JACOB FELDMAN
MEIR SMORODINSKY

Simple examples of non-generating Girsanov processes

Séminaire de probabilités (Strasbourg), tome 31 (1997), p. 247-251
<http://www.numdam.org/item?id=SPS_1997__31__247_0>

Simple examples of non-generating Girsanov processes

J.Feldman*and M.Smorodinsky
University of California, Berkeley and Tel-Aviv University

Let $B(t), 0 \leq t < \infty$ be a Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$ with $B_0 = 0$. Let $\mathcal{F}(t), 0 \leq t \leq \infty$ be its filtration, with $\mathcal{F}(\infty) = \mathcal{F}$. We construct simple examples of probability measures $P' \sim P$ for which this filtration is not generated by the corresponding Girsanov process, but is nevertheless generated by *some* process which is a Brownian motion for the measure P'.

1. Introduction. Given a Brownian motion and $P' \sim P$ as above, the corresponding Radon-Nikodym derivative may be written in the form

$$dP'/dP = exp\{\int_0^\infty \Phi(t)dB(t) - (1/2)\int_0^\infty |\Phi(t)|^2 dt\},$$

where Φ is a process on $[0, \infty)$ adapted to the filtration of B and satisfying certain other conditions which, in particular, cause the expression to make sense and have expectation 1. This is the *Cameron-Martin-Girsanov formula.* $\Phi(t)$ is uniquely determined *a.e.* in t. While it is not easy to characterize exactly those processes Φ which arise in this manner(See Kazamaki's recent monograph [K]), we note that if Φ is adapted to the filtration of B, and $\int_0^\infty |\Phi|^2(t)dt$ is bounded by a fixed constant, then Φ arises in such a way. The associated *Girsanov Process G* defined by

$$G(t) = B(t) - \int_0^t \Phi(s)ds, 0 \leq t < \infty$$

is a Brownian motion with respect to the measure P' (*"Girsanov's Theorem"*). Let $\mathcal{G}(t), 0 \leq t \leq \infty$ be its filtration. Because Φ is adapted to the filtration of the original Brownian motion, we have always $\mathcal{G}(t) \subset \mathcal{F}(t)$ for all t. A good reference for these matters is [RY].

The question then arises, whether the Girsanov Process always generates the filtration of B, i.e. whether $\mathcal{G}(t) = \mathcal{F}(t)$ for all t in $[0, \infty)$; note that this will follow for all such t if it holds for $t = \infty$. This question is of relevance for Stochastic Differential Equations. In 1975 B.Tsirelson showed that the answer is *no*: in [T] he constructed a P' for which the Girsanov Process does *not* generate this filtration. For further discussion of this important example see [Y].

An obvious next question, explicitly asked in [RY], is whether at least there is for

*Supported in part by NSF Grant #DMS 9113642.

every $P' \sim P$ *some* process which is a Brownian motion for P' and whose filtration coincides with that of the original Brownian motion. The answer again is *no*: in [DFST] there are constructed measures $P' \sim P$ for which no such Brownian motion for P' exists, i.e for which the filtration $\mathcal{F}(t), 0 \le t \le \infty$ is not Brownian with respect to P'.

This leaves open the following question: can it happen that $\mathcal{F}(t), 0 \le t \le \infty$ is Brownian for P' but the corresponding Girsanov process does not generate the filtration of B? One would expect so, and this is what we show here:

Theorem: *Given a Brownian motion $B(t), 0 \le t < \infty$ on $(\Omega, \mathcal{F}, \mathcal{P})$, there exist probability measures $P' \sim P$ for which the filtration of B is Brownian with respect to P' as well, but for which the corresponding Girsanov Process does not generate.*

Our examples are simple, and the described properties are easy to demonstrate. What about the P' of Tsirelson's 1975 example? We have not determined whether the filtration of B is Brownian for this P'. However, A.M.Vershik tells us he can show that it is.

It should be remarked that although this paper is in no way dependent on it, our examples were motivated by considerations coming from the study of decreasing sequences of sigma fields, for which see [V]. We also note that questions of this type for discrete time processes were studied by M.Rosenblatt [R], and one of his constructions there may be viewed as analogous to ours.

We thank Marc Yor for his careful reading and helpful suggestions.

2. A class of examples. Choose $\infty > t_1 > t_2 > \cdots \to 0$, and a sequence $a_1, a_2, \cdots$ of positive numbers with $\sum_{n=0}^{\infty} a_n^2 (t_n - t_{n+1}) < \infty$; for example, any *bounded* sequence will do. For a subset S of the reals, let $\chi_S = -1$ on S and 1 elsewhere. Let $S_1, S_2, \cdots$ be measurable subsets of the reals, and $\sigma_n = \chi_{S_n}(B(t_n) - B(t_{n+1}))$, i.e. -1 if $B(t_n) - B(t_{n+1})$ lies in S_n and 1 otherwise. Let $\Phi(t) = \sigma_{n+1} a_n$ if $t_{n+1} < t \le t_n$ and zero if $t_1 < t$. The following lemma is just a small calculation:

Lemma 1: *$\int_0^\infty |\Phi|^2(t)dt$ is bounded by a constant, in fact equals $\sum_{n=1}^{\infty} a_n^2 (t_n - t_{n+1})$, so it defines a probability measure $P' \sim P$ with Radon-Nikodym derivative*

$$dP'/dP = exp\{\sum_{n=1}^{\infty} \sigma_{n+} a_n (B(t_n) - B(t_{n+1})) - (1/2) \sum_{n=1}^{\infty} a_n^2 (t_n - t_{n+1})\}.$$

Denote by $N(m, v)$ the normal distribution with mean m and variance v. The following remarks follow from Lemma 1 and Girsanov's Theorem.

Remark 2: The random variables $y_1, y_2, \cdots$ defined by

$$y_n = G(t_n) - G(t_{n+1}) = B(t_n) - B(t_{n+1}) - \sigma_{n+1} a_n (t_n - t_{n+1}).$$

form an independent sequence of random variables with respect to P', y_n having distribution $N(0, t_n - t_{n+1})$.

Remark 3: The sequence of random variables $x_1, x_2, \cdots$ defined by

$$x_n = B(t_n) - B(t_{n+1})$$

forms, with respect to P', a Markov stochastic process; the conditional measure $P'(\cdot|x_{n+1})$ is either $N(b_n, t_n - t_{n+1})$ or $N(-b_n, t_n - t_{n+1})$, depending on whether x_{n+1} is in S_{n+1} or not, where $b_n = a_n(t_n - t_{n+1})$. The total distribution of x_n, call it μ_n, is just their average, weighted by $(\mu_{n+1}(S_{n+1}), 1 - \mu_{n+1}(S_{n+1}))$. Let us further define

$$y(t) = G(t) - G(t_{n(t)}), 0 \leq t < \infty,$$

where $n(t) = min\{n : t_n \leq t\}$, and let $\mathcal{G}_{n-1}$ be the sigma-field generated by $y(t), t_n < t \leq t_{n-1}$. Then the following stronger Markov relation holds: x_n is P' conditionally independent of $\mathcal{F}(t_{n+1}) \vee \mathcal{G}_n$ given y_n.

3. Proof of Theorem.

Lemma 4: *For each nonatomic probability measure μ on the real line and each $b > 0$ there exists a measurable subset S of the real line, in fact a countable union of intervals, with $\mu(S) = 1/2$, and a.e. symmetric about zero, so that for each real number η exactly one of the two numbers $\eta + b, \eta - b$ lies in S.*

Proof: Let $I(n)$ be the half-open interval $[(n-1)b, (n+1)b)$,and for $0 \leq r \leq b$ let $J(n,r)$ be the half-open interval $[nb - r, nb + r)$. Let $S(r)$ be the union of the sets $J(4n+2, r)$ and $I(4n) \cap J(4n, r)^c$ over all integers n. This set is *a.e.* symmetric: the only differences between $S(r)$ and $-S(r)$ occur at end points of the constituent intervals. The sets $S(0)$ and $S(b)$ form a partition of the real line, so $\mu(S(0)) + \mu(S(b)) = 1$. The map $r \mapsto \mu(S(r))$ is continuous, so there is some r in $(0, b)$ with $\mu(S(r)) = 1/2$. Setting $S = S(r)$, we are done.

Now choose the sets S_n by setting $b = b_n$ in Lemma 4 and μ equal to the $(1/2, 1/2)$ average of $N(b_n, t_n - t_{n+1})$ and $N(-b_n, t_n - t_{n+1})$. Then $P'[x_n \in S_n]$ will be 1/2 for each n. Each value η of y_n could have come from either of the two values $\eta + b_n$ or $\eta - b_n$ for x_n; and S_n has been so chosen that for each η exactly one of these lies in S_n. Additionally, each S_n is *a.e.* symmetric about zero. We proceed to prove that the Girsanov process constructed by means of this sequence of sets does not generate the filtration of B.

Denote by $E'(\cdot|\cdot)$ conditional expectation with respect to P'. Then $E'(\sigma_1|y_1; x_2) = E'(\sigma_1|y_1; \sigma_2) = 1$ or -1 with equal P' probability. Integrating out x_2 gives $E'(\sigma_1|y_1) = 0$ *a.e.* Repeating this argument inductively with $E'(\sigma_1|y_1, y_2, ..., y_n; \sigma_{n+1})$ gives with probability one:

$$E'(\sigma_1|y_1, y_2, ..., y_n) = 0$$

for all integers $n > 0$. It follows from the Markov relation in Remark 3 that $E'(\sigma_1|\mathcal{G}(t_1)) = E'(\sigma_1|y_1, y_2, ...)$, so σ_1 is P' independent of $\mathcal{G}(t_1)$. But σ_1 is measurable with respect to $\mathcal{F}(t_1)$. So $\mathcal{G}(t_1)$ is not all of $\mathcal{F}(t_1)$.

Next we show that there is a process $B'(t), 0 \leq t < \infty$ which is a Brownian motion under P' and whose filtration is precisely that of B. First introduce the random variables $y'_n = \sigma_{n+1} y_n$ and $y'(t) = \sigma_{n(t)+1} y(t)$. Then put

$$B'(t) = y'(t) + \sum_{i=n(t)}^{\infty} y'_i = \int_0^t \sigma_{n(s)+1} dy(s).$$

It is clear that B' is a Brownian motion with respect to P'.
Let $x(t) = B(t) - B(t_{n(t)})$. To prove our claim it suffices to show that $x(t)$ is $\mathcal{F}'(t)$ measurable for all $t > 0$, where $\mathcal{F}'(t), 0 \leq t \leq \infty$ is the filtration generated by B'.

We claim that for any $t > 0$, $x(t)$ is *a.e* a function of the variables $y'(t), y'_{n(t)}, y'_{n(t)+1}$. For if $y'_{n(t)+1}$ takes on the value η then either:

(1) $\sigma_{n(t)+2} = -1$, so $y_{n(t)+1} = -\eta$, and $x_{n(t)+1} = -\eta + b_n$, or

(2) $\sigma_{n(t)+2} = 1$, so $y_{n(t)+1} = \eta$, and $x_{n(t)+1} = \eta - b_n$.

Thus $y'_{n(t)+1}$ completely determines $|x_{n(t)+1}|$. Since $S_{n(t)+1}$ is *a.e.* symmetric, $y'_{n(t)+1}$ determines with probability one the distribution of $x_{n(t)}$, so $y'_{n(t)}$ and $y'_{n(t)+1}$ together determine $x_{n(t)}$ with probability one. But clearly the pair $(y'(t), x_{n(t)})$ determines $x(t)$ with probability one. This completes the proof of the theorem.

Remark 5: Even without the symmetry assumption on the sets $S_1, S_2, \cdots$ the process B' generates the filtration of B, but the argument is a little more involved.

References:

[DFST] L. Dubins, J.Feldman, M.Smorodinsky, and B.S.Tsirelson(1995): *Decreasing sequences of σ-fields and a measure change for Brownian motion*, to appear, Ann.Prob.

[K] N.Kazamaki (1994): *Continuous exponential martingales and BMO*, Lec.Notes Math.1579, Springer-Verlag.

[R] M.Rosenblatt (1959): *Stationary processes as shifts of functions of independent random variables*, Jour.Math.Mech. 8, pp.665-681.

[RY] D.Revuz, and M.Yor (1991): *Continuous Martingales and Brownian Motion*, Springer, Berlin.

[T] B.S.Tsirelson, (1975): *An example of a stochastic differential equation having no strong solution*, Theor.Prob.Appl. 20, pp.416-418.

[V] A.M.Vershik, (1994): *Theory of decreasing sequences of measurable partitions*, Alg. Anal. 6, pp.1-68,in Russian; English version to appear in St.Petersburg Math. Jour. 6.

[Y] M.Yor, (1992): *Tsirelson's equation in discrete time*, Prob.Th. Related Fields, 91, pp.135-152.