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On continuous conditional Gaussian martingales and stable convergence in law

Jean Jacod

In this paper, we start with a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0,1]}, P)$, the time interval being $[0, 1]$, on which are defined a “basic” continuous local martingale M and a sequence Z^n of martingales or semimartingales, asymptotically “orthogonal to all martingales orthogonal to M ”. Our aim is to give some conditions under which Z^n converges “stably in law” to some limiting process which is defined on a suitable extension of $(\Omega, \mathcal{F}, \mathbb{F}, P)$.

In the first section we study systematically some, more or less known, properties of extensions of filtered spaces and of $\mathcal{F}$ -conditional Gaussian martingales and so-called M -biased $\mathcal{F}$ -conditional Gaussian martingales. Then we explain our limit results: in Section 2 we give a fairly general result, and in Section 3 we specialize to the case when Z^n is some “discrete-time” process adapted to the discretized filtration $\mathbb{F}^n = (\mathcal{F}_t^n)_{t \in [0,1]}$, where $\mathcal{F}_t^n = \mathcal{F}_{[nt]/n}$. Finally, Section 4 is devoted to studying the limit of a sequence of M -biased $\mathcal{F}$ -conditional Gaussian martingales.

1 Extension of filtered spaces and conditionally Gaussian martingales

We begin with some general conventions. Our filtrations will always be assumed to be right-continuous. All local martingales below are supposed to be 0 at time 0, and we write $\langle M, N \rangle$ for the predictable quadratic variation between M and N if these are locally square-integrable martingales. When M and N are respectively d - and r -dimensional, then $\langle M, N^* \rangle$ is the $d \times r$ dimensional process with components $\langle M, N^* \rangle^{i,j} = \langle M^i, N^j \rangle$ (N^* stands for the transpose of N).

In all these notes, we have a basic filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$.

1-1. Let us start with some definitions. We call *extension* of $(\Omega, \mathcal{F}, \mathbb{F}, P)$ another filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{P})$ constructed as follows: starting with an auxiliary filtered space $(\Omega, \mathcal{F}', \mathbb{F}' = (\mathcal{F}'_t)_{t \in [0,1]})$ such that each σ -field $\mathcal{F}'_{t-}$ is separable, and a transition probability $Q_\omega(d\omega')$ from $(\Omega, \mathcal{F})$ into $(\Omega', \mathcal{F}')$, we set

$$\tilde{\Omega} = \Omega \times \Omega', \quad \tilde{\mathcal{F}} = \mathcal{F} \otimes \mathcal{F}', \quad \tilde{\mathcal{F}}_t = \cap_{s>t} \mathcal{F}_s \otimes \mathcal{F}'_s, \quad \tilde{P}(d\omega, d\omega') = P(d\omega)Q_\omega(d\omega'). \quad (1.1)$$

According to ([3], Lemma 2.17), the extension is called *very good* if all martingales

on the space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ are also martingales on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{P})$, or equivalently, if $\omega \rightsquigarrow Q_\omega(A')$ is $\mathcal{F}_t$ -measurable whenever $A' \in \mathcal{F}'_t$.

A process Z on the extension is called an $\mathcal{F}$ -conditional martingale (resp. $\mathcal{F}$ -Gaussian process) if for P -almost all ω the process $Z(\omega, \cdot)$ is a martingale (resp. a centered Gaussian process) on the space $(\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t \in [0,1]}, Q_\omega)$.

Let us finally denote by $\mathcal{M}_b$ the set of all bounded martingales on $(\Omega, \mathcal{F}, \mathbb{F}, P)$.

Proposition 1-1: *Let Z be a continuous adapted q -dimensional process on the very good extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{P})$, with $Z_0 = 0$. The following statements are equivalent:*

- (i) *Z is a local martingale on the extension, orthogonal to all elements of $\mathcal{M}_b$, and the bracket $\langle Z, Z^* \rangle$ is $(\mathcal{F}_t)$ -adapted.*
- (ii) *Z is an $\mathcal{F}$ -conditional Gaussian martingale.*

In this case, the $\mathcal{F}$ -conditional law of Z is characterized by the process $\langle Z, Z^* \rangle$ (i.e., for P -almost all ω , the law of $Z(\omega, \cdot)$ under Q_ω depends only on the function $t \rightsquigarrow \langle Z, Z^* \rangle_t(\omega)$).

Proof. a) We first prove that, if each Z_t is $\tilde{P}$ -integrable, then Z is an $\mathcal{F}$ -conditional martingale iff it is an $\tilde{\mathbb{F}}$ -martingale orthogonal to all bounded $\mathbb{F}$ -martingales. For this, we can and will assume that Z is 1-dimensional.

Let $t \leq s$ and let U, U' be bounded measurable function on $(\Omega, \mathcal{F}_t)$ and $(\Omega', \mathcal{F}'_t)$ respectively. Let also $M \in \mathcal{M}_b$. We have

$$\tilde{E}(UU' M_s Z_s) = \int P(d\omega) U(\omega) M_s(\omega) \int Q_\omega(d\omega') U'(\omega') Z_s(\omega, \omega'), \quad (1.2)$$

$$\tilde{E}(UU' M_t Z_t) = \int P(d\omega) U(\omega) M_t(\omega) \int Q_\omega(d\omega') U'(\omega') Z_t(\omega, \omega'). \quad (1.3)$$

Assume first that Z is an $\mathcal{F}$ -conditional martingale. Then for P -almost all ω we have

$$\int Q_\omega(d\omega') U'(\omega') Z_s(\omega, \omega') = \int Q_\omega(d\omega') U'(\omega') Z_t(\omega, \omega'),$$

and the latter is $\mathcal{F}_t$ -measurable as a function of ω because the extension is very good. Since M is an $\mathbb{F}$ -martingale, we deduce that (1.2) and (1.3) are equal: thus MZ is a martingale on the extension: then Z is a martingale (take $M \equiv 1$), orthogonal to all bounded $\mathbb{F}$ -martingales.

Next we prove the sufficient condition. Take V bounded and $\mathcal{F}_s$ -measurable, and consider the martingale $M_r = E(V|\mathcal{F}_r)$. With the notation above we have equality between (1.2) and (1.3), and further in (1.3) we can replace $M_t(\omega)$ by $M_s(\omega) = V(\omega)$ because the last integral is $\mathcal{F}_t$ -measurable in ω . Then taking $U = 1$ we get

$$\int P(d\omega) V(\omega) \int Q_\omega(d\omega') U'(\omega') Z_s(\omega, \omega') = \int P(d\omega) V(\omega) \int Q_\omega(d\omega') U'(\omega') Z_t(\omega, \omega').$$

Hence for P -almost ω , $Q_\omega(U' Z_s(\omega, \cdot)) = Q_\omega(U' Z_t(\omega, \cdot))$. Using the separability of the σ -field $\mathcal{F}'_{t-}$ and the continuity of Z , we have this relation P -almost surely in

ω , simultaneously for all $t \leq s$ and all $\mathcal{F}'_{t-}$ -measurable variable U' : this gives the $\mathcal{F}$ -conditional martingality for Z .

b) Assume that (i) holds. If $Y = \langle Z, Z^* \rangle$, a simple application of Ito's formula and the fact that Z is continuous show that, since Z is orthogonal to all $M \in \mathcal{M}_b$, the same holds for Y . Each $T_n = \inf(t : |\langle Z, Z^* \rangle_t| > n)$ is an $\mathbb{F}$ -stopping time, and $T_n \uparrow \infty$ as $n \rightarrow \infty$. Then $Z(n)_t = Z_t \wedge_{T_n}$ and $Y(n)_t = Y_t \wedge_{T_n}$ are continuous $\tilde{\mathbb{F}}$ -martingale, orthogonal to all $M \in \mathcal{M}_b$, and obviously $|Z(n)_t|$ and $|Y(n)_t|$ are integrable: by (a), and by letting $n \uparrow \infty$, we deduce that for P -almost all ω , under Q_ω the process $Z(n)(\omega, \cdot)$ is a continuous martingale with deterministic bracket $\langle Z, Z^* \rangle(\omega)$, hence it is an $\mathcal{F}$ -Gaussian martingale, so we have (ii). Furthermore, it is well-known that the law of $Z(\omega)$ under Q_ω is then entirely determined by $\langle Z, Z^* \rangle(\omega)$.

c) Assume now (ii). There is a P -full set $A \in \mathcal{F}$ such that for all $\omega \in A$, under Q_ω , the process $Z(\omega, \cdot)$ is both centered Gaussian and an $\mathbb{F}'$ -martingale. Therefore if $F_t(\omega) = \int Q_\omega(d\omega') Z_t(\omega, \omega')$, the process $(ZZ^*)(\omega, \cdot) - F(\omega)$ is an $\mathbb{F}'$ -martingale under Q_ω for $\omega \in A$: that is, $ZZ^* - F$ is an $\mathcal{F}$ -conditional martingale. By localizing at the $\mathbb{F}$ -stopping times $T_n = \inf(t : |F_t| > n)$ and by (a), we deduce that Z and $ZZ^* - F$ are local martingales on the extension, orthogonal to all $M \in \mathcal{M}_b$. Since F is continuous, $\mathbb{F}$ -adapted, and of bounded variation (since it is non-decreasing for the strong order in the set of nonnegative symmetric matrices), it follows that it is a version of $\langle Z, Z^* \rangle$, hence we have (i). $\square$

1-2. Let now M be a continuous d -dimensional local martingale, and $\mathcal{M}_b(M^\perp)$ be the class of all elements of $\mathcal{M}_b$ which are orthogonal to M (i.e., to all components of M).

A q -dimensional process Z on the extension is called an *M -biased $\mathcal{F}$ -conditional Gaussian martingale* if it can be written as

$$Z_t = Z'_t + \int_0^t u_s dM_s, \quad (1.4)$$

where Z' is an $\mathcal{F}$ -conditional Gaussian martingale and u is a predictable $\mathbb{R}^q \otimes \mathbb{R}^d$ on $(\Omega, \mathcal{F}, \mathbb{F}, P)$.

Proposition 1-2: *Let Z be a continuous adapted q -dimensional process on the very good extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{P})$, with $Z_0 = 0$. The following statements are equivalent:*

- (i) *Z is a local martingale on the extension, orthogonal to all elements of $\mathcal{M}_b(M^\perp)$, and the brackets $\langle Z, Z^* \rangle$ and $\langle Z, M^* \rangle$ are $\mathbb{F}$ -adapted.*
- (ii) *Z is an M -biased $\mathcal{F}$ -conditional Gaussian martingale.*

In this case, the $\mathcal{F}$ -conditional law of Z is characterized by the processes M , $\langle Z, Z^ \rangle$ and $\langle Z, M^* \rangle$.*

Proof. Under either (i) or (ii), Z and M are continuous local martingales (use the fact that the extension is very good, and use (1.4) under (ii)). We write $F = \langle Z, Z^* \rangle$, $G = \langle Z, M^* \rangle$ and $H = \langle M, M^* \rangle$.

If (ii) holds, (1.4) and Proposition 1-1 yield for all $N \in \mathcal{M}_b$:

$$G_t = \int_0^t u_s^* dH_s, \quad F_t = \langle Z', Z'^* \rangle_t + \int_0^t u_s^* dH_s u_s^*, \quad \langle Z, N \rangle_t = \int_0^t u_s^* d\langle M, N \rangle_s. \quad (1.5)$$

Then (i) readily follows. Further, (1.5) implies that u and $\langle Z', Z'^* \rangle$ are determined by F , G and H . Since $\int_0^t u_s dM_s$ is $\mathcal{F}$ -measurable, the last claim follows from (1.4) and Proposition 1-1 again.

Assume conversely (i). There are a continuous increasing process A and predictable processes f, g, h with values in $\mathbb{R}^q \otimes \mathbb{R}^q$, $\mathbb{R}^q \otimes \mathbb{R}^d$ and $\mathbb{R}^d \otimes \mathbb{R}^d$ respectively, such that $F_t = \int_0^t f_s dA_s$, $G_t = \int_0^t g_s dA_s$ and $H_t = \int_0^t h_s dA_s$.

The process (M, Z) is a continuous local martingale on the extension, with bracket $K_t = \int_0^t k_s dA_s$, where $k = \begin{pmatrix} h & g^* \\ g & f \end{pmatrix}$. By triangularization we may write $k = zz^*$, where

$$z = \begin{pmatrix} v & 0 \\ uv & w \end{pmatrix}, \quad (1.6)$$

so that $h = vv^*$, $g = uvv^*$ and $f = uvv^*u^* + ww^*$. Let us put $Y_t = \int_0^t u_s dM_s$ and $Z' = Z - Y$. Then since the extension is very good, Z' is a local martingale on the extension, and $\langle Z', Z'^* \rangle_t = \int_0^t w_s w_s^* dA_s$ is $\mathbb{F}$ -adapted. Further, $\langle Z', N \rangle_t = \langle Z, N \rangle_t - \int_0^t u_s d\langle M, N \rangle_s$: first this implies that $\langle Z', N \rangle = 0$ if $N \in \mathcal{M}_b(M^\perp)$ (since then $\langle Z, N \rangle = 0$ by hypothesis), second this implies that when $N_t = \int_0^t \alpha_s dM_s$ we have $\langle Z', N \rangle_t = \int_0^t (g_s \alpha_s^* - u_s v_s v_s^* \alpha_s) dA_s = 0$. Thus Z' is orthogonal to all $N \in \mathcal{M}_b$, and it is an $\mathcal{F}$ -conditional Gaussian martingale by Proposition 1-1. $\square$

1-3. Let us denote by $\mathcal{S}_r$ the set of all symmetric nonnegative $r \times r$ -matrices. In Proposition 1.1, the process $\langle Z, Z^* \rangle$ is a continuous adapted non-decreasing $\mathcal{S}_q$ -valued process, null at 0. In Proposition 1-2, the bracket of (M, Z) is a continuous adapted non-decreasing $\mathcal{S}_{d+q}$ -valued process, null at 0. Conversely we have:

Proposition 1-3: a) Let F be a continuous adapted nondecreasing $\mathcal{S}_q$ -valued process, with $F_0 = 0$, on the basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$. There exists a continuous $\mathcal{F}$ -conditional Gaussian martingale Z on a very good extension, such that $\langle Z, Z^* \rangle = F$.

b) Let K be a continuous adapted nondecreasing $\mathcal{S}_{d+q}$ -valued process, with $K_0 = 0$, and M be a continuous d -dimensional local martingale with $\langle M^i, M^j \rangle = K^{ij}$ for $1 \leq i, j \leq d$, on the basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$. There exists a continuous M -biased $\mathcal{F}$ -conditional Gaussian martingale Z on a very good extension, such that $\langle Z^i, M^j \rangle = K^{d+i, j}$ for $1 \leq i \leq q$, $1 \leq j \leq d$, and $\langle Z^i, Z^j \rangle = K^{d+i, d+j}$ for $1 \leq i, j \leq q$.

Of course (a) is a particular case of (b) (take $M = 0$), but in the proof below (b) is obtained as a consequence of (a).

Proof. a) Take $(\Omega', \mathcal{F}', \mathbb{F}')$ to be the canonical space of all $\mathbb{R}^d$ -valued continuous functions on $[0, 1]$, with the usual filtration and the canonical process $Z_t(\omega') = \omega'(t)$. For each ω , denote by Q_ω the unique probability measure on $(\Omega', \mathcal{F}')$ under which Z is a centered Gaussian process with covariance $\int Z_t Z_s^* dQ_\omega = F_s \wedge_t(\omega)$. This structure

of the covariance implies that Z has independent increments and thus is a martingale under each Q_ω : Defining $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{P})$ by (1.1) gives the result.

b) As in the previous proof, we can write $K_t = \int_0^t k_s dA_s$ for a continuous adapted increasing process A and a predictable process $k = zz^*$ with z as in (1.6). By (a) we have a continuous $\mathcal{F}$ -conditional Gaussian martingale Z' on a very good extension, with $\langle Z', Z'^* \rangle_t = \int_0^t w_s w_s^* dA_s$. We can set $Z_t = Z'_t + \int_0^t u_s dM_s$, and some computations yields that Z satisfies our requirements. $\square$

We even have a more “concrete” way of constructing Z above, when K is absolutely continuous w.r.t. Lebesgue measure on $[0, 1]$. Let $(\Omega^W, \mathcal{F}^W, \mathbb{F}^W, P^W)$ be the q -dimensional Wiener space with the canonical Wiener process W . Then $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{P})$ defined by

$$\tilde{\Omega} = \Omega \times \Omega^W, \quad \tilde{\mathcal{F}} = \mathcal{F} \otimes \mathcal{F}^W, \quad \tilde{\mathcal{F}}_t = \cap_{s>t} \mathcal{F}_s \otimes \mathcal{F}_s^W, \quad \tilde{P} = P \otimes P^W. \quad (1.7)$$

is a very good extension of $(\Omega, \mathcal{F}, \mathbb{F}, P)$, called the *canonical q -dimensional Wiener extension* of $(\Omega, \mathcal{F}, \mathbb{F}, P)$. Note that W is also a Wiener process on the extension.

Proposition 1-4: *Let K and M be as in Proposition 1-3(b), and assume that $K_t = \int_0^t k_s ds$ with k predictable $\mathcal{S}_{d+q}$ -valued. Then we can choose a version of k of the form $k = zz^*$ with $z = \begin{pmatrix} v & 0 \\ uv & w \end{pmatrix}$, and on the canonical q -dimensional Wiener extension of $(\Omega, \mathcal{F}, \mathbb{F}, P)$ the process*

$$Z_t = \int_0^t u_s dM_s + \int_0^t w_s dW_s \quad (1.8)$$

is a continuous M -biased $\mathcal{F}$ -conditional Gaussian martingale, such that $\langle Z^i, M^j \rangle = K^{d+i,j}$ for $1 \leq i \leq q$ and $1 \leq j \leq d$, and $\langle Z^i, Z^j \rangle = K^{d+i,d+j}$ for $1 \leq i, j \leq q$.

Proof. The first claim has already been proved. (1.8) defines a continuous q -dimensional local martingale on the canonical Wiener extension and a simple computation shows that it has the required brackets. $\square$

2 Stable convergence to conditionally Gaussian martingales

2-1. First we recall some facts about stable convergence. Let X_n be a sequence of random variables with values in a metric space E , all defined on $(\Omega, \mathcal{F}, P)$. Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ be an extension of $(\Omega, \mathcal{F}, P)$ (as in Section 1, except that there is no filtration here), and let X be an E -valued variable on the extension. Let finally $\mathcal{G}$ be a sub σ -field of $\mathcal{F}$. We say that X_n $\mathcal{G}$ -stably converges in law to X , and write $X_n \xrightarrow{\mathcal{G}-\mathcal{L}} X$, if

$$E(Yf(X_n)) \rightarrow \tilde{E}(Yf(X)) \quad (2.1)$$

for all $f : E \rightarrow \mathbb{R}$ bounded continuous and all bounded variable Y on $(\Omega, \mathcal{G})$. This property, introduced by Renyi [6] and studied by Aldous and Eagleson [1], is (slightly)

stronger than the mere convergence in law. It applies in particular when X_n, X are $\mathbb{R}^q$ -valued càdlàg processes, with $E = \mathcal{D}([0, 1], \mathbb{R}^q)$ the Skorokhod space.

If X'_n are some other E -valued variables, then (with δ denoting a distance on E):

$$\delta(X'_n, X_n) \xrightarrow{P} 0, \quad X_n \xrightarrow{\mathcal{G}-\mathcal{L}} X \Rightarrow X'_n \xrightarrow{\mathcal{G}-\mathcal{L}} X. \quad (2.2)$$

Also, if U_n, U are on $(\Omega, \mathcal{F})$, with values in another metric space E' , then

$$U_n \xrightarrow{P} U, \quad X_n \xrightarrow{\mathcal{G}-\mathcal{L}} X \Rightarrow (U_n, X_n) \xrightarrow{\mathcal{G}-\mathcal{L}} (U, X). \quad (2.3)$$

When $\mathcal{G} = \mathcal{F}$ we simply say that X_n stably converges in law to X , and we write $X_n \xrightarrow{s-\mathcal{L}} X$.

2-2. Now we describe a rather general setting for our convergence results. We start with a continuous d -dimensional local martingale M on the basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$: this will be our “reference” process. The set $\mathcal{M}_b$ is as in Section 1.

Next, for each integer n we are given a filtration $\mathbb{F}^n = (\mathcal{F}_t^n)_{t \in [0, 1]}$ on $(\Omega, \mathcal{F})$ with the following property:

Property (F): We have a d -dimensional square-integrable $\mathbb{F}^n$ -martingale $M(n)$ and, for each $N \in \mathcal{M}_b$, a bounded $\mathbb{F}^n$ -martingale $N(n)$, such that

$$\sup_{n, t, \omega} |N(n)_t(\omega)| < \infty, \quad (2.4)$$

$$\langle M(n), M(n)^* \rangle_t \xrightarrow{P} \langle M, M^* \rangle_t, \quad \forall t \in [0, 1], \quad (2.5)$$

(the bracket above in the predictable quadratic variation relative to $\mathbb{F}^n$) and that, for any finite family $(N^1, \dots, N^m)$ in $\mathcal{M}_b$,

$$(M(n), N^1(n), \dots, N^m(n)) \xrightarrow{P} (M, N^1, \dots, N^m) \quad \text{in } \mathcal{D}([0, 1], \mathbb{R}^{d+m}). \square \quad (2.6)$$

In practice we encounter two situations: first, $\mathcal{F}_t^n = \mathcal{F}_t$, for which (F) is obvious with $M(n) = M$ and $N(n) = N$. Second, $\mathcal{F}_t^n = \mathcal{F}_{[nt]/n}$, a situation which will be examined in Section 3.

2-3. For stating our main result we need some more notation. We are interested in the behaviour of a sequence (Z^n) of q -dimensional processes, each Z^n being an $\mathbb{F}^n$ -semimartingale, and we denote by (B^n, C^n, ν^n) its characteristics, relative to a given continuous truncation function h_q on $\mathbb{R}^q$ (i.e. a continuous function $h_q : \mathbb{R}^q \rightarrow \mathbb{R}^q$ with compact support and $h_q(x) = x$ for $|x|$ small enough): see [5]. If $h'_q(x) = x - h_q(x)$, we can write

$$Z_t^n = B_t^n + X_t^n + \sum_{s \leq t} h'_q(\Delta Z_s^n) \quad (2.7)$$

where X^n is an $(\mathcal{F}_t^n)$ -local martingale with bounded jumps, and $\Delta Y_t = Y_t - Y_{t-}$.

Here is the main result:

Theorem 2-1: Assume Property (F). Assume also that there are two continuous processes F and G and a continuous process B of bounded variation on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ such that (the brackets below being the predictable quadratic variations relative to the filtration $\mathbb{F}^n$):

$$\sup_t |B_t^n - B_t| \xrightarrow{P} 0, \quad (2.8)$$

$$F_t^n := \langle X^n, X^{n*} \rangle_t \xrightarrow{P} F_t, \quad \forall t \in [0, 1], \quad (2.9)$$

$$G_t^n := \langle X^n, M(n)^* \rangle_t \xrightarrow{P} G_t, \quad \forall t \in [0, 1], \quad (2.10)$$

$$U(\varepsilon)^n := \nu^n([0, 1] \times \{x : |x| > \varepsilon\}) \xrightarrow{P} 0, \quad \forall \varepsilon > 0, \quad (2.11)$$

$$V(N)_t^n := \langle X^n, N(n) \rangle_t \xrightarrow{P} 0, \quad \forall t \in [0, 1], \quad \forall N \in \mathcal{M}_b(M^\perp). \quad (2.12)$$

Then

- (i) There is a very good extension of $(\Omega, \mathcal{F}, \mathbb{F}, P)$ and an M -biased continuous $\mathcal{F}$ -conditional Gaussian martingale Z' on this extension with

$$\langle Z', Z'^* \rangle = F, \quad \langle Z', M^* \rangle = G, \quad (2.13)$$

such that $Z^n \xrightarrow{s-\mathcal{L}} Z := B + Z'$.

- (ii) Assuming further that $d\langle M^i, M^i \rangle_t \ll dt$ and $dF_t^{ii} \ll dt$, there are predictable processes u, v, w with values in $\mathbb{R}^q \otimes \mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^d$ and $\mathbb{R}^q \otimes \mathbb{R}^q$ respectively, such that

$$\left. \begin{aligned} \langle M, M^* \rangle_t &= \int_0^t u_s u_s^* ds, \quad G_t = \int_0^t u_s v_s v_s^* ds, \\ F_t &= \int_0^t (u_s v_s v_s^* u_s^* + w_s w_s^* ds, \end{aligned} \right\} \quad (2.14)$$

and the limit of Z^n can be realized on the canonical q -dimensional Wiener extension of $(\Omega, \mathcal{F}, \mathbb{F}, P)$, with the canonical Wiener process W , as

$$Z_t = B_t + \int_0^t u_s dM_s + \int_0^t w_s dW_s. \quad (2.15)$$

The proof will be divided in a number of steps.

Step 1. Let $H^n = \langle M(n), M(n)^* \rangle$ and $H = \langle M, M^* \rangle$. Consider the following processes with values in the set of symmetric $(d+q) \times (d+q)$ matrices:

$$K^n = \begin{pmatrix} H^n & G^{n*} \\ G^n & F^n \end{pmatrix}, \quad K = \begin{pmatrix} H & G^* \\ G & F \end{pmatrix}.$$

By (2.9), (2.10) and (F), we have $K_t^n \xrightarrow{P} K_t$ for all t , while K^n is a nondecreasing process with values in $\mathcal{S}_{d+q}$. So there is a version of K which is also a nondecreasing $\mathcal{S}_{d+q}$ -valued process. Further K is continuous in time, so by a classical result we even have

$$\sup_t |K_t^n - K_t| \xrightarrow{P} 0. \quad (2.16)$$

Further we can write $K_t = \int_0^t k_s dA_s$ for some continuous adapted increasing process A and some predictable $\mathcal{S}_{d+q}$ -valued process k , and as seen in the proof of Proposition 1-2 we have $k = zz^*$ with z given by (1.6): under the additional assumption of (ii), we can take $A_t = t$, so we have (2.14), and the last claim of (ii) will follow from (i) and from Proposition 1-4.

Step 2. In this step we prove (2.12) can be strenghtened as such:

$$\sup_t |V(N)_t^n| \xrightarrow{P} 0. \quad (2.17)$$

In view of (2.12) it suffices to prove that

$$\forall \varepsilon, \eta > 0, \exists \theta > 0, \exists n_0 \in \mathbb{N}^*, \forall n \geq n_0 \Rightarrow P(w^n(\theta) > \eta) \leq \varepsilon, \quad (2.18)$$

where $w^n(\theta) = \sup_{0 \leq s \leq \theta, 0 \leq t \leq 1-\theta} |V(N)_{t+s}^n - V(N)_t^n|$ is the θ -modulus of continuity of $V(N)^n$. Denoting by $w^n(\theta)$ the θ -modulus of continuity of F^n , (2.16) and the continuity of K yield

$$\forall \varepsilon, \eta > 0, \exists \theta > 0, \exists n_0 \in \mathbb{N}^*, \forall n \geq n_0 \Rightarrow P(w^n(\theta) > \eta) \leq \varepsilon. \quad (2.19)$$

On the other hand, a classical inequality on quadratic covariations yields that for all $u > 0$ we have $2|V(N)_t^n - V(N)_s^n| \leq |F_t^n - F_s^n|/u + u(\langle N, N \rangle_t - \langle N, N \rangle_s)$ if $s < t$, so that $2w^n(\theta) \leq w^n(\theta)/u + \langle N, N \rangle_1$, hence

$$P(w^n(\theta) > \eta) \leq P(w^n(\theta) > u\eta) + \frac{u}{\eta} E(N(n)_1^2).$$

Then (2.18) readily follows from (2.19), $\sup_n E(N(n)_1^2) < \infty$ and from the arbitrariness of $u > 0$.

Step 3. Here we prove that, instead of proving $Z^n \xrightarrow{s-\mathcal{L}} Z$ with $Z = B + Z'$ as in (i), it is enough to prove that

$$X^n \xrightarrow{s-\mathcal{L}} Z' \quad (2.20)$$

Indeed, set $Z_t^{\prime\prime n} = \sum_{s \leq t} h'_q(\Delta Z_s^n)$. By ([5], VI-4.22), (2.11) implies $\sup_t |\Delta Z_t^n| \xrightarrow{P} 0$; since $h'_q(x) = 0$ for $|x|$ small enough, we have $\sup_t |Z_t^{\prime\prime n}| \xrightarrow{P} 0$. On the other hand $\Delta B_t^n = \int h_q(x) \nu^n(\{t\}, dx)$, so (2.11) again yields $\sup_t |\Delta B_t^n| \xrightarrow{P} 0$, hence B is continuous by (2.8). Hence the claim follows from (2.3).

Step 4. Here we prove (2.20) under the additional assumption that $\mathcal{F}$ is separable.

a) There is a sequence of bounded variables $(Y_m)_{m \in \mathbb{N}}$ which is dense in $\mathbb{L}^1(\Omega, \mathcal{F}, P)$. We set $N_t^m = E(Y_m | \mathcal{F}_t)$, so $N^m \in \mathcal{M}_b$, and we have two important properties:

(A) Every bounded martingale is the limit in $\mathbb{L}^2$, uniformly in time, of a sequence of sums of stochastic integrals w.r.t. a finite number of N^m 's: see (4.15) of [2].

(B) $(\mathcal{F}_t)$ is the smallest filtration, up to P -null sets, w.r.t. which all N^m 's are adapted: indeed let $(\mathcal{G}_t)$ be the above-described filtration, and $A \in \mathcal{F}_t$; there is a sequence $Y_{m(n)} \rightarrow 1_A$ in $\mathbb{L}^1$, so $N_t^{m(n)} = E(Y_{m(n)} | \mathcal{F}_t)$ is $\mathcal{G}_t$ -measurable and converges in $\mathbb{L}^1$ to $E(1_A | \mathcal{F}_t) = 1_A$.

b) Introduce some more notation. First $\mathcal{N} = (N^m)_{m \in \mathbb{N}}$ and $\mathcal{N}(n) = (N^m(n))_{m \in \mathbb{N}}$ (recall Property (F)) can be considered as processes with paths in $\mathcal{D}([0, 1], \mathbb{R}^{\mathbb{N}})$. Then (2.6) and (2.16) yield

$$(M(n), \mathcal{N}(n), K^n) \xrightarrow{P} (M, \mathcal{N}, K) \text{ in } \mathcal{D}([0, 1], \mathbb{R}^d \times \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{(d+q)^2}). \quad (2.21)$$

On the other hand, VI-4.18 and VI-4.22 in [5] and (2.11) and (2.16) imply that the sequence (X^n) is C-tight. It follows from (2.21) that the sequence $(X^n, M(n), \mathcal{N}(n))$ is tight and that any limiting process $(\hat{X}, \hat{M}, \hat{\mathcal{N}})$ has $\mathcal{L}(\hat{M}, \hat{\mathcal{N}}) = \mathcal{L}(M, \mathcal{N})$.

c) Choose now any subsequence, indexed by n' , such that $(X^{n'}, M(n'), \mathcal{N}(n'))$ converges in law. From what precedes one can realize the limit as such: consider the canonical space $(\Omega', \mathcal{F}', \mathbb{F}')$ of all continuous functions from $[0, 1]$ into $\mathbb{R}^q$, with the canonical process Z' , and define $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \in [0, 1]})$ by (1.1); since $\mathcal{F} = \sigma(Y_m : m \in \mathbb{N})$ up to P -null sets, there is a probability measure $\tilde{P}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ whose Ω -marginal is P , and such that the laws of $(X^{n'}, M(n'), \mathcal{N}(n'))$ converge to the law of $(X, M, \mathcal{N})$ under $\tilde{P}$.

Therefore we have an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{P})$ of $(\Omega, \mathcal{F}, \mathbb{F}, P)$ (the existence of a disintegration of $\tilde{P}$ as in (1.1) is obvious, due to the definition of $(\Omega', \mathcal{F}')$, and up to $\tilde{P}$ -null sets the filtrations $\mathbb{F}$ and $\tilde{\mathbb{F}}$ are generated by $(M, \mathcal{N})$ and $(Z', M, \mathcal{N})$ respectively (use Property (B) of (a)).

Set $Y^n = (M(n), X^n)$ and $Y = (M, Z')$. By construction, all components of Y^n , $\mathcal{N}(n)$, $Y^n Y^{n*} - K^n$ are $\mathbb{F}^n$ -local martingales with uniformly bounded jumps. Then IX-1.17 of [5] (applied to processes with countably many components, which does not change the proof) yields that all components of Y , $\mathcal{N}$ and $Y Y^* - K$ are $\tilde{\mathbb{F}}$ -local martingales under $\tilde{P}$. This implies first that on our extension we have

$$F = \langle Z', Z'^* \rangle, \quad G = \langle Z', M^* \rangle \quad (2.22)$$

(since K is continuous increasing in $\mathcal{S}_{d+q}$), and second that all N^m are $\tilde{\mathbb{F}}$ -martingales. Then by (9.21) of [2] any stochastic integral $\int_0^\cdot a_s dN_s^m$ with a $\mathbb{F}$ -predictable is also an $(\tilde{\mathbb{F}}$ -martingale: Property (A) of (a) yields that all elements of $\mathcal{M}_b$ are $\tilde{\mathbb{F}}$ -martingales, hence our extension is very good.

d) Let now $N \in \mathcal{M}_b(M^\perp)$. We could have included N in the sequence (N^m) : what precedes remains valid, with the same limit, for a suitable subsequence (n'') of (n') . Moreover $X^n N(n) - V(N)^n$ is an $\mathbb{F}^n$ -local martingale with bounded jumps, while by (2.17) the sequence $(X^{n''}, \mathcal{N}(n''), (n''), V(N)^{n''})$ converges in law to $(Z', \mathcal{N}, N, 0)$. The same argument as above yields that $Z'N$ is a local martingale on the extension, so Z' is orthogonal to all elements of $\mathcal{M}_b(M^\perp)$.

Therefore Z' satisfies (i) of Proposition 1-2: hence Z' is an M -biased continuous $\mathcal{F}$ -conditional Gaussian martingale, whose law under Q_ω , which is Q_ω itself, is determined by the processes M , F , G , and in particular it does not depend on the subsequence (n') chosen above.

In other words all convergent subsequence of $(X^n, \mathcal{N}(n))$ have the same limit $(Z', \mathcal{N})$ in law, with the same measure $\tilde{P}$, and thus the original sequence $(X^n, \mathcal{N}(n))$ converges in law to $(Z', \mathcal{N})$. In particular if f is a bounded continuous function on

$\mathcal{D}([0, 1], \mathbb{R}^q)$ and since $N(n)^m$ is a component of $\mathcal{N}(n)$ bounded uniformly in n , we get

$$E(f(X^n)N(n)_1^m) \rightarrow \tilde{E}(f(Z')N_1^m).$$

Now (2.4) and (2.6) yield that $N(n)_1^m \rightarrow N_1^m$ in $\mathbb{L}^1$, hence

$$E(f(X^n)N_1^m) \rightarrow \tilde{E}(f(Z')N_1^m).$$

Since $\tilde{E}(UN_1^m) = \tilde{E}(UY_m)$ for any bounded $\tilde{\mathcal{F}}$ -measurable variable U , we deduce

$$E(f(X^n)Y_m) \rightarrow \tilde{E}(f(Z')Y_m).$$

Finally any bounded $\mathcal{F}$ -measurable variable Y is the $\mathbb{L}^1$ -limit of a subsequence of (Y_m) , hence one readily deduces that

$$E(f(X^n)Y) \rightarrow \tilde{E}(f(Z')Y), \quad (2.23)$$

which is (2.20).

Step 5. It remains to remove the separability assumption on $\mathcal{F}$. Denote by $\mathcal{H}$ the σ -field generated by the random variables $(M_t, K_t, B_t, X_t^n : t \in [0, 1], n \geq 1)$, and let $\mathcal{G}$ be any separable σ -field containing $\mathcal{H}$. Let $(Y_m)_{m \in \mathbb{N}}$ be a dense sequence of bounded variables in $\mathbb{L}^1(\Omega, \mathcal{G}, P)$, and $N_t^m = E(Y_m | \mathcal{F}_t)$, and set $\mathcal{G} = (\mathcal{G}_t)_{t \in [0, 1]}$ for the filtration generated by the processes $(N^m)_{m \in \mathbb{N}}$.

We have $E(Y_m | \mathcal{F}_t) = E(Y_m | \mathcal{G}_t)$ for all m , so by a density argument $E(Y | \mathcal{F}_t) = E(Y | \mathcal{G}_t)$ for all $Y \in \mathbb{L}^1(\Omega, \mathcal{G}, P)$: this implies that any $\mathcal{G}$ -martingale is an $\mathcal{F}$ -martingale, and in particular each N^m is in $\mathcal{M}_b$, and also that every $\mathcal{F}$ -adapted and $\mathcal{G}$ -measurable process (like K , B and M) is $\mathcal{G}$ -adapted. Thus M is a $\mathcal{G}$ -local martingale. Finally, any bounded $\mathcal{G}$ -martingale which is orthogonal w.r.t. $\mathcal{G}$ to M is also orthogonal to M w.r.t. $\mathcal{F}$.

In other words, Property (F) is satisfied by $\mathcal{G}$ and the same filtration $\mathcal{F}^n$ and processes $M(n)$, $N(n)$, and (2.8)-(2.12) are satisfied as well with $\mathcal{G}$ instead of $\mathcal{F}$. We can thus apply Step 4 with the same space $(\Omega', \mathcal{F}', \mathbb{F}')$ and process Z' , and $\tilde{\Omega} = \Omega \times \Omega'$, $\tilde{\mathcal{G}} = \mathcal{G} \otimes \mathcal{F}'$, $\tilde{\mathcal{G}}_t = \cap_{s > t} \mathcal{G}_s \otimes \mathcal{F}'_s$. We have a transition probability $Q_{\mathcal{G}, \omega}(d\omega')$ from $(\Omega, \mathcal{G})$ into $(\Omega', \mathcal{F}')$, such that if $\tilde{P}_{\mathcal{G}}(d\omega, d\omega') = P_{\mathcal{G}}(d\omega)Q_{\mathcal{G}, \omega}(d\omega')$ (where $P_{\mathcal{G}}$ is the restriction of P to $\mathcal{G}$), then

$$E_{\mathcal{G}}(f(X^n)Y) \rightarrow \tilde{E}_{\mathcal{G}}(f(Z')Y) \quad (2.24)$$

for all bounded continuous function f on $\mathcal{D}([0, 1], \mathbb{R}^q)$ and all bounded $\mathcal{G}$ -measurable variable Y .

Further, $Q_{\mathcal{G}, \omega}$ only depends on M , F , G and so is indeed a transition from $(\Omega, \mathcal{H})$ into $(\Omega', \mathcal{F}')$ not depending on $\mathcal{G}$ and written Q_{ω} .

It remains to define $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{P})$ by (1.1): since $\omega \rightsquigarrow Q_{\omega}(A)$ is $\mathcal{F}_t$ -measurable for $A \in \mathcal{F}'_t$ it is a very good extension of $(\Omega, \mathcal{F}, \mathbb{F}, P)$. Furthermore $E_{\mathcal{G}}(f(X^n)Y) = E(f(X^n)Y)$ and $\tilde{E}_{\mathcal{G}}(f(Z')Y) = \tilde{E}(f(Z')Y)$ for all bounded $\mathcal{G}$ -measurable Y : hence (2.24) yields (2.23) for all such Y . Since any $\mathcal{F}$ -measurable variable Y is also $\mathcal{G}$ -measurable for some separable σ -field $\mathcal{G}$ containing $\mathcal{H}$, we deduce that (2.23) holds for all bounded $\mathcal{F}$ -measurable Y , and we are finished. $\square$

2-4. When each Z^n is $\mathbb{F}^n$ -locally square integrable, i.e. when we can write

$$Z^n = B^n + X^n, \quad (2.25)$$

with B^n a $\mathbb{F}^n$ -predictable with finite variation and X^n a $\mathbb{F}^n$ -locally square-integrable martingale, we have another version, involving a Lindeberg-type condition instead of (2.11), namely:

Theorem 2-2: *Assume Property (F). Assume also that Z^n is as in (2.25), and that there are two continuous processes F and G and a continuous process B of bounded variation on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ satisfying (2.8), (2.9), (2.10), (2.12) and*

$$W(\varepsilon)^n := \int_{|x|>\varepsilon} |x|^2 \nu^n([0, 1] \times dx) \xrightarrow{P} 0, \quad \forall \varepsilon > 0. \quad (2.26)$$

Then all results of Theorem 2-1 hold true.

Proof. We have (2.25), and also the decomposition (2.7), i.e.:

$$Z_t^n = B_t^n + X_t^n + \sum_{s \leq t} h'_q(\Delta Z_s^n) \quad (2.27)$$

We will denote by F_t^n , G_t^n and $V'(N)_t^n$ the quantities defined in (2.9), (2.10) and (2.12) with X^n instead of X^n . We will prove that the assumptions of Theorem 2-1 are met, i.e. we have (2.11) and

$$\sup_t |B_t^n - B_t| \xrightarrow{P} 0, \quad (2.28)$$

$$F_t^n \xrightarrow{P} F_t, \quad \forall t \in [0, 1], \quad (2.29)$$

$$G_t^n \xrightarrow{P} G_t, \quad \forall t \in [0, 1], \quad (2.30)$$

$$V'(N)_t^n \xrightarrow{P} 0, \quad \forall t \in [0, 1], \quad \forall N \in \mathcal{M}_b \text{ orthogonal to } M. \quad (2.31)$$

First (2.11) readily follows from (2.26). Next, comparing (2.25) and (2.27), and if μ^n denotes the jump measure of Z^n , we get

$$B_t^n = B_t^n + \int h'_q(x) \nu^n([0, t] \times dx), \quad X''^n := X^n - X'^n = h'_q \star (\mu^n - \nu^n).$$

We have $|h'_q(x)| \leq C|x|1_{\{|x|>\theta\}}$ for some constants $\theta > 0$ and C . This implies first that (2.28) follows from (2.8) and (2.26). It also implies

$$\sum_{i=1}^q \langle X^{m_i, n}, X^{m_i, n} \rangle_t \leq \int |h'_q(x)|^2 \nu^n((0, t] \times dx) \leq C^2 W^n(\theta). \quad (2.32)$$

We have

$$|F_t^n - F_t^m| \leq |\langle X''^n, X''^{m*} \rangle_t| + \sqrt{|\langle X^n, X^{n*} \rangle_t| |\langle X^m, X^{m*} \rangle_t|},$$

so (2.9), (2.26) and (2.32) yield (2.29). Similarly, (2.30) follows from (2.5), (2.10), (2.26), (2.32) and from the following inequality:

$$|G_t^n - G_t^m| \leq \sqrt{|\langle M(n), M(n)^* \rangle_t| |\langle X''^n, X''^{n*} \rangle_t|}.$$

Finally we have

$$|V(N)_t^n - V'(N)_t^n| \leq \sqrt{\langle N(n), N(n) \rangle_t} |\langle X''^n, X''^{n*} \rangle_t|,$$

while $E(\langle N(n), N(n) \rangle_t^2) \leq E(N(n)_1^2)$, which is bounded by a constant by (2.4): hence (2.31) follows as above. $\square$

3 Convergence of discretized processes

In this section we specialize the previous results to the case when the filtration $\mathbb{F}^n$ is the “discretized” filtration defined by $\mathcal{F}_t^n = \mathcal{F}_{[nt]/n}$. For every càdlàg process Y write

$$Y_t^n = Y_{[nt]/n}, \quad \Delta_i^n Y = Y_{i/n} - Y_{(i-1)/n}. \quad (3.1)$$

Here again we have a continuous d -dimensional local martingale M on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$. We denote by h_d a continuous truncation function on $\mathbb{R}^d$. We also consider for each n an $\mathbb{F}^n$ -semimartingale, i.e. a process of the form

$$Z_t^n = \sum_{i=1}^{[nt]} \chi_i^n \quad (3.2)$$

where each χ_i^n is $\mathcal{F}_{i/n}$ -measurable. We then have:

Theorem 3-1: *Assume that there are two continuous processes F and G and a continuous process B of bounded variation on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ such that*

$$\sup_t \left| \sum_{i=1}^{[nt]} E(h_q(\chi_i^n) | \mathcal{F}_{\frac{i-1}{n}}) - B_t \right| \xrightarrow{P} 0, \quad (3.3)$$

$$\sum_{i=1}^{[nt]} (E(h_q(\chi_i^n) h_q(\chi_i^n)^* | \mathcal{F}_{\frac{i-1}{n}}) - E(h_q(\chi_i^n) | \mathcal{F}_{\frac{i-1}{n}}) E(h_q(\chi_i^n)^* | \mathcal{F}_{\frac{i-1}{n}})) \xrightarrow{P} F_t, \quad \forall t \in [0, 1], \quad (3.4)$$

$$\begin{aligned} \sum_{i=1}^{[nt]} (E(h_q(\chi_i^n) h_d(\Delta_i^n M)^* | \mathcal{F}_{\frac{i-1}{n}}) - E(h_q(\chi_i^n) | \mathcal{F}_{\frac{i-1}{n}}) E(h_d(\Delta_i^n M)^* | \mathcal{F}_{\frac{i-1}{n}})) \\ \xrightarrow{P} G_t, \quad \forall t \in [0, 1], \end{aligned} \quad (3.5)$$

$$\sum_{i=1}^n P(|\chi_i^n| > \varepsilon | \mathcal{F}_{\frac{i-1}{n}}) \xrightarrow{P} 0, \quad \forall \varepsilon > 0, \quad (3.6)$$

$$\sum_{i=1}^{[nt]} E(h_q(\chi_i^n) \Delta_i^n N | \mathcal{F}_{\frac{i-1}{n}}) \xrightarrow{P} 0, \quad \forall t \in [0, 1], \quad \forall N \in \mathcal{M}_b(M^\perp). \quad (3.7)$$

Then all results of Theorem 2-1 hold true.

Proof. We will prove that the assumptions of Theorem 2-1 are in force.

a) First we check Property (F). We will take $N(n) = N^n$, as defined in (3.1), for all $N \in \mathcal{M}_b$, so (2.4) is obvious. Note also that if $N^1, \dots, N^m$ are in $\mathcal{M}_b$, then

$$(M^n, N(n)^1, \dots, N(n)^m) \xrightarrow{P} (M, N^1, \dots, N^m) \text{ in } \mathcal{D}([0, 1], \mathbb{R}^{d+m}). \quad (3.8)$$

Next, $M(n)$ is:

$$M(n)_t = \sum_{i=1}^{[nt]} (h_d(\Delta_i^n M) - E(h_d(\Delta_i^n M) | \mathcal{F}_{\frac{i-1}{n}})), \quad (3.9)$$

so $M^n - M(n) = A^n + A'^n$, where we have put $A_t^n = \sum_{i=1}^{[nt]} E(h_d(\Delta_i^n M) | \mathcal{F}_{\frac{i-1}{n}})$ and $A'_t^n = \sum_{i=1}^{[nt]} h'_d(\Delta_i^n M)$ (with $h'_d(x) = x - h_d(x)$). Then (2.5) follows from combining the results (1.15) and (2.12) in [4] (since M is continuous). These results also yield $\sup_t |A_t^n| \xrightarrow{P} 0$, and for all $\varepsilon > 0$:

$$\sum_{i=1}^n P(|\Delta_i^n M| > \varepsilon | \mathcal{F}_{\frac{i-1}{n}}) \xrightarrow{P} 0.$$

This and VI-4.22 of [5], together with the fact that $h'_d(x) = 0$ for $|x|$ small enough, imply that $\sup_t |A_t^n| \xrightarrow{P} 0$, so finally $\sup_t |M_t^n - M(n)_t| \xrightarrow{P} 0$ and (2.6) follows from (3.9): we thus have (F).

b) The decomposition (2.7) of Z^n has $B_t^n = \sum_{i=1}^{[nt]} E(h_g(\chi_i^n) | \mathcal{F}_{\frac{i-1}{n}})$ and $X_t^n = \sum_{i=1}^{[nt]} (h_g(\chi_i^n) - E(h_g(\chi_i^n) | \mathcal{F}_{\frac{i-1}{n}}))$. Hence (3.3) is (2.8), and the left-hand sides of (3.4), (3.5) and (3.7) are those of (2.9), (2.10) and (2.12). Finally the left-hand sides of (3.6) and of (2.11) are also the same, so we are finished. $\square$

Finally, we could state the “discrete” version of Theorem 2-2. We will rather specialize a little bit more, by supposing that M is square-integrable and that each χ_i^n is square-integrable. This reads as:

Theorem 3-2: Assume that M is a square-integrable continuous martingale, and that each χ_i^n is square-integrable. Assume also that there are two continuous processes F and G and a continuous process B of bounded variation on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ such that

$$\sup_t \left| \sum_{i=1}^{[nt]} E(\chi_i^n | \mathcal{F}_{\frac{i-1}{n}}) - B_t \right| \xrightarrow{P} 0, \quad (3.10)$$

$$\sum_{i=1}^{[nt]} (E(\chi_i^n \chi_i^{n*} | \mathcal{F}_{\frac{i-1}{n}}) - E(\chi_i^n | \mathcal{F}_{\frac{i-1}{n}}) E(\chi_i^{n*} | \mathcal{F}_{\frac{i-1}{n}})) \xrightarrow{P} F_t, \quad \forall t \in [0, 1]; \quad (3.11)$$

$$\sum_{i=1}^{[nt]} E(\chi_i^n \Delta_i^n M^* | \mathcal{F}_{\frac{i-1}{n}}) \xrightarrow{P} G_t, \quad \forall t \in [0, 1]; \quad (3.12)$$

$$\sum_{i=1}^n E(|\chi_i^n|^2 1_{\{|\chi_i^n| > \varepsilon\}} | \mathcal{F}_{\frac{i-1}{n}}) \xrightarrow{P} 0, \quad \forall \varepsilon > 0, \quad (3.13)$$

$$\sum_{i=1}^{[nt]} E(\chi_i^n \Delta_i^n N | \mathcal{F}_{i-1}^n) \rightarrow^P 0, \quad \forall t \in [0, 1], \quad \forall N \in \mathcal{M}_b(M^\perp). \quad (3.14)$$

Then all results of Theorem 2-1 hold true.

Proof. If we write the decomposition (2.26) for Z^n , the left-hand sides of (3.10), (3.11), (3.12), (3.13) and (3.14) are the left-hand sides of (2.8), (2.9), (2.10) with M^n instead of $M(n)$, (2.26) and (2.12). By Theorem 2-2 it thus suffices to prove that (F) is satisfied if $N(n) = N^n$ and $M(n) = M^n$. We have seen (2.4) and (2.6) in the proof of Theorem 3-1, so it remains to prove that $\langle M^n, M^{n*} \rangle_t \rightarrow^P \langle M, M^* \rangle_t$ for all t .

Let us consider $M(n)$ as in (3.9): we have seen that it has (2.5), so it is enough to prove that if $Y^n = M^n - M(n)$, then

$$\langle Y^n, Y^{n*} \rangle_1 \rightarrow^P 0. \quad (3.15)$$

The process $\langle Y^n, Y^{n*} \rangle_t$ is L-dominated by $D_t^n = \sup_{s \leq t} |Y_s^n|$, and $W = \sup_{n,t} |\Delta D_t^n|$ satisfies $W \leq 2C + 2 \sup_t |M_t|$ where $C = \sup |h_d|$: hence $E(W) < \infty$. We have seen in the proof of Theorem 3-1 that $D_1^n \rightarrow^P 0$, so the “optional” Lenglart inequality I-3.32 of [5] yields (3.15), and the proof is finished. $\square$

4 Convergence of conditionally Gaussian martingales

Here we still have our basic continuous d -dimensional local martingale M on the basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$, and a sequence Z^n of M -biased continuous $\mathcal{F}$ -conditional Gaussian martingales: each one is defined on its own very good extension $(\Omega^n, \mathcal{F}^n, \mathbb{F}^n, \tilde{P}^n)$. Note that $\mathcal{F}$ can be considered as a sub σ -field of $\mathcal{F}^n$ for each n .

Theorem 4-1: Assume that there are two continuous processes F and G on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ such that

$$F_t^n := \langle Z^n, Z^{n*} \rangle_t \rightarrow^P F_t, \quad \forall t \in [0, 1], \quad (4.1)$$

$$G_t^n := \langle Z^n, M(n)^* \rangle_t \rightarrow^P G_t, \quad \forall t \in [0, 1], \quad (4.2)$$

Then there is a very good extension of $(\Omega, \mathcal{F}, \mathbb{F}, P)$ and an M -biased $\mathcal{F}$ -conditional Gaussian martingale Z on this extension with

$$\langle Z, Z^* \rangle = F, \quad \langle Z, M^* \rangle = G, \quad (4.3)$$

such that $Z^n \rightarrow^{\mathcal{F}\text{-}\mathcal{L}} Z$.

Proof. Set $H^n = H = \langle M, M^* \rangle$, and define K^n and K as in Step 1 of the proof of Theorem 2-1. (4.1) and (4.2) imply that $K_t^n \rightarrow^P K_t$ for all t , and since K^n is continuous in time the same holds for K , and we have (2.16). Further, if $V(N)^n = \langle Z^n, N \rangle$, by assumption on Z^n we know that $V(N)^n = 0$ for all $N \in \mathcal{M}_b(M^\perp)$.

We can then reproduce Step 4 of the proof of Theorem 2-1, with $M(n) = M$ and $N^n(n) = N^n$ and Z^n and Z instead of X^n and Z' . In place of (2.23), we get

$$\tilde{E}^n(f(Z^n)Y) \rightarrow \tilde{E}(f(Z)Y)$$

for all bounded $\mathcal{F}$ -measurable variables Y and all bounded continuous functions f on $\mathcal{D}([0, 1], \mathbb{R}^q)$: this is the desired convergence result when $\mathcal{F}$ is separable. Finally, Step 5 of the same proof may be reproduced here, to relax the separability assumption on $\mathcal{F}$, and the proof is complete. $\square$

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