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ANNE ESTRADE

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A characterization of Markov solutions for stochastic differential equations with jumps

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Introduction

It is well known that solutions of stochastic differential equations such as

$$X_0 = x \; ; \; dX_t = f(X_{t-})\, dZ_t \;, \tag{1}$$

where Z is a Lévy process, are Markov processes. A converse result has been obtained by Jacod and Protter [6] as follows : consider the stochastic differential equations $(1)_x$ driven by the same semimartingale Z with initial conditions x and never-vanishing coefficient f. It is proved that, if the solutions X^x of $(1)_x$ are time homogeneous Markov processes with the same transition semigroup for all x, then Z is a Lévy process.

The present work is in the spirit of Jacod and Protter's converse problem. We obtain a converse result for stochastic differential equations with jumps between manifolds. More precisely we will look at the equations studied by Cohen [4] for which it is already known that solutions are Markov processes provided the driving semimartingale is a Lévy process.

The main interest of this paper is in the consequences of this converse result. In fact we are able to establish a characterization of diffusions with jumps : usually, diffusions are constructed as Markov solutions of stochastic differential equations. What we prove here is that the only time homogeneous Markov processes obtained as solutions of stochastic differential equations are those arising from equations driven by Lévy processes.

The method is an extension of [6]. The principle consists in "inverting" the stochastic differential equation and writing the driving process as an additive functional of the solution; the Markov property of the solution then yields the conclusion. To invert the stochastic differential equation, some inverting assumptions are required, similar to the "never-vanishing coefficient f" assumption in [6].

The paper is divided into two main sections. In section 1, we establish the method to prove that the driving process is Lévy. In section 2, we characterize the diffusions with jumps, first in a manifold and then, as a special case, in $\mathbf{R}^d$.

In the following $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ will be a filtered probability space with $(\mathcal{F}_t)_{t\geq 0}$ a right continuous filtration containing all $\mathbf{P}$-zero measure sets of $\mathcal{F}$.

1 A criterion to be a Lévy process

Let M be a finite dimensional manifold and $(X^x)_{x\in M}$ a collection of $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$-adapted càdlàg semimartingales with values in M such that $X_0^x = x$ for all x in M.

Let $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_t)$ be the canonical space of càdlàg M-valued functions, equipped with the canonical process $\hat{X}$ ($\hat{X}_t(\omega) = \omega(t)$, for $t \geq 0$ and ω in $\hat{\Omega}$) and the natural filtration $(\hat{\mathcal{F}}_t)_{t\geq 0}$ of $\hat{X}$. We will also denote by $(\theta_t)_{t\geq 0}$ the semigroup of translations on $\hat{\Omega}$ ($\theta_t(\omega)(.) = \omega(t+.)$, for $t \geq 0$ and ω in $\hat{\Omega}$) and by P^x the probability measure on $(\hat{\Omega}, \hat{\mathcal{F}})$ which is the law of X^x for all x in M.

Finally let Z be an $\mathbf{R}^d$-valued càdlàg semimartingale adapted to $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ with $Z_0 = 0$. We recall the usual definition of a Lévy process.

Definition 1 *A process Z on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ is called a Lévy process if it is a càdlàg adapted process such that $\mathbf{P}(Z_0 = 0) = 1$, and for all $s, t \geq 0$, the variable $Z_{t+s} - Z_t$ is independent from the $(Z_u\,;\, 0 \leq u \leq t)$ and has the same distribution as Z_s.*

We are now able to give the main result of this section.

Proposition 1 *Assume that there exists an $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_t)$-adapted process $(A_t)_{t\geq 0}$ with values in $\mathbf{R}^d$ such that*
(i) $\forall x \in M$, $P^x(A_0 = 0$ and $A_{s+t} = A_t + A_s \circ \theta_t\,,\, \forall s, t \geq 0) = 1$;
(ii) $\forall x \in M$, $\mathbf{P}(Z_t = A_t(X^x)\,,\, \forall t \geq 0) = 1$.
If the X^x are time homogeneous Markov processes with transition semigroup independent of x, then Z is a Lévy process.

This proposition is very similar to the result in [6]. The generalization consits in replacing the explicit formula giving Z in terms of X by a condition assuring that Z is an additive functional of X. It is also close in spirit to theorem 6.27 of [2] where the local characteristics of an additive semimartingale based on a Markov process are described.

Proof of proposition 1 : Take a bounded Borel function f on $\mathbf{R}^d$ and compute $E^x(f(A_{t+s} - A_t)/\hat{\mathcal{F}}_t)$ for $s, t \geq 0$ and some x in M. Using the additive property (i) of A and the Markov property of $\hat{X}$ on $(\hat{\Omega}, \hat{\mathcal{F}}_t, P^x)$, we get

$$E^x(f(A_{t+s} - A_t)/\hat{\mathcal{F}}_t) = E^x(f(A_s) \circ \theta_t)/\hat{\mathcal{F}}_t) = E^{\hat{X}_t}(f(A_s)).$$

By (ii), the $\mathbf{P}$-distribution of Z equals the P^x-distribution of A, for all x in M. Then we get

$$E^x(f(A_{t+s} - A_t)/\hat{\mathcal{F}}_t) = \mathbf{E}(f(Z_s)).$$

This proves that under P^x, $A_{t+s} - A_t$ is independent from $\hat{\mathcal{F}}_t$ and hence from $(A_u\,;\, 0 \leq u \leq t)$, and has the same distribution as A_s. Finally, use (ii) again and the proposition follows. □

2 Stochastic differential equations in manifolds

We will be concerned with stochastic differential equations driven by d-dimensional càdlàg semimartingales, whose solutions live in a d-dimensional manifold ($\mathbf{R}^d$ included !). We will use the formalism introduced by Cohen and studied with respect to Markov property in [4]. Such equations can also be studied with the formalism of Kurtz, Pardoux and Protter in [5] but with restricted possibilities for the jumps of the solution (at a jump time s, in [5] the X_s term is given as the end point of an ordinary differential equation starting at X_{s-} with a coefficient linearly depending on ΔZ_s, whereas in [4], X_s is given by any function of X_{s-} and ΔZ_s).

2.1 Definitions and properties

Let us first recall some of Cohen's results. In the following, M will be a smooth manifold of dimension d.

Definition 2 *A map $\psi : M \times \mathbf{R}^d \longrightarrow M$ is called a jump coefficient if*
(i) $\forall x \in M$, $\psi(x,0) = x$;
(ii) ψ is C^3 in a neighborhood of $M \times \{0\}$ in $M \times \mathbf{R}^d$.

Suppose we are given a d-dimensional càdlàg semimartingale Z, a jump coefficient ψ according to the previous definition and a fixed point $x \in M$. In [4], a meaning is given to the following stochastic differential equation

$$X_0 = x \; ; \; \overset{\triangle}{d}X = \psi(X, \overset{\triangle}{d}Z) \tag{2}$$

by the prescription that the process X is a solution of (2) if X is an M-valued semimartingale such that, for any embedding $(x^\alpha)_{1\leq\alpha\leq m}$ of M in $\mathbf{R}^m$, one has

$$\begin{aligned} \forall \alpha = 1,...,m \qquad X_t^\alpha &= x^\alpha + \int_0^t \frac{\partial \psi^\alpha}{\partial z^i}(X_{s-},0)\, dZ_s^i \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 \psi^\alpha}{\partial z^i \partial z^j}(X_s,0)\, d < Z^{ic}, Z^{jc} >_s \\ &+ \sum_{s\leq t} \left(\psi^\alpha(X_{s-},\Delta Z_s) - X_{s-}^\alpha - \frac{\partial \psi^\alpha}{\partial z^i}(X_{s-},0)\, \Delta Z_s^i \right) \end{aligned} \tag{3}$$

In the following, the summation convention on repeated indices will be in force; sums on i and j will run from 1 to d and sums on α and β from 1 to m. Also Z^c will denote the continuous martingale part of any real semimartingale Z.

It is established in [4] that the equation (2) admits a unique, possibly exploding solution X^x. Moreover, if Z is a Lévy process then X^x is an homogeneous Markov process with transition semigroup independent of x.

2.2 Converse result

To obtain a converse to this result, we need some inverting assumptions. The first one deals with the jump coefficient ψ of the stochastic differential equation (2).

Definition 3 *A jump coefficient ψ is said to be invertible if for all x in M, the differential at 0 of $\psi(x,.) : z \in \mathbf{R}^d \longrightarrow \psi(x,z) \in M$, which we denote by $d_z\psi(x,0)$, is an isomorphism from $\mathbf{R}^d$ onto T_xM.*

As promised, we now give a characterization of jump diffusions in manifolds.

Theorem 2 *Let Z be a d-dimensional semimartingale, let ψ be an invertible jump coefficient and, for all x in M, let X^x be the unique solution of the equation*

$$X_0 = x \; ; \; \overset{\triangle}{d}X = \psi(X, \overset{\triangle}{d}Z).$$

Suppose that

$$\forall x \in M \;,\; \mathbf{P}((X^x_{t-}, X^x_t) \in \mathcal{V}_\psi, \forall t > 0) = 1 \tag{4}$$

where

$$\mathcal{V}_\psi = \{(x,y) \in M \times M \; ; \; \textit{there exists a unique } z \textit{ in } \mathbf{R}^d \textit{ such that } y = \psi(x,z)\}.$$

Then, the X^x are time homogeneous Markov processes with the same transition semigroup for all x in M if and only if Z is a Lévy process.

Proof : If Z is a Lévy process we already know by prop.1 of [4] that the X^x are time homogeneous Markov processes. We will prove the converse result.

The procedure is to write Z as an additive functional of X^x in order to show that the hypothesis of proposition 1 is valid and then the result follows immediatly.

For $x \in M$, by definition 3, $d_z\psi(x,0)$ is an isomorphism from $\mathbf{R}^d$ onto T_xM; denote by $\Phi(x)$ the inverse isomorphism and for $(x,y) \in \mathcal{V}_\psi$ denote by $\Gamma(x,y)$ the unique z in $\mathbf{R}^d$ such that $y = \psi(x,z)$.

Recall the notations introduced in the first part concerning the canonical space $\hat{\Omega}$ of all càdlàg M-valued functions. We choose as a candidate for our additive process on $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_t, P^x)$ the following :

$$\begin{aligned} A_t &= \int_0^t \Phi_\alpha(\hat{X}_{s-})\, d\hat{X}^\alpha_s \\ &- \frac{1}{2}\int_0^t \Phi(\hat{X}_s) \circ d^2_{zz}\psi(\hat{X}_s, 0) \circ (\Phi_\alpha(\hat{X}_s) \otimes \Phi_\beta(\hat{X}_s))\, d < \hat{X}^{\alpha c}, X^{\beta c} >_s \\ &+ \sum_{s \leq t} \left(\Gamma(\hat{X}_{s-}, \hat{X}_s) - \Phi_\alpha(\hat{X}_{s-})\, \Delta\hat{X}^\alpha_s\right) \end{aligned} \tag{5}$$

for any embedding $(x^\alpha)_{1 \leq \alpha \leq m}$ of M into $\mathbf{R}^m$.
Following [2] th.3.12, there exists a version of A such that (5) is valid for every probability P^x, $x \in M$.

Take any x in M. An easy computation based on the stochastic differential equation (2) solved by X^x yields

$$\begin{aligned} A_t(X^x) &= \int_0^t dZ_s + \sum_{s \leq t} (\Gamma(X^x_{s-}, X^x_s) - \Delta Z_s) \\ &= Z_t \end{aligned}$$

since $\forall s \geq 0, \Delta Z_s = \Gamma(X^x_{s-}, X^x_s)$. This proves that the process A satisfies condition (ii) of prop.1.

On the other hand, A is clearly additive and so also satisfies condition (i) of prop.1. □

Before we study the geometrical aspect of the inverting assumptions, let us look at the special case where the manifold M is the whole of $\mathbf{R}^d$.

2.3 The vectorial case

We now take $M = \mathbf{R}^d$ and choose $f \in \mathcal{C}^3(\mathbf{R}^d, \mathbf{R}^{d\times d})$ such that $f(x) \in GL(d)$, $\forall x \in \mathbf{R}^d$. We define a jump coefficient ψ as follows :

$$\forall x, z \in \mathbf{R}^d \ ; \ \psi(x,z) = \psi(x,z,1)$$

where

$$(\psi(x,z,u))_{0\leq u\leq 1} = (y(u))_{0\leq u\leq 1}$$

is the (possibly exploding) unique solution of the ordinary differential equation

$$y(0) = x \ ; \ \frac{dy}{du}(u) = f(y(u)).z \tag{6}$$

Since $d_z\psi(x,0) = f(x)$, ψ is an invertible jump coefficient and the stochastic differential equation (2) becomes :

$$\begin{aligned} \forall \alpha = 1,...,d \qquad X_t^\alpha &= x + \int_0^t f_i^\alpha(X_{s-})\, dZ_s^i \\ &+ \frac{1}{2}\int_0^t \frac{\partial f_i^\alpha}{\partial x^\beta}(X_s) f_j^\beta(X_s)\, d < Z^{ic}, Z^{jc} >_s \\ &+ \sum_{s\leq t}\left(\psi^\alpha(X_{s-},\Delta Z_s) - X_{s-}^\alpha - f_i^\alpha(X_{s-})\,\Delta Z_s^i\right). \end{aligned} \tag{7}$$

It is of the type introduced by Kurtz, Pardoux and Protter in [5]. To be able to apply theorem 2, we must verify the condition (4). A sufficient condition for this is :

$$\exists F \in \mathcal{C}^1(\mathbf{R}^d, \mathbf{R}^d) \text{ such that } \forall x \in \mathbf{R}^d \ , \ dF(x) = (f(x))^{-1}. \tag{8}$$

In fact, if (8) holds, at every jump time s, there will be exactly one z in $\mathbf{R}^d$ such that $X^x_s = \psi(X^x_{s-}, z)$, which is given by $z = \Delta Z_s = F(X^x_s) - F(X^x_{s-})$.

Remarks :

a) In the one-dimensional case, condition (8) reduces to "$\forall x \in \mathbf{R}^d$, $f(x) \neq 0$" since existence of a primitive F of $1/f$ is then assured.

b) One can use the definition of a closed 1-form (see [1] p.207) to give the following equivalent form of condition (8), where $g(x)$ denotes the inverse matrix of $f(x)$:

$$\forall i, \alpha, \beta \in \{1,...,d\}, \forall x \in \mathbf{R}^d \ , \ \frac{\partial g^i_\alpha(x)}{\partial x^\beta} = \frac{\partial g^i_\beta(x)}{\partial x^\alpha}.$$

2.4 Comments on the inverting assumptions

As a conclusion let us comment about the inverting assumptions we have required. First of all, we give an example.

An example of invertible jump coefficient. Suppose there exist $(L_1, ..., L_d)$ vector fields on M such that for all x in M, $(L_i(x))_{1\leq i\leq d}$ is a basis of T_xM. Define the map ψ by

$$(x,z) \in M \times \mathbf{R}^d \ \mapsto \ \psi(x,z) = Exp_x(\sum_i z^i L_i(x))$$

where Exp_x denotes the exponential mapping at x. Then ψ is clearly a jump coefficient as defined in definition 2. Moreover, for all x in M, the differential of $\psi(x,.)$ at 0 is given by

$$h \in \mathbf{R}^d \ \mapsto \ d_z\psi(x,0).h = \sum_i h^i L_i(x)$$

and therefore is an isomorphism. According to definition 3, ψ is an invertible jump coefficient. Next we look at condition (4) of theorem 2.

Condition (4) is a "usual" condition. Let us recall a paper with M.Pontier [7] where the horizontal lift of a càdlàg manifold valued semimartingale is defined. This lift exists provided the semimartingale X satisfies some hypothesis (H) very similar to condition (4).

(H) : *with probability one, there exists one and only one geodesic curve between X_{t-} and X_t for all $t \geq 0$.*

Note that this hypothesis is stronger than that given in [7], but it is actually the right one. We have just become aware of this error in [7]. However all the results theirein are valid under the correct hypothesis (H).

Let us also mention that in [3] another horizontal lift is defined for all càdlàg manifold valued semimartingale without any hypothesis (H). However, as we will see, we cannot deal here without condition (4).

Condition (4) is essential for th.2. We give an example where (4) is not fulfilled and theorem 2 does not apply.

Let M be the unit circle : $M = \{e^{i\theta} \ ; \ \theta \in [0, 2\pi[\}$ and take for jump coefficient $\psi : (x,z) \in M \times \mathbf{R} \ \mapsto \ \psi(x,z) = xe^{i\pi z}$.

Let $(T_n)_{n\geq 1}$ be a sequence of random exponential times and, independently, $(Y_n)_{n\geq 1}$ be a sequence of independent random variables with the same Poisson distribution. We define the processes Z and $\tilde{Z}$ by

$$\begin{aligned} Z_t &= \sum_{n\geq 1} \mathbb{1}_{T_n \leq t} Y_n \\ \tilde{Z}_t &= \sum_{n\geq 1} \mathbb{1}_{T_n \leq t} (Y_n + 2n\mathbb{1}_{Y_{n+1}=0}). \end{aligned}$$

Process Z is a Lévy process (actually a compound Poisson process) whereas $\tilde{Z}$ is not a Lévy process (the variables $(Y_n + 2n\mathbb{1}_{Y_{n+1}=0})_n$ are neither independent nor are they identically distributed).

Now let X^x be the solution of

$$X_0 = x \; ; \; \overset{\triangle}{d}X = \psi(X, \overset{\triangle}{d}\tilde{Z}). \tag{9}$$

By the definition of $\tilde{Z}$ and ψ, it is clear that $X_t^x = x\, e^{i\pi\tilde{Z}_t}$. But, since $\pi\tilde{Z}_t - \pi Z_t \in 2\pi\mathbb{N}$, one also has $X_t^x = x\, e^{i\pi Z_t}$. This proves that X^x is the solution of (9) where $\tilde{Z}$ has been replaced by Z, and since Z is Lévy, X^x is a Markov process with transition semigroup independent of x. Hence we obtain a Markov solution of a stochastic differential equation driven by a non-Lévy process.

Of course, the assumption (4) of th.2 is not valid since $\mathcal{V}_\psi$ is empty !

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ANNE ESTRADE
URA MAPMO - DÉPARTEMENT DE MATHÉMATIQUES
UNIVERSITÉ D'ORLÉANS - BP 6759
45067 ORLEANS CEDEX 2