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1 Introduction

We consider the following problem: for a positive integer $n \geq 1$, let $U_1, ..., U_n$ be n independent, integrable, centered, non-degenerate random variables. We are looking for conditions on a family of n càdlàg functions $f_1, ..., f_n$ on $\mathbb{R}_+$ with $f_i(0) = 0$, under which the following process:

$$X_t = \sum_{i=1}^{n} f_i(t) U_i \tag{1}$$

is a martingale, with respect to its own filtration $(\mathcal{F}_t)_{t \geq 0}$.

This (apparently) simple problem has a general solution given in Section 1. However, the answer is not quite satisfactory, since for example it does not allow to recognize whether there is a unique (up to the obvious multiplication by constants and time-changes) set (f_i) meeting our condition.

To get more insight, we specialize in Section 3 to the case where $n = 2$ and (for the most interesting results) with U_1 and U_2 having the same law. In this very particular situation we are able to give a complete description of all martingales of the form (1). This description emphasizes the particular role played by the stable distributions.

For the case $n \geq 3$, we have been unable to provide any interesting result of the same kind as for $n = 2$.

2 A general result

Here is a general theorem solving (in principle) our problem.

Theorem 1. *The process X is a martingale if and only if it satisfies the following:*

Condition [M]: *There are an integer p, $0 \leq p \leq n$, and deterministic times $0 = T_0 < T_1 < ... < T_p < T_{p+1} = \infty$, and p linearly independent vectors $a_j = (a_j^i)_{1 \leq i \leq n}$ in $\mathbb{R}^n$ (when $p \geq 1$), such that, with $V_0 = 0$ and $V_j = \sum_{1 \leq i \leq n} a_j^i U_i$ for $j \geq 1$,*

(M1) $(V_j)_{0\leq j\leq p}$ *is a discrete-time martingale;*

(M2) $X_t = \sum_{1\leq j\leq p} V_j 1_{[T_j,T_{j+1})}(t)$.

Before proving this theorem, we state some remarks on the conditions. First, Condition (M2) implies that $f_i(t) = \sum_{1\leq j\leq p} a^i_j 1_{[T_j,T_{j+1})}(t)$, because of the following property:

$$\alpha_i, \beta_i \in I\!R, \quad \sum_{i=1}^n \alpha_i U_i = \sum_{i=1}^n \beta_i U_i \quad a.s. \quad \Rightarrow \quad \alpha_i = \beta_i \quad \forall i. \tag{2}$$

Second, Condition (M1) is obviously difficult to verify, except when $p = 0$ (it is void) and $p = 1$ (it is obvious because V_1 is centered). Below we give an equivalent condition based on the characteristic functions φ_i of U_i. We recall that each function φ_i is C^1 with $\varphi_i'(0) = 0$. Then, when $p \geq 2$, (M1) is equivalent to the following:

Condition (M'1). For all $1 \leq l \leq p-1$ and all v_j in $I\!R$,

$$\sum_{i=1}^n (a^i_{l+1} - a^i_l)\varphi_i'(\sum_{j=1}^l a^i_j v_j) \prod_{k\neq i} \varphi_k(\sum_{j=1}^l a^k_j v_j) = 0. \tag{3}$$

We observe that (3) is the same as $E((V_{l+1} - V_l)\exp i \sum_{j=1}^l v_j V_j) = 0$. When the φ_i's do not vanish (so $\varphi_i = \exp \psi_i$ with ψ_i of class C^1 and $\psi_i'(0) = 0$) this condition is also equivalent to:

Condition (M"1). For all $1 \leq l \leq p-1$ and all v_j in $I\!R$,

$$\sum_{i=1}^n (a^i_{l+1} - a^i_l)\psi_i'(\sum_{j=1}^l a^i_j v_j) = 0. \tag{4}$$

Proof. The sufficient condition is obvious. For the necessary condition, we suppose that X is a martingale and let $F(t)$ be the vector with components $(f_i(t))_{1\leq i\leq n}$. Denote by E_t the linear space spanned by $(F(s) : s \leq t)$, let $d_t = \dim(E_t)$, $T_{-1} = -1$, $T_j = \inf(t : d_t \geq j)$ for $0 \leq j \leq n$, and $T_{n+1} = \infty$. Thus $T_{-1} < 0 = T_0 \leq T_1 \leq ... \leq T_p < T_{p+1} = \infty$ for some $0 \leq p \leq n$, and $d_0 = 0$.

Let $0 \leq i \leq p$ with $T_i < T_{i+1}$ and consider s, t such that $T_i < s < t < T_{i+1}$. Then $E_t = E_s$ is spanned by the linearly independent vectors $F(s_1), ..., F(s_i)$ with $s_j \leq s$ (if $i = 0$, then $E_t = E_s = \{0\}$). Therefore, X_s and X_t are $\sigma(X_{s_1}, ..., X_{s_i})$-measurable and thus $\mathcal{F}_s = \mathcal{F}_t = \sigma(X_{s_1}, ..., X_{s_i})$ (which is the trivial σ-field when $i = 0$). The martingale property $E(X_t|\mathcal{F}_s) = X_s$ yields $X_t = X_s$ a.s., and (2) gives $F(s) = F(t)$. It follows that $F(.)$ is constant on (T_i, T_{i+1}) as well as on $[T_i, T_{i+1})$ by right-continuity. Thus

$$T_i < T_{i+1} \quad \Rightarrow \quad d_r = i \quad \forall r \in [T_i, T_{i+1}). \tag{5}$$

In fact $0 < T_1 < ... < T_p$; otherwise we would be in one of the following two situations:

a) $0 = T_j < T_{j+1}$ for some $1 \le j \le p$, and therefore $d_{T_j} = d_0 = 0$, which contradicts (5);

b) $T_{i-1} < T_i = T_j < T_{j+1}$ for i, j with $1 \le i < j \le p$, in which case $d_r = i - 1$ on $[T_{i-1}, T_j)$ by (3). This implies that $d_{T_j} \le i$; being also impossible since $d_{T_j} \ge j$.

Since $0 < T_1 < \dots < T_p$ holds, we trivially have (M2) with $a_j = F(T_j)$. Finally, (M2) and the martingale property of X yield (M1). □

3 The case n=2

Let φ_i be the characteristic function of U_i, and when φ_i never vanishes we use the notation $\varphi_i = \exp \psi_i$ without further comment. In this section we always assume that $n = 2$.

Theorem 2. *The process X is a martingale if and only if it has one of the following two (mutually exclusive) representations:*

a) For some $\alpha, \beta \in \mathbb{R}$, $S_1, S_2 \in (0, \infty]$

$$X_t = \alpha U_1 1_{[S_1,\infty)}(t) + \beta U_2 1_{[S_2,\infty)}(t). \tag{6}$$

b) For some $0 < T_1 < T_2 < \infty$, $\alpha, \alpha', \gamma, \gamma' \in \mathbb{R}^*$ with $\gamma \ne \gamma'$ and

$$\varphi_1'(v)\varphi_2(\gamma v) + \gamma'\varphi_1(v)\varphi_2'(\gamma v) = 0 \quad \forall v \in \mathbb{R}, \tag{7}$$

$$X_t = \alpha(U_1 + \gamma U_2)1_{[T_1,\infty)}(t) + \alpha'(U_1 + \gamma' U_2)1_{[T_2,\infty)}(t). \tag{8}$$

Remark. Since the coefficients in (8) do not vanish, the form (8) is indeed symmetric in (U_1, U_2). When φ_1 and φ_2 do not vanish, (7) is equivalent to $\psi_1'(v) + \gamma'\psi_2'(\gamma v) = 0$, which is the same as $\psi_1(v) + \frac{\gamma'}{\gamma}\psi_2(\gamma v) = 0$, which in turn is equivalent to

$$\varphi_1(v) = \varphi_2(\gamma v)^{-\gamma'/\gamma} \quad \forall v \in \mathbb{R}. \tag{9}$$

Proof. Sufficient condition: That (a) gives a martingale is obvious. Condition (b) implies (M2) with $a_1^1 = \alpha$, $a_1^2 = \alpha\gamma$, $a_2^1 = \alpha' + a_1^1$, $a_2^2 = \alpha'\gamma' + a_1^2$ and then (7) gives (M'1).

Necessary condition: We assume (M'1) and (M2). If $T_1 = \infty$, then (a) holds with $\alpha = \beta = 0$ and S_i arbitrary. If $T_1 < T_2 = \infty$, then (a) holds with $\alpha = a_1^1$, $\beta = a_1^2$ and $S_1 = S_2 = T_1$.

Suppose now that $T_1 < T_2 < \infty$. We have $a_1 \ne 0$, and since both (a) and (b) are symmetric in (U_1, U_2), without lost of generality we assume that $a_1^1 \ne 0$. Let $\alpha = a_1^1$ and $\gamma = a_1^2/\alpha$ and write $a_2^i = a_1^i + \beta^i$. Then the linear independence between a_1 and a_2 gives

$$\beta^2 \ne \gamma\beta^1, \tag{10}$$

while (M'1) is

$$\beta^1\varphi_1'(v)\varphi_2(\gamma v) + \beta^2\varphi_1(v)\varphi_2'(\gamma v) = 0 \quad \forall v \in \mathbb{R}. \tag{11}$$

We assume first that $\gamma = 0$. Recalling that $\varphi_i(0) = 1$, $\varphi_i'(0) = 0$ and φ_i' is not identically 0 in any neighborhood of 0 (because $P(U_i = 0) < 1$), (11) yields $\beta^1 = 0$, that is, we have (a) with $S_1 = T_1$, $S_2 = T_2$, $\beta = \beta^2$.

Next, assume that $\gamma \neq 0$. Then there exists $\theta \in \mathbb{R}^*$ with $\varphi_1'(\theta) \neq 0$, $\varphi_1(\theta) \neq 0$ and $\varphi_2(\gamma\theta) \neq 0$. Suppose for the time being that $\varphi_2'(\gamma\theta) = 0$. Then (11) yields $\beta^1 = 0$ and since there is another $\theta' \in \mathbb{R}^*$ with $\varphi_1(\theta') \neq 0$ and $\varphi_2(\gamma\theta') \neq 0$, we also have $\beta^2 = 0$, which contradicts (10). Thus $\varphi_2'(\gamma\theta) \neq 0$ and (10) and (11) yield $\beta^1 \neq 0$ and $\beta^2 \neq 0$. Hence we have (b) with $\gamma' = \beta^2/\beta^1$ and $\alpha' = \beta^1$ (note that $\gamma \neq \gamma'$ follows from (10), and (7) is the same as (11)). □

When U_1 and U_2 are arbitrary, it seems there is not much more to say. From now on we concentrate on the case where $U_1 =^d U_2$, i.e. $\varphi_1 = \varphi_2 = \varphi$. In this situation, the existence of a martingale X of the form (b) above depends on the existence of constants $\gamma, \gamma' \in \mathbb{R}^*$ with $\gamma \neq \gamma'$ and

$$\varphi'(v)\varphi(\gamma v) + \gamma'\varphi(v)\varphi'(\gamma v) = 0 \quad \forall v \in \mathbb{R}. \tag{12}$$

Let D denote the set of all $\gamma \in \mathbb{R}^*$ for which (12) holds for some $\gamma' \in \mathbb{R}^*$ with $\gamma' \neq \gamma$. If $\gamma \in D$ there is a unique $\gamma' = \delta(\gamma)$ satisfying (12), because we have seen before that for each $\gamma \neq 0$ there is $v \in \mathbb{R}$ with $\varphi(v) \neq 0$ and $\varphi'(\gamma v) \neq 0$.

Theorem 3. **a)** *If* U_1 *is symmetric about* 0, *then one of the following three cases is satisfied:*

(Cs-1) $D = \{-1, 1\}$.

(Cs-2) $D = \{r^n, -r^n : n \in \mathbb{Z}\}$ *for some* $r > 1$ *and* φ *never vanishes.*

(Cs-3) $D = \mathbb{R}^*$. *This is the case if and only if* U_1 *is stable with index* $\rho \in (1, 2]$, *i.e.* $\varphi(u) = e^{-a|u|^\rho}$ *for some* $a > 0$.

b) *If* U_1 *is not symmetric about* 0, *we are in one of the following five situations:*

(Ca-1) $D = \{1\}$.

(Ca-2) $D = \{-1, 1\}$. *This is the case if and only if* $\varphi = \rho e^{\eta}$, *where* ρ *and* η *are real-valued,* $\eta(0) = 0$, *and* η *is constant on each open interval on which* ρ *(or* φ*) does not vanish (necessarily* φ *vanishes somewhere, and* η *is not identically* 0, *otherwise we would be in the symmetric case).*

(Ca-3) $D = \{r^n : n \in \mathbb{Z}\}$ *for some* $r > 1$ *and* φ *never vanishes.*

(Ca-4) $D = \{r^n, -r^{n+1/2} : n \in \mathbb{Z}\}$ *for some* $r > 1$ *and* φ *never vanishes.*

(Ca-5) $D = (0, \infty)$. *This is the case if and only if* U_1 *is asymmetric strictly stable with index* $\rho \in (1, 2)$, *i.e.,* $\varphi(u) = e^{-a|u|^\rho(1+ib\,sign(u))}$ *for some* $a > 0$, $b \neq 0$, $|b| \leq \tan(\frac{\pi}{2(2-\rho)})$.

c) *There is a constant* $\theta \in (1, 2]$ *such that* $\delta(\gamma) = -\gamma/|\gamma|^\theta$ *(so* $\delta(1) = -1$, *and* $\delta(-1) = 1$ *if* $-1 \in D$*), and* $\theta = \rho$ *in cases (Cs-3) and (Ca-5).*

Therefore the martingales X *of the form (8) are indeed represented as*

$$X_t = \alpha(U_1 + \gamma U_2)1_{[T_1,\infty)}(t) + \alpha'(U_1 - \gamma U_2/|\gamma|^\theta)1_{[T_2,\infty)}(t), \tag{13}$$

where $\alpha, \alpha' \in I\!R^*$, $0 < T_1 < T_2 < \infty$, *and* $\gamma \in D$.

Remark. There are of course examples of variables satisfying (Cs-1) or (Cs-3) in the symmetrical case, (Ca-1) in the asymmetrical case. We presume that (Cs-2) and (Ca-3) are not empty, and believe that (Ca-2) is empty (but we have been unable to prove these facts).

Before giving the proof of Theorem 3 we present some useful lemmas. First we note that $\gamma = 1$ and $\gamma' = -1$ always satisfy (12), so $1 \in D$ and $\delta(1) = -1$.

Lemma 4. *We have* $-1 \in D$ *if and only if* $\varphi = \rho e^{\eta}$, *where* ρ *and* η *are real-valued and* $\eta(0) = 0$ *and* η *is constant on each open interval on which* ρ *(or* φ*) does not vanish. Moreover,* $\delta(-1) = 1$.

Proof. Let (x, y) be a maximal interval on which φ does not vanish, so φ does not vanish either on $(-y, -x)$ (we may have $(x, y) = I\!R$, of course). We can write $\varphi = e^{\psi}$ with ψ of class C^1 on (x, y) and $(-y, -x)$, and since $\psi(-v) = \overline{\psi(v)}$ the property $-1 \in D$ and (12) yield

$$\psi'(v) = \gamma' \overline{\psi'(v)} \quad \forall v \in (x, y).$$

Since $\gamma' \in I\!R^*$, we deduce that $\psi'(v) \in I\!R$ and thus $\gamma' = 1$ (because ψ' cannot be identically 0). Therefore, if $v_0 \in (x, y)$, we have $\psi(v) - \psi(v_0) \in I\!R$ for all $v \in (x, y)$ and hence $\varphi = \rho e^{\eta}$ with $\eta(v) = \eta(v_0) \in I\!R$ for all $v \in (x, y)$. The converse is obvious.□

Lemma 5. *Let* $\gamma \in I\!R^*$ *with* $|\gamma| \neq 1$. *Then* $\gamma \in D$ *if and only if* φ *does not vanish, and satisfies for some* $C(\gamma) > 0$

$$\varphi(v) = \varphi(\gamma v)^{C(\gamma)} \quad \forall v \in I\!R. \tag{14}$$

Moreover,

- **a)** $\mathcal{R}e\psi(v) < 0$ *for all* $v \in I\!R^*$.
- **b)** $\delta(\gamma) = -\gamma C(\gamma)$.
- **c)** *For all* $n \in Z\!\!Z$ *we have* $\gamma^n \in D$ *and* $C(\gamma^n) = C(\gamma)^n$.
- **d)** $-\gamma \in D$ *if and only if* φ *is real-valued, and then* $C(-\gamma) = C(\gamma)$.

Proof. The sufficient condition is obvious, as well as (b).

Conversely, assume that $\gamma \in D$. Let $(-x, x)$ be the maximal interval on which φ does not vanish. We have $\varphi = e^{\psi}$ with ψ of class C^1 on $(-x, x)$. For simplicity we set $\psi_r = \mathcal{R}e\psi$, and we have $\psi_r(u) \to -\infty$ as $|u| \uparrow x$ if $x < \infty$. On $(-x, x)$, (12) yields $\psi'(v) + \gamma'\psi'(\gamma v) = 0$, so $\psi(v) + \frac{\gamma'}{\gamma}\psi(\gamma v) = 0$, since $\psi(0) = 0$.

If $|\gamma| > 1$ and $x < \infty$, then $|\psi_r(v)| = |\frac{\gamma'}{\gamma}||\psi_r(\gamma v)| \to \infty$ as $|v| \uparrow x/|\gamma|$, contradicting the fact that ψ is continuous on $(-x, x)$. Similarly, if $|\gamma| > 1$ and $x < \infty$, $|\psi_r(\gamma v)| = |\frac{\gamma}{\gamma'}||\psi_r(v)| \to \infty$ as $|v| \uparrow x$, bringing up the same contradiction; therefore $x = \infty$, and φ does not vanish. It follows that $\varphi = e^{\psi}$ everywhere and, with $C(\gamma) = -\gamma'/\gamma$,

$$\psi(v) = C(\gamma)\psi(\gamma v) \quad \forall v \in I\!R, \tag{15}$$

that is, we have (14). Since U_1 is non-degenerate, ψ is not identically 0 and thus $C(\gamma) \neq 0$. Note also that (c) is obvious from (14).

We always have that $\psi_r \leq 0$ and that ψ_r is even. Assume that $\psi_r(v) = 0$ for some $v > 0$. Then (15) and (c) imply $\psi_r(v|\gamma|^n) = 0$ for all $n \in \mathbb{Z}$. It follows that the characteristic fonction of the symmetrized random variable $U = U_1 - U_2$ equals 1 for all $v|\gamma|^n$, $n \in \mathbb{Z}$, so U is supported by $\{2k\pi/v|\gamma|^n : k \in \mathbb{Z}\}$, for all $n \in \mathbb{Z}$, which implies that $U = 0$ a.s., contradicting again the non-degeneracy assumption. Thus (a) holds and (15) yields $C(\gamma) > 0$.

Finally, it only remains to prove (d). If φ is real-valued, it is even and (14) is satisfied with $-\gamma$ and $C(-\gamma) = C(\gamma)$. Suppose conversely that $-\gamma \in D$, then (15) gives $\overline{\psi(v)} = C(\gamma)\psi(-\gamma v)$, while $-\gamma \in D$ yields $\psi(v) = C(-\gamma)\psi(-\gamma v)$. Comparing the real parts of these two equalities and using (a) we obtain $C(-\gamma) = C(\gamma)$. Then $\overline{\psi} = \psi$ and φ is real-valued. □

Lemma 6. *With $D_+ = D \cap \mathbb{R}_+$, one of the following three cases is satisfied:*

(C$_+$1) $D_+ = \{1\}$.

(C$_+$2) $D_+ = \{r^n : n \in \mathbb{Z}\}$ *for some* $r > 1$.

(C$_+$3) $D_+ = \mathbb{R}_+^*$.

Moreover, we are in case (C$_+$3) if and only if either $\varphi(u) = e^{-a|u|^2}$ *for some* $a > 0$ *or* $\varphi(u) = e^{-a|u|^\rho(1+ib\,sign(u))}$ *for some* $a > 0$, $\rho \in (1,2)$, $|b| \leq \tan(\frac{\pi}{2(2-\rho)})$.

Proof. Due to the fact that $1 \in D$ and to Lemma 5, if we are not in case (C$_+$1), D_+ contains at least a $\gamma > 0$, $\gamma \neq 1$, and then $\varphi = e^\psi$ satisfies (14). Indeed, D_+ is the set of all $\gamma > 0$ such that (15) holds for some $C(\gamma) > 0$. Then D_+ is clearly a multiplicative group, therefore it is closed since ψ is continuous and thus it is of the form (C$_+$2) or (C$_+$3).

Assuming (C$_+$3), for each $\gamma > 0$ there is $C(\gamma) > 0$ such that, if f denotes either the real or the imaginary part of ψ, we have $f(0) = 0$ and

$$f(v) = C(\gamma)f(\gamma v) \quad \forall v \geq 0.$$

Then f is either identically 0, or everywhere positive, or everywhere negative, on $(0,\infty)$. In the last two cases, $g(u) = \log|f(e^u)/f(1)|$ satisfies $g(u + \log\gamma) = g(u) + g(\log\gamma)$ for all $u \in \mathbb{R}$, $\gamma > 0$, i.e., $g(u + u') = g(u) + g(u')$ for all $u, u' \in \mathbb{R}$. Since g is continuous, we obtain $g(u) = Ku$. Thus, in all cases we have $f(v) = \eta v^\rho$ for some $\eta, \rho \in \mathbb{R}$, and furthermore $\gamma^\rho C(\gamma) = 1$ for all $\gamma > 0$ (hence ρ is the same for both the real and imaginary parts of ψ). We then deduce that $\psi(v) = (\alpha + i\beta)v^\rho$ for some $\alpha, \beta, \rho \in \mathbb{R}$, if $v > 0$. By (a) of Lemma 5 we have $\alpha < 0$ and since $\psi(-v) = \overline{\psi(v)}$, we also have $\psi(v) = (\alpha - i\beta)|v|^\rho$ for $v < 0$. Then $\psi(v) = -a|v|^\rho(1 + ib\,sign(v))$ for $a > 0$, $b \in \mathbb{R}$, $\rho \in \mathbb{R}$. Conversely, each such ψ satisfies (15) for all $\gamma > 0$, with $C(\gamma 1) = \gamma^{-\rho}$, implying $D_+ = \mathbb{R}_+^*$.

It remains to examine under which conditions on (a, b, ρ) the function $\varphi = e^\psi$ with ψ as above is a characteristic function. Observe that for all $\alpha, \alpha' > 0$ we have $\psi(\alpha v) + \psi(\alpha' v) = \psi(\alpha'' v)$ with $\alpha''^\rho = \alpha^\rho + \alpha'^\rho$. Then, if it is the case, the corresponding distribution will be strictly stable, with a first moment equal to 0. As is well known, this will be the case if and only if either $\rho = 2$ and $b = 0$ (normal case), or $\rho \in (1,2)$ and $|b| \leq \tan(\frac{\pi}{2(2-\rho)})$. □

Proof of Theorem 3. a) When U_1 is symmetric, so is D, and (Cs-i) = (C_+i). Therefore Lemma 5 yields that one of (Cs-1), (Cs-2) or (Cs-3) is satisfied. Moreover, (Cs-2) implies that φ never vanishes (by Lemma 5), and (Cs-3) holds if and only if $\varphi(v) = e^{-a|v|^\rho}$ (because here φ is real-valued).

b) Now we suppose that U_1 is not symmetric. It suffices to prove that if $D \neq \{1\}$, then we are in one of the cases (Ca-i) for i=2,3,4,5.

First, by Lemma 4, $-1 \in D$ if and only if the necessary and sufficient condition in (Ca-2) is satisfied. Then φ vanishes somewhere, and D contains no γ with $|\gamma| \neq 1$ by Lemma 5. Thus $-1 \in D$ if and only if (Ca-2) holds.

Next, suppose that we are not in any of the cases (Ca-1) and (Ca-2). If $D = D_+$, we are then in cases (Ca-3) or (Ca-5) by Lemma 5. Otherwise there exists $\gamma > 0$ with $\gamma \neq 1$ and $-\gamma \in D$. Then $\gamma^2 \in D$ and $\gamma^2 \neq 1$ and by Lemma 5 either (C_+2) or (C_+3) holds. However, under (C_+3) we also have $\gamma \in D$, hence Lemma 5(d) contradicts the assumption that U_1 is non-symmetric and indeed we have (C_+2) with some $r > 1$. It then follows that $\gamma^2 = r^k$ for some $k \in \mathbb{N}^*$, while Lemma 5(c) gives $C(r^n) = C(r)^n$ and $C(\gamma) = C(r)^{k/2}$. Furthermore if k were even we would have $r^{k/2} \in D$ and $-r^{k/2} = -\gamma \in D$, again a contradiction by Lemma 5(d), so $k = 2p+1$ with $p \in \mathbb{Z}$ and $\gamma = r^{p+1/2}$. In order to obtain (Ca-4), it thus remains to prove that $-r^{n+1/2} \in D$ for all $n \in \mathbb{Z}$. For this, a repeated use of (15) yields

$$\psi(v) = C(r^{n-p})\psi(r^{n-p}v) = C(r^{n-p})C(\gamma)\psi(-\gamma r^{n-p}v) = C(r)^{n+1/2}\psi(-r^{n+1/2}),$$

and the result follows.

c) Since $\delta(1) = -1$ and $\delta(-1) = 1$ (Lemma 4), (c) is obvious in cases (Cs-1), (Ca-1) and (Ca-2). Also, (c) with $\theta = \rho$ follows from Lemma 5(b) and from a comparison between (14) and the explicit form of φ in cases (Cs-3) and (Ca-5).

Under (Ca-4) we have seen that $C(r^n) = C^n$ and $C(-r^{n+1/2}) = C^{n+1/2}$, where $C = C(r)$. Thus, for all $\gamma \in D$, $C(\gamma) = C^{\log(|\gamma|)/\log(r)} = |\gamma|^{-\theta}$ with $\theta = -\log(C)/\log(r)$ (so $\delta(\gamma) = -\gamma/|\gamma|^\theta$). The same holds for (Cs-2) and (Ca-3). Now (15) yields $\psi'(v) = Cr\psi'(rv)$, hence $\psi'(r^{-n}v) = (Cr)^n\psi'(v)$ and since $\psi'(0) = 0$ and $\psi'(v) \neq 0$ for some $v \neq 0$ we must have $Cr < 1$ and thus $\theta > 1$.

Suppose that $\theta > 2$, i.e. $A := Cr^2 < 1$. Then $\psi'(r^{-n}v)/(r^{-n}v) = A^n\psi'(v)/v$ and if $|w| \leq r^{-n}$ there is $m \geq n$ and $v \in (1/r, 1]$ with $w = vr^{-m}$ or $w = -vr^{-m}$. Therefore $\sup_{|w|\leq r^{-n}} |\psi'(w)/w| \leq A^n \sup_{1/r<|v|\leq 1} |\psi'(v)|$. It follows that ψ' is differentiable at 0, with $\psi''(0) = 0$. Hence U_1 is square-integrable, with variance 0, which contradicts once more the non-degeneracy assumption and therefore $\theta \leq 2$. □

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