2$. Since the random walk X is transient, there exists a flow $\tilde{I}^\alpha$ starting at α with finite minimal energy (see [8] and [10]) $\tilde{E}^\alpha$ which we call the *resistance* of $\mathcal{A}$ *at* α and we denote it by $R_{\mathcal{A}}(\alpha)$. We think of $R_{\mathcal{A}}(\alpha)$ as the inverse of the ordinary capacity. If $\mathcal{B}$ is a subtree of $\mathcal{A}$ rooted at α, we define in the same way the resistance $R_{\mathcal{B}}(\alpha)$ of $\mathcal{B}$ at α if $\mathcal{B}$ is transient, and we put $R_{\mathcal{B}}(\alpha) = \infty$ if $\mathcal{B}$ is recurrent. Finally, if $\eta = (x_n)_{n\in\mathbb{N}}$ is a geodesic ray we denote by $\mathcal{A}_\eta(x_n)$ the subtree of $\mathcal{A}$ which has

$$\{x_n\} \cup \{x \in A,\ x_n <_{x_{n+1}} x,\ x_n <_0 x\},$$

as vertices and we denote by $R_\eta(k)$ the resistance $R_{\mathcal{A}_\eta(x_k)}(x_k)$ of $\mathcal{A}_\eta(x_k)$ at x_k.

We now give the analogue of Wiener's test for the tree [11]

Theorem 1 *Suppose that $R[x,y] \geq 1$ for all $[x,y] \in \mathcal{U}$. Then a geodesic ray $\eta = (x_n)$ is non regular if and only if*

$$\sum_{n=1}^{\infty} \frac{1}{R_\eta(n)} \sum_{k=1}^{n} R[x_k, x_{k-1}] < \infty;$$

in particular, if we have $R[x_k, x_{k+1}] = 1$ for all $k \in \mathbb{N}$, the geodesic ray $\eta = (x_n)_n$ is non regular if and only if

$$\sum_{n=1}^{\infty} \frac{n}{R_\eta(n)} < \infty.$$

We give an example before proving our theorem.

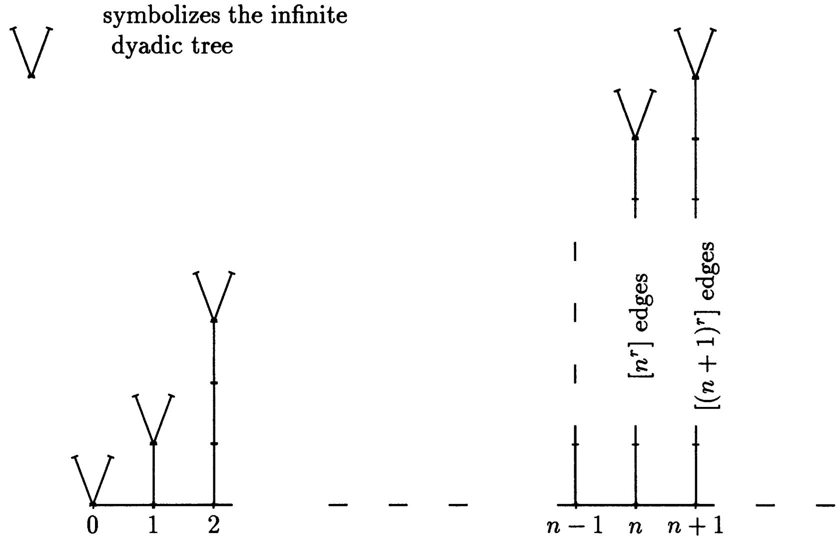

figure 1

Let us consider the simple random walk X in the tree $\mathcal{A}$ depending on the parameter r. Using the symmetry of $\mathcal{A}_\eta(n)$ we obtain

$$R_\eta(n) = [n^r] + 1.$$

According to Theorem 1 $\eta = (k)_{k\in N}$ is non regular if and only if $\sum_{n\in\mathbb{N}} n/R_\eta(n) < \infty$, which is equivalent to $r > 2$. Furthermore if $\alpha \leq 2$ then $\lim_{n\to\infty} P_n[T_0 < \infty] = 0$ but if $r > 2$ then $\lim_{n\to\infty} P_n[T_0 < \infty] > 0$.

We prove Theorem 1 in several steps.

First, if $R_\eta(k)$ is infinite for all k then the ray η is non regular because $R \geq 1$. Hence we can suppose that there exists k such that $R_\eta(k)$ is finite and then we can change x_k to 0. For simplification we suppose $R_\eta(0)$ is finite, which implies $\tilde{I}^{x_n}([x_k, x_{k-1}]) > 0$ for all $n \geq k \geq 1$.

Proposition 1 *Let $\eta = (x_n)_{n\in\mathbb{N}}$ be a geodesic ray.*

1) For all $k > 0$ the quantity

$$c_k = \frac{\tilde{I}^{x_n}([x_k, x_{k-1}])}{\tilde{I}^{x_n}([x_{k+1}, x_k])}$$

is independent of $n > k$.

2) A geodesic ray η is non regular if and only if $\prod_{n\in\mathbb{N}} c_n > 0$.

Proof of proposition 1 Part 1) is trivial. To prove 2) note that the flow starting at α defined by

$$I^\alpha[x,y] = \sum_{k\in\mathbb{N}} P_\alpha[X_k = x, X_{k+1} = y]\, P_y[T_x = \infty] \text{ if } x <_\alpha y,$$

is the flow of minimal energy $\tilde{E}^{x_n}$. This means that

$$\tilde{I}^{x_n}([x_1, x_0]) \leq P_{x_n}[T_0 < \infty].$$

Combining this inequality and the transience of X we easily deduce 2).

Proof of Theorem 1 Suppose $R_\eta(k) < \infty$ and $R_\eta(k+1) < \infty$. Using the minimality of the energy $\tilde{I}^{x_{k+2}}$, we obtain the equation of equilibrium

$$\begin{aligned} R[x_{k+1}, x_k]\tilde{I}^{x_{k+2}}([x_k, x_{k+1}]) \; &+ \; R_\eta(k)\{\tilde{I}^{x_{k+2}}([x_{k+1}, x_k]) - \tilde{I}^{x_{k+2}}([x_k, x_{k-1}])\} \\ &= \; R_\eta(k+1)\{\tilde{I}^{x_{k+2}}([x_{k+2}, x_{k+1}]) - \tilde{I}^{x_{k+2}}([x_{k+1}, x_k])\}. \end{aligned}$$

Dividing each term by $\tilde{I}^{x_{k+2}}([x_{k+2}, x_{k+1}])$ we obtain

$$(1 - c_{k+1})R_\eta(k+1) = c_{k+1}R[x_{k+1}, x_k] + c_{k+1}(1 - c_k)R_\eta(k).$$

Multiplying by $\prod_{i=k+2}^{n} c_i$ for $n \geq k+1$, we obtain

$$R_\eta(k+1)(1 - c_{k+1}) \prod_{i=k+2}^{n} c_i = R[x_{k+1}, x_k] \prod_{i=k+1}^{n} c_i + R_\eta(k)(1 - c_k) \prod_{i=k+1}^{n} c_i$$

and finally

$$(1 - c_n)R_\eta(n) = \sum_{k=0}^{n-1} R[x_{k+1}, x_k] \prod_{i=k+1}^{n} c_i + R_\eta(0) \prod_{i=1}^{n} c_i, \tag{1}$$

if $R_\eta(k) < \infty$ for $k = 0, \ldots, n$. We show easily that (1) is true if we suppose only $R_\eta(n)$ is finite and $R_\eta(k)$, for $k = 1, \ldots n-1$ are finite or infinite. This implies the inequality

$$1 - c_n \geq \frac{1}{R_\eta(n)} \sum_{k=1}^{n} (R[x_k, x_{k-1}] \prod_{i=k}^{n} c_i)$$

and therefore

$$\sum_{n=1}^{\infty} (1 - c_n) \geq \sum_{n=1}^{\infty} \frac{1}{R_\eta(n)} \sum_{k=1}^{n} (R[x_k, x_{k-1}] \prod_{i=k}^{n} c_i). \tag{2}$$

If $\eta = (x_n)$ is irregular, by Proposition 1 $\prod_{n=1}^{\infty} c_n$ is finite, and so the series

$$\sum_{n=1}^{\infty} \frac{1}{R_\eta(n)} \sum_{k=1}^{n} R[x_k, x_{k-1}] \prod_{i=k}^{n} c_i$$

converges, and by inequality (2), we have

$$\sum_{n=1}^{\infty} \frac{1}{R_\eta(n)} \sum_{k=1}^{n} R[x_k, x_{k-1}] < \infty.$$

If η is regular, by Proposition 1, $R[x_1, x_0] \geq 1$ and (1), we have

$$(1 - c_n)\frac{1}{R_\eta(0) + 1} \leq \frac{1}{R_\eta(n)} \sum_{k=1}^{n} (R[x_k, x_{k-1}] \prod_{i=k}^{n} c_i) \tag{3}$$

for all n. Since $c_i \in]0,1]$ and $\lim_{n\to\infty} \prod_{i=1}^{n} c_i = 0$, $\sum_{n\in\mathbb{N}}(1-c_n)$ diverges, and by using inequality (3) we obtain

$$\sum_{1}^{\infty} \frac{1}{R_\eta(n)} \sum_{k=1}^{n} (R[x_k, x_{k-1}] \prod_{i=k}^{n} c_i) = \infty.$$

This completes the proof because $c_i \in]0,1]$.

The consequence following Theorem 1 has an interesting physical interpretation. Let us denote by $\tilde{I}^0$ and $\tilde{E}^0$ the flow and the energy at equilibrium starting at 0 i.e. the flow with minimal energy starting at 0.

Proposition 2 *Let $\eta = (x_n)_{n\in\mathbb{N}}$ be a geodesic ray and define the equilibrium potential $\tilde{V}^0(\eta)$ at η by*

$$\tilde{V}^0(\eta) = \sum_{k=0}^{\infty} R[x_k, x_{k+1}]\tilde{I}^0[x_k, x_{k+1}].$$

The ray η is irregular if and only if $\tilde{V}^0(\eta) < \tilde{E}^0$.

Proof First let us define, for $n > 0$, the variation potential at equilibrium $\tilde{V}^0_\eta(n)$ in $\mathcal{A}_\eta(x_n)$ by

$$\tilde{V}^0_\eta(n) = R_\eta(n)\{\tilde{I}^0([x_{n-1}, x_n]) - \tilde{I}^0([x_n, x_{n+1}])\}.$$

Suppose η is irregular. Using the equilibrium for $n < m$ at $R_\eta(m)$ and $R_\eta(n)$ we obtain

$$\tilde{V}^0_\eta(n) = \sum_{k=n}^{m-1} R[x_k, x_{k+1}]\tilde{I}^0[x_k, x_{k+1}] + \tilde{V}^0_\eta(m),$$

if $R_\eta(m) < \infty$ and $R_\eta(n) < \infty$. Since $\tilde{I}^0[x_k, x_{k+1}] \to 0$ if $k \to \infty$ we obtain

$$\tilde{I}^0_\eta[x_{k+1}, x_k] = \sum_{i=k+1}^{\infty} \frac{\tilde{V}^0_\eta(i)}{R_\eta(i)}.$$

For simplification, we put

$$\tilde{V}^0_\eta(p_n + k) = \tilde{V}^0_\eta(p_{n+1}) \text{ for } k = 1, \cdots, p_{n+1} - p_n$$

where $R_\eta(p_n) < \infty$, $R_\eta(p_n + 1) = \infty, \cdots$, $R_\eta(p_{n+1} - 1) = \infty$, $R_\eta(p_{n+1}) < \infty$. Thus we have

$$\tilde{V}^0_\eta(p_n) = \tilde{V}^0_\eta(p_{n+1}) + \sum_{k=p_n}^{p_{n+1}-1} R[x_k, x_{k+1}] \sum_{i=k+1}^{\infty} \frac{\tilde{V}^0_\eta(i)}{R_\eta(i)}; \tag{4}$$

since η is irregular the series of general term

$$R[x_k, x_{k+1}] \sum_{i=k+1}^{\infty} \frac{1}{R_\eta(i)}$$

is convergent by Theorem 1. Therefore we have

$$\prod_{n\geq 1} \left(1 + \sum_{k=p_n}^{p_{n+1}-1} R[x_k, x_{k+1}] \sum_{i=k+1}^{\infty} \frac{1}{R_\eta(i)}\right) < \infty;$$

by applying the inequality (4) and the nonincrease of $V_\eta^0(n)_n$ we obtain $\lim_{n\to\infty} \tilde{V}_\eta^0(n) > 0$, which proves the first implication.

Conversely, $\tilde{V}^0(\eta) < \tilde{E}$ implies $\inf_n \tilde{V}_\eta^0(n) \geq \tilde{E} - \tilde{V}^0(\eta) > 0$ and equation (3) gives

$$\tilde{V}_\eta^0(p_n) \geq \inf_k \tilde{V}_\eta^0(k) \left(1 + \sum_{k=p_n}^{p_{n+1}-1} R[x_k, x_{k+1}] \sum_{i=k+1}^{\infty} \frac{1}{R_\eta(i)}\right)$$

and therefore the result follows.

3 A third criterion of irregularity

In this section we assume that

$$0 < \inf_{y\in A} \sum_{\{z:y\sim z\}} R[y,z]^{-1} \leq \sup_{y\in A} \sum_{\{z:y\sim z\}} R[y,z]^{-1} < \infty$$

and we denote by G the potential kernel of the transient random walk X. Let $\eta = (x_n)$ be a geodesic ray.

Since $n \mapsto P_{x_n}[T_x < \infty]$ is nonincreasing for large value of n, and since $G(x_n, x) = P_{x_n}[T_x < \infty]\, G(x,x)$ then $\lim_{n\to\infty} G(x_n, x)$ exists. We denote it by $G(\eta, x)$.

For a subset B of A we define its *capacity* $\mathrm{Cap}(B)$ as in [9] by

$$\mathrm{Cap}(B) = \sum_{x\in B} C(x) P_x[S_B = \infty],$$

which is equivalent to other classical definitions.

Theorem 2 *Let $\eta = (x_n)_{n\in\mathbb{N}}$ be a geodesic ray and put for all $k \in \mathbb{N}$*

$$A_k = \{x \in A : 2^k \leq \lim_{n\to\infty} G(x_n, x) \leq 2^{k+1}\}.$$

If η is irregular we have

$$\limsup_{n\to\infty} 2^n \mathrm{Cap}(A_n) > 0.$$

If η is regular we have $G(\eta, x) = 0$ for all $x \in A$.

We begin the proof with two lemmas.

Lemma 1 *If $\lim_{n\to\infty} P_{x_n}[T_0 < \infty] > 0$ then $\lim_{n\to\infty} P_0[T_{x_n} < \infty] = 0$.*

Proof of lemma 1 Suppose the result is not true, i.e. $\lim_{n\to\infty} P_0[T_{x_n} < \infty] \neq 0$. By proposition 2.6 of [3] (strong Markov property) we have

$$P_{x_n}[T_0 < \infty] = P_{x_n}[T_{x_{n-1}} < \infty]\, P_{x_{n-1}}[T_0 < \infty]$$

and

$$P_0[T_{x_n} < \infty] = P_0[T_{x_{n-1}} < \infty]\, P_{x_{n-1}}[T_{x_n} < \infty].$$

By this equality we have

$$\lim_{n\to\infty} P_{x_{n-1}}[T_{x_n} < \infty] = P_{x_n}[T_{x_{n-1}} < \infty] = 1$$

which gives $\lim_{n\to\infty} P_{x_n}[S_{x_n} < \infty] = 1$ and so $\lim_{n\to\infty} G(x_n, x_n) = \infty$. By [3] we have

$$C(x_n)P_{x_n}[T_0 < \infty]\, G(0,0) = C(0)P_0[T_{x_n} < \infty]\, G(x_n, x_n) \tag{5}$$

because

$$C(x_n)\, G(x_n, 0) = C(0)\, G(0, x_n)$$

and for all x, y in A we have

$$G(x,y) = P_x[T_y < \infty]G(y,y).$$

Since the graph is bounded, this gives $\lim_{n\to\infty} P_0[T_{x_n} < \infty] = 0$, which contradicts the hypotheses. Hence the lemma is proven.

Lemma 2 *If η is irregular, then, for all $\epsilon > 0$, the subsets $\{x \in A : P_x[T_0 < \infty] \geq \epsilon\}$ and $\{x \in A : \lim_{n\to\infty} G(x_n, x) \geq \epsilon\}$ are non recurrent.*

Proof of lemma 2 Let us denote by $\mathcal{A}_{\eta,\epsilon}$ the tree induced by $\{x \in A : P_x[T_0 < \infty] \geq \epsilon\}$. If $\mathcal{A}_{\eta,\epsilon}$ is finite, $\{x \in A : P_x[T_0 < \infty] \geq \epsilon\}$ is non recurrent ; if $\mathcal{A}_{\eta,\epsilon}$ is infinite, every geodesic ray of $\mathcal{A}_{\eta,\epsilon}$ is irregular, and hence applying Proposition 4.3 of [8], we obtain that $\mathcal{A}_{\eta,\epsilon}$ is recurrent, and so $\{x \in A : P_x[T_0 < \infty] \geq \epsilon\}$ is non recurrent.

To prove the non recurrence of the second subset in lemma 1, we apply the first part to each element of the decomposition of $\{x \in A : \lim_{n\to\infty} G(x_n, x) \geq \epsilon\}$ in the subgraph $\mathcal{A}_\eta(x_k)$, $k \in \mathbb{N}$, and so we easily obtain the conclusion.

Proof of theorem 2 Suppose the geodesic ray is irregular ; by Lemma 1 and equality (6) we have

$$\lim_{n\to\infty} G(x_n, x_n) = \infty.$$

Let $n \in \mathbb{N}^*$ such that, in A_n we have a vertex x_k of η and i_n the largest $k \in \mathbb{N}$ such that $x_k \in A_n$. Let u_n be the equilibrium measure of A_n, i.e. the non negative function u_n such that $Gu_n = 1$ in A_n and vanishes in the complement of A_n. In fact A_n is non recurrent by Lemma 1, and we have $u_n(x) = P_x[S_{A_n} = \infty]$ for $x \in A_n$. By definition we have

$$2^{n+1}\text{Cap }(A_n) = 2^{n+1} \sum_{x\in A_n} u_n(x)C(x)\,;$$

since $G(\eta, x) \in [2^n, 2^{n+1}]$ for $x \in A_n$, we have

$$2^{n+1}\text{Cap }(A_n) \geq \sum_{x\in A_n} \lim_{k\to\infty} G(x_k, x)u_n(x)C(x),$$

and

$$\sum_{x\in A_n} \lim_{k\to\infty} G(x_k, x)u_n(x)C(x) \geq 2^n\text{Cap }(A_n).$$

On the other hand, since x_{i_n} is in the geodesic ray between x_k and x for large values of k, we have

$$G(x_k, x) = P_{x_k}[T_{x_{i_n}} < \infty]\, G(x_{i_n}, x).$$

This implies

$$2^{n+1}\text{Cap }(A_n) \geq \sum_{x\in A_n} \lim_{k\to\infty} P_{x_k}[T_{x_{i_n}} < \infty]\, G(x_{i_n}, x)\, u_n(x)\, C(x)$$

and

$$\sum_{x\in A_n} \lim_{k\to\infty} P_{x_k}[T_{x_{i_n}} < \infty]\, G(x_{i_n}, x)\, u_n(x)\, C(x) \geq 2^n \mathrm{Cap}\ (A_n)$$

so

$$2^{n+1}\mathrm{Cap}\ (A_n) \geq \lim_{k\to\infty} P_{x_k}[T_{x_{i_n}} < \infty] \sum_{x\in A_n} G(x_{i_n}, x)\, u_n(x)\, C(x)$$

therefore

$$2^{n+1}\mathrm{Cap}\ (A_n) \geq \lim_{k\to\infty} P_{x_k}[T_{x_{i_n}} < \infty].$$

With the same argument we have

$$\lim_{k\to\infty} P_{x_k}[T_{x_{i_n}} < \infty] \geq 2^n \mathrm{Cap}(A_n)/[\max_{x\in A} C(x)].$$

which finishes the result and the theorem.

Remark Here we use a geodesic ray for the determination of A_n. We have an analogous result if we replace the geodesic by a vertex : we obtain in the case of a tree Wiener's test for Markov chains in [7].

4 Appendix

We give an example in which there are infinitely many non countable irregular points which are in the support of the harmonic measure starting at the root.

Let Γ be the dyadic tree which has vertices, root and edges denoted respectively by $(x_n)_{n\in\mathbb{N}}$, x_0 and (i,j) if $x_i \sim x_j$. Let (n_p) be a increasing sequence of $\mathbb{N}^*$ such that the series $\sum_{p\geq 1} n_p/n_{p+1}$ is convergent. We construct the tree Λ (see figure 2) as follows. We introduce $n_p - 1$ vertices, in each edge (i,j) such $d(x_0, x_i) = p$ and $d(x_0, x_j) = p+1$ and we attach at each vertex x_i of Γ a tree $\Gamma_{2,p}$ where $\Gamma_{2,p}$ is the tree obtained by attaching at the root of a dyadic tree a geodesic ray formed with $n_p - 1$ edges.

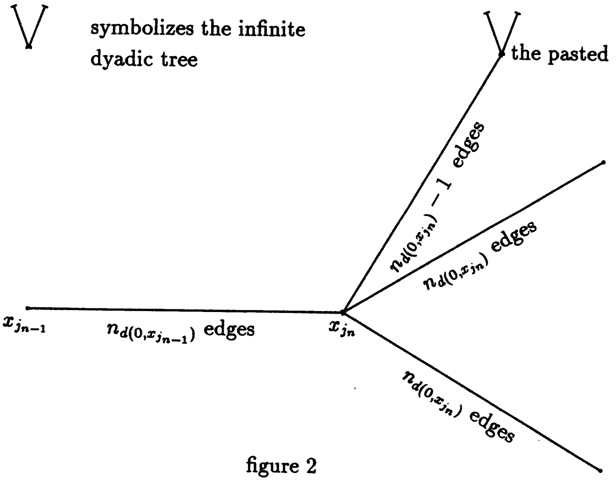

figure 2

We show easily by applying theorem 1 that a geodesic ray of Λ which contains an infinite mumber of x_i's and passes through x_0, is irregular and is in the support of the harmonic measure $\mu(\cdot) = P_{x_0}(\cdot)$. Then we have an uncountable set of irregular points in the support of the harmonic measure

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