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In this paper, we start with a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,1]}, P)$, the time interval being $[0,1]$, on which are defined a "basic" continuous local martingale M and a sequence Z^n of martingales or semimartingales, asymptotically "orthogonal to all martingales orthogonal to M". Our aim is to give some conditions under which Z^n converges "stably in law" to some limiting process which is defined on a suitable extension of $(\Omega, \mathcal{F}, \mathbb{F}, P)$.

In the first section we study systematically some, more or less known, properties of extensions of filtered spaces and of $\mathcal{F}$-conditional Gaussian martingales and so-called M-biased $\mathcal{F}$-conditional Gaussian martingales. Then we explain our limit results: in Section 2 we give a fairly general result, and in Section 3 we specialize to the case when Z^n is some "discrete-time" process adapted to the discretized filtration $\mathbb{F}^n = (\mathcal{F}^n_t)_{t\in[0,1]}$, where $\mathcal{F}^n_t = \mathcal{F}_{[nt]/n}$. Finally, Section 4 is devoted to studying the limit of a sequence of M-biased $\mathcal{F}$-conditional Gaussian martingales.

1 Extension of filtered spaces and conditionally Gaussian martingales

We begin with some general conventions. Our filtrations will always be assumed to be right-continuous. All local martingales below are supposed to be 0 at time 0, and we write $\langle M, N\rangle$ for the predictable quadratic variation between M and N if these are locally square-integrable martingales. When M and N are respectively d- and r-dimensional, then $\langle M, N^*\rangle$ is the $d \times r$ dimensional process with components $\langle M, N^*\rangle^{i,j} = \langle M^i, N^j\rangle$ (N^* stands for the transpose of N).

In all these notes, we have a basic filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$.

1-1. Let us start with some definitions. We call *extension* of $(\Omega, \mathcal{F}, \mathbb{F}, P)$ another filtered probability space $(\tilde\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde P)$ constructed as follows: starting with an auxiliary filtered space $(\Omega, \mathcal{F}', \mathbb{F}' = (\mathcal{F}'_t)_{t\in[0,1]})$ such that each σ-field $\mathcal{F}'_{t-}$ is separable, and a transition probability $Q_\omega(d\omega')$ from $(\Omega, \mathcal{F})$ into $(\Omega', \mathcal{F}')$, we set

$$\tilde\Omega = \Omega \times \Omega', \quad \tilde{\mathcal{F}} = \mathcal{F} \otimes \mathcal{F}', \quad \tilde{\mathcal{F}}_t = \cap_{s>t} \mathcal{F}_s \otimes \mathcal{F}'_s, \quad \tilde P(d\omega, d\omega') = P(d\omega) Q_\omega(d\omega'). \tag{1.1}$$

According to ([3], Lemma 2.17), the extension is called *very good* if all martingales on the space $(\Omega, \mathcal{F}, I\!\!F, P)$ are also martingales on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{I\!\!F}, \tilde{P})$, or equivalently, if $\omega \rightsquigarrow Q_\omega(A')$ is $\mathcal{F}_t$-measurable whenever $A' \in \mathcal{F}'_t$.

A process Z on the extension is called an *$\mathcal{F}$-conditional martingale* (resp. *$\mathcal{F}$-Gaussian process*) if for P-almost all ω the process $Z(\omega, .)$ is a martingale (resp. a centered Gaussian process) on the space $(\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\in[0,1]}, Q_\omega)$.

Let us finally denote by $\mathcal{M}_b$ the set of all bounded martingales on $(\Omega, \mathcal{F}, I\!\!F, P)$.

Proposition 1-1: *Let Z be a continuous adapted q-dimensional process on the very good extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{I\!\!F}, \tilde{P})$, with $Z_0 = 0$. The following statements are equivalent:*

(i) Z is a local martingale on the extension, orthogonal to all elements of $\mathcal{M}_b$, and the bracket $\langle Z, Z^* \rangle$ is $(\mathcal{F}_t)$-adapted.

(ii) Z is an $\mathcal{F}$-conditional Gaussian martingale.

In this case, the $\mathcal{F}$-conditional law of Z is characterized by the process $\langle Z, Z^* \rangle$ (i.e., for P-almost all ω, the law of $Z(\omega, .)$ under Q_ω depends only on the function $t \rightsquigarrow \langle Z, Z^* \rangle_t(\omega)$).

Proof. a) We first prove that, if each Z_t is $\tilde{P}$-integrable, then Z is an $\mathcal{F}$-conditional martingale iff it is an $\tilde{I\!\!F}$-martingale orthogonal to all bounded $I\!\!F$-martingales. For this, we can and will assume that Z is 1-dimensional.

Let $t \le s$ and let U, U' be bounded measurable function on $(\Omega, \mathcal{F}_t)$ and $(\Omega', \mathcal{F}'_t)$ respectively. Let also $M \in \mathcal{M}_b$. We have

$$\tilde{E}(UU'M_sZ_s) = \int P(d\omega)U(\omega)M_s(\omega) \int Q_\omega(d\omega')U'(\omega')Z_s(\omega, \omega'), \qquad (1.2)$$

$$\tilde{E}(UU'M_tZ_t) = \int P(d\omega)U(\omega)M_t(\omega) \int Q_\omega(d\omega')U'(\omega')Z_t(\omega, \omega'). \qquad (1.3)$$

Assume first that Z is an $\mathcal{F}$-conditional martingale. Then for P-almost all ω we have

$$\int Q_\omega(d\omega')U'(\omega')Z_s(\omega, \omega') = \int Q_\omega(d\omega')U'(\omega')Z_t(\omega, \omega'),$$

and the latter is $\mathcal{F}_t$-measurable as a function of ω because the extension is very good. Since M is an $I\!\!F$-martingale, we deduce that (1.2) and (1.3) are equal: thus MZ is a martingale on the extension: then Z is a martingale (take $M \equiv 1$), orthogonal to all bounded $I\!\!F$-martingales.

Next we prove the sufficient condition. Take V bounded and $\mathcal{F}_s$-measurable, and consider the martingale $M_r = E(V|\mathcal{F}_r)$. With the notation above we have equality between (1.2) and (1.3), and further in (1.3) we can replace $M_t(\omega)$ by $M_s(\omega) = V(\omega)$ because the last integral is $\mathcal{F}_t$-measurable in ω. Then taking $U = 1$ we get

$$\int P(d\omega)V(\omega) \int Q_\omega(d\omega')U'(\omega')Z_s(\omega, \omega') = \int P(d\omega)V(\omega) \int Q_\omega(d\omega')U'(\omega')Z_t(\omega, \omega').$$

Hence for P-almost ω, $Q_\omega(U'Z_s(\omega, .)) = Q_\omega(U'Z_t(\omega, .))$. Using the separability of the σ-field $\mathcal{F}'_{t-}$ and the continuity of Z, we have this relation P-almost surely in

ω, simultaneously for all $t \leq s$ and all $\mathcal{F}'_{t-}$-measurable variable U': this gives the $\mathcal{F}$-conditional martingality for Z.

b) Assume that (i) holds. If $Y = \langle Z, Z^* \rangle$, a simple application of Ito's formula and the fact that Z is continuous show that, since Z is orthogonal to all $M \in \mathcal{M}_b$, the same holds for Y. Each $T_n = \inf(t : |\langle Z, Z^* \rangle_t| > n)$ is an $\mathbb{F}$-stopping time, and $T_n \uparrow \infty$ as $n \to \infty$. Then $Z(n)_t = Z_{t \bigwedge T_n}$ and $Y(n)_t = Y_{t \bigwedge T_n}$ are continuous $\tilde{\mathbb{F}}$-martingale, orthogonal to all $M \in \mathcal{M}_b$, and obviously $|Z(n)_t|$ and $|Y(n)_t|$ are integrable: by (a), and by letting $n \uparrow \infty$, we deduce that for P-almost all ω, under Q_ω the process $Z(n)(\omega, .)$ is a continous martingale with deterministic bracket $\langle Z, Z^* \rangle(\omega)$, hence it is an $\mathcal{F}$-Gaussian martingale, so we have (ii). Furthermore, it is well-known that the law of $Z(\omega)$ under Q_ω is then entirely determined by $\langle Z, Z^* \rangle(\omega)$.

c) Assume now (ii). There is a P-full set $A \in \mathcal{F}$ such that for all $\omega \in A$, under Q_ω, the process $Z(\omega, .)$ is both centered Gaussian and an $\mathbb{F}'$-martingale. Therefore if $F_t(\omega) = \int Q_\omega(d\omega') Z_t(\omega, \omega')$, the process $(ZZ^*)(\omega, .) - F(\omega)$ is an $\mathbb{F}'$-martingale under Q_ω for $\omega \in A$: that is, $ZZ^* - F$ is an $\mathcal{F}$-conditional martingale. By localizing at the $\mathbb{F}$-stopping times $T_n = \inf(t : |F_t| > n)$ and by (a), we deduce that Z and $ZZ^* - F$ are local martingales on the extension, orthogonal to all $M \in \mathcal{M}_b$. Since F is continuous, $\mathbb{F}$-adapted, and of bounded variation (since it is non-decreasing for the strong order in the set of nonnegative symmetric matrices), it follows that it is a version of $\langle Z, Z^* \rangle$, hence we have (i). □

1-2. Let now M be a continous d-dimensional local martingale, and $\mathcal{M}_b(M^\perp)$ be the class of all elements of $\mathcal{M}_b$ which are orthogonal to M (i.e., to all components of M).

A q-dimensional process Z on the extension is called an *M-biased $\mathcal{F}$-conditional Gaussian martingale* if it can be written as

$$Z_t = Z'_t + \int_0^t u_s dM_s, \tag{1.4}$$

where Z' is an $\mathcal{F}$-conditional Gaussian martingale and u is a predictable $\mathbb{R}^q \otimes \mathbb{R}^d$ on $(\Omega, \mathcal{F}, \mathbb{F}, P)$.

Proposition 1-2: *Let Z be a continuous adapted q-dimensional process on the very good extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{P})$, with $Z_0 = 0$. The following statements are equivalent:*

(i) Z is a local martingale on the extension, orthogonal to all elements of $\mathcal{M}_b(M^\perp)$, and the brackets $\langle Z, Z^* \rangle$ and $\langle Z, M^* \rangle$ are $\mathbb{F}$-adapted.

(ii) Z is an M-biased $\mathcal{F}$-conditional Gaussian martingale.

In this case, the $\mathcal{F}$-conditional law of Z is characterized by the processes M, $\langle Z, Z^* \rangle$ and $\langle Z, M^* \rangle$.

Proof. Under either (i) or (ii), Z and M are continous local martingales (use the fact that the extension is very good, and use (1.4) under (ii)). We write $F = \langle Z, Z^* \rangle$, $G = \langle Z, M^* \rangle$ and $H = \langle M, M^* \rangle$.

If (ii) holds, (1.4) and Proposition 1-1 yield for all $N \in \mathcal{M}_b$:

$$G_t = \int_0^t u_s^* dH_s, \quad F_t = \langle Z', Z'^* \rangle_t + \int_0^t u_s^* dH_s u_s^*, \quad \langle Z, N\rangle_t = \int_0^t u_s^* d\langle M, N\rangle_s. \tag{1.5}$$

Then (i) readily follows. Further, (1.5) implies that u and $\langle Z', Z'^*\rangle$ are determined by F, G and H. Since $\int_0^\cdot u_s dM_s$ is $\mathcal{F}$-measurable, the last claim follows from (1.4) and Proposition 1-1 again.

Assume conversely (i). There are a continuous increasing process A and predictable processes f, g, h with values in $\mathbb{R}^q \otimes \mathbb{R}^q$, $\mathbb{R}^q \otimes \mathbb{R}^d$ and $\mathbb{R}^d \otimes \mathbb{R}^d$ respectively, such that $F_t = \int_0^t f_s dA_s$, $G_t = \int_0^t g_s dA_s$ and $H_t = \int_0^t h_s dA_s$.

The process (M, Z) is a continuous local martingale on the extension, with bracket $K_t = \int_0^t k_s dA_s$, where $k = \begin{pmatrix} h & g^* \\ g & f \end{pmatrix}$. By triangularization we may write $k = zz^*$, where

$$z = \begin{pmatrix} v & 0 \\ uv & w \end{pmatrix}, \tag{1.6}$$

so that $h = vv^*$, $g = uvv^*$ and $f = uvv^*u^* + ww^*$. Let us put $Y_t = \int_0^t u_s dM_s$ and $Z' = Z - Y$. Then since the extension is very good, Z' is a local martingale on the extension, and $\langle Z', Z'^*\rangle_t = \int_0^t w_s w_s^* dA_s$ is $\mathbb{F}$-adapted. Further, $\langle Z', N\rangle_t = \langle Z, N\rangle_t - \int_0^t u_s d\langle M, N\rangle_s$: first this implies that $\langle Z', N\rangle = 0$ if $N \in \mathcal{M}_b(M^\perp)$ (since then $\langle Z, N\rangle = 0$ by hypothesis), second this implies that when $N_t = \int_0^t \alpha_s dM_s$ we have $\langle Z', N\rangle_t = \int_0^t (g_s \alpha_s^* - u_s v_s v_s^* \alpha_s) dA_s = 0$. Thus Z' is orthogonal to all $N \in \mathcal{M}_b$, and it is an $\mathcal{F}$-conditional Gaussian martingale by Proposition 1-1. □

1-3. Let us denote by $\mathcal{S}_r$ the set of all symmetric nonnegative $r \times r$-matrices. In Proposition 1.1, the process $\langle Z, Z^*\rangle$ is a continuous adapted non-decreasing $\mathcal{S}_q$-valued process, null at 0. In Proposition 1-2, the bracket of (M, Z) is a continuous adapted non-decreasing $\mathcal{S}_{d+q}$-valued process, null at 0. Conversely we have:

Proposition 1-3: *a) Let F be a continuous adapted nondecreasing $\mathcal{S}_q$-valued process, with $F_0 = 0$, on the basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$. There exists a continuous $\mathcal{F}$-conditional Gaussian martingale Z on a very good extension, such that $\langle Z, Z^*\rangle = F$.*

b) Let K be a continuous adapted nondecreasing $\mathcal{S}_{d+q}$-valued process, with $K_0 = 0$, and M be a continuous d-dimensional local martingale with $\langle M^i, M^j\rangle = K^{ij}$ for $1 \le i, j \le d$, on the basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$. There exists a continuous M-biased $\mathcal{F}$-conditional Gaussian martingale Z on a very good extension, such that $\langle Z^i, M^j\rangle = K^{d+i,j}$ for $1 \le i \le q$, $1 \le j \le d$, and $\langle Z^i, Z^j\rangle = K^{d+i,d+j}$ for $1 \le i, j \le q$.

Of course (a) is a particular case of (b) (take $M = 0$), but in the proof below (b) is obtained as a consequence of (a).

Proof. a) Take $(\Omega', \mathcal{F}', \mathbb{F}')$ to be the canonical space of all $\mathbb{R}^d$-valued continuous functions on $[0,1]$, with the usual filtration and the canonical process $Z_t(\omega') = \omega'(t)$. For each ω, denote by Q_ω the unique probability measure on $(\Omega', \mathcal{F}')$ under which Z is a centered Gaussian process with covariance $\int Z_t Z_s^* dQ_\omega = F_{s \wedge t}(\omega)$. This structure of the covariance implies that Z has independent increments and thus is a martingale under each Q_ω: Defining $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{P})$ by (1.1) gives the result.

b) As in the previous proof, we can write $K_t = \int_0^t k_s dA_s$ for a continuous adapted increasing process A and a predictable process $k = zz^*$ with z as in (1.6). By (a) we have a continuous $\mathcal{F}$-conditional Gaussian martingale Z' on a very good extension, with $\langle Z', Z'^* \rangle_t = \int_0^t w_s w_s^* dA_s$. We can set $Z_t = Z'_t + \int_0^t u_s dM_s$, and some computations yileds that Z satisfies our requirements. □

We even have a more "concrete" way of constructing Z above, when K is absolutely continuous w.r.t. Lebesgue measure on $[0,1]$. Let $(\Omega^W, \mathcal{F}^W, \mathbb{F}^W, P^W)$ be the q-dimensional Wiener space with the canonical Wiener process W. Then $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{P})$ defined by

$$\tilde{\Omega} = \Omega \times \Omega^W, \quad \tilde{\mathcal{F}} = \mathcal{F} \otimes \mathcal{F}^W, \quad \tilde{\mathcal{F}}_t = \cap_{s>t} \mathcal{F}_s \otimes \mathcal{F}_s^W, \quad \tilde{P} = P \otimes P^W. \tag{1.7}$$

is a very good extension of $(\Omega, \mathcal{F}, \mathbb{F}, P)$, called the *canonical q-dimensional Wiener extension* of $(\Omega, \mathcal{F}, \mathbb{F}, P)$. Note that W is also a Wiener process on the extension.

Proposition 1-4: *Let K and M be as in Proposition 1-3(b), and assume that $K_t = \int_0^t k_s ds$ with k predictable $\mathcal{S}_{d+q}$- valued. Then we can choose a version of k of the form $k = zz^*$ with $z = \begin{pmatrix} v & 0 \\ uv & w \end{pmatrix}$, and on the canonical q-dimensional Wiener extension of $(\Omega, \mathcal{F}, \mathbb{F}, P)$ the process*

$$Z_t = \int_0^t u_s dM_s + \int_0^t w_s dW_s \tag{1.8}$$

is a continuous M-biased $\mathcal{F}$-conditional Gaussian martingale, such that $\langle Z^i, M^j \rangle = K^{d+i,j}$ for $1 \le i \le q$ and $1 \le j \le d$, and $\langle Z^i, Z^j \rangle = K^{d+i,d+j}$ for $1 \le i,j \le q$.

Proof. The first claim has already been proved. (1.8) defines a continuous q-dimensional local martingale on the canonical Wiener extension and a simple computation shows that it has the required brackets. □

2 Stable convergence to conditionally Gaussian martingales

2-1. First we recall some facts about stable convergence. Let X_n be a sequence of random variables with values in a metric space E, all defined on $(\Omega, \mathcal{F}, P)$. Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ be an extension of $(\Omega, \mathcal{F}, P)$ (as in Section 1, except that there is no filtration here), and let X be an E-valued variable on the extension. Let finally $\mathcal{G}$ be a sub σ-field of $\mathcal{F}$. We say that X_n *$\mathcal{G}$-stably converges in law* to X, and write $X_n \to^{\mathcal{G}\text{-}\mathcal{L}} X$, if

$$E(Yf(X_n)) \to \tilde{E}(Yf(X)) \tag{2.1}$$

for all $f: E \to \mathbb{R}$ bounded continuous and all bounded variable Y on $(\Omega, \mathcal{G})$. This property, introduced by Renyi [6] and studied by Aldous and Eagleson [1], is (slightly) stronger than the mere convergence in law. It applies in particular when X_n, X are $\mathbb{R}^q$-valued càdlàg processes, with $E = \mathbb{D}([0,1], \mathbb{R}^q)$ the Skorokhod space.

If X'_n are some other E-valued variables, then (with δ denoting a distance on E):

$$\delta(X'_n, X_n) \to^P 0, \quad X_n \to^{\mathcal{G}\text{-}\mathcal{L}} X \quad \Rightarrow \quad X'_n \to^{\mathcal{G}\text{-}\mathcal{L}} X. \tag{2.2}$$

Also, if U_n, U are on $(\Omega, \mathcal{F})$, with values in another metric space E', then

$$U_n \to^P U, \quad X_n \to^{\mathcal{G}\text{-}\mathcal{L}} X \quad \Rightarrow \quad (U_n, X_n) \to^{\mathcal{G}\text{-}\mathcal{L}} (U, X). \tag{2.3}$$

When $\mathcal{G} = \mathcal{F}$ we simply say that X_n stably converges in law to X, and we write $X_n \to^{s\text{-}\mathcal{L}} X$.

2-2. Now we describe a rather general setting for our convergence results. We start with a continuous d-dimensional local martingale M on the basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$: this will be our "reference" process. The set $\mathcal{M}_b$ is as in Section 1.

Next, for each integer n we are given a filtration $\mathbb{F}^n = (\mathcal{F}^n_t)_{t\in[0,1]}$ on $(\Omega, \mathcal{F})$ with the following property:

Property (F): We have a d-dimensional square-integrable $\mathbb{F}^n$-martingale $M(n)$ and, for each $N \in \mathcal{M}_b$, a bounded $\mathbb{F}^n$-martingale $N(n)$, such that

$$\sup_{n,t,\omega} |N(n)_t(\omega)| < \infty, \tag{2.4}$$

$$\langle M(n), M(n)^* \rangle_t \to^P \langle M, M^* \rangle_t, \quad \forall t \in [0,1], \tag{2.5}$$

(the bracket above in the predictable quadratic variation relative to $\mathbb{F}^n$) and that, for any finite family $(N^1, .., N^m)$ in $\mathcal{M}_b$,

$$(M(n), N^1(n), .., N^m(n)) \to^P (M, N^1, .., N^m) \quad \text{in } \mathbb{D}([0,1], \mathbb{R}^{d+m}).\square \tag{2.6}$$

In practice we encounter two situations: first, $\mathcal{F}^n_t = \mathcal{F}_t$, for which (F) is obvious with $M(n) = M$ and $N(n) = N$. Second, $\mathcal{F}^n_t = \mathcal{F}_{[nt]/n}$, a situation which will be examined in Section 3.

2-3. For stating our main result we need some more notation. We are interested in the behaviour of a sequence (Z^n) of q-dimensional processes, each Z^n being an $\mathbb{F}^n$-semimartingale, and we denote by (B^n, C^n, ν^n) its characteristics, relative to a given continuous truncation function h_q on $\mathbb{R}^q$ (i.e. a continuous function $h_q : \mathbb{R}^q \to \mathbb{R}^q$ with compact support and $h_q(x) = x$ for $|x|$ small enough): see [5]. If $h'_q(x) = x - h_q(x)$, we can write

$$Z^n_t = B^n_t + X^n_t + \sum_{s\leq t} h'_q(\Delta Z^n_s) \tag{2.7}$$

where X^n is an $(\mathcal{F}^n_t)$-local martingale with bounded jumps, and $\Delta Y_t = Y_t - Y_{t-}$.

Here is the main result:

Theorem 2-1: *Assume Property (F). Assume also that there are two continuous processes F and G and a continuous process B of bounded variation on* $(\Omega, \mathcal{F}, \mathbb{F}, P)$ *such that (the brackets below being the predictable quadratic variations relative to the filtration* $\mathbb{F}^n$*):*

$$\sup_t |B_t^n - B_t| \to^P 0, \tag{2.8}$$

$$F_t^n := \langle X^n, X^{n*}\rangle_t \to^P F_t, \quad \forall t \in [0,1], \tag{2.9}$$

$$G_t^n := \langle X^n, M(n)^*\rangle_t \to^P G_t, \quad \forall t \in [0,1], \tag{2.10}$$

$$U(\varepsilon)^n := \nu^n([0,1] \times \{x : |x| > \varepsilon\}) \to^P 0, \quad \forall \varepsilon > 0, \tag{2.11}$$

$$V(N)_t^n := \langle X^n, N(n)\rangle_t \to^P 0, \quad \forall t \in [0,1], \quad \forall N \in \mathcal{M}_b(M^\perp). \tag{2.12}$$

Then

(i) There is a very good extension of $(\Omega, \mathcal{F}, \mathbb{F}, P)$ *and an M-biased continuous* $\mathcal{F}$*-conditional Gaussian martingale* Z' *on this extension with*

$$\langle Z', Z'^*\rangle = F, \quad \langle Z', M^*\rangle = G, \tag{2.13}$$

such that $Z^n \to^{s\text{-}\mathcal{L}} Z := B + Z'$.

(ii) Assuming further that $d\langle M^i, M^i\rangle_t \ll dt$ *and* $dF_t^{ii} \ll dt$*, there are predictable processes u, v, w with values in* $\mathbb{R}^q \otimes \mathbb{R}^d$, $\mathbb{R}^d \otimes \mathbb{R}^d$ *and* $\mathbb{R}^q \otimes \mathbb{R}^q$ *respectively, such that*

$$\left.\begin{aligned} \langle M, M^*\rangle_t &= \textstyle\int_0^t u_s u_s^* ds, \quad G_t = \int_0^t u_s v_s v_s^* ds, \\ F_t &= \textstyle\int_0^t (u_s v_s v_s^* u_s^* + w_s w_s^* ds, \end{aligned}\right\} \tag{2.14}$$

and the limit of Z^n *can be realized on the canonical q-dimensional Wiener extension of* $(\Omega, \mathcal{F}, \mathbb{F}, P)$*, with the canonical Wiener process W, as*

$$Z_t = B_t + \int_0^t u_s dM_s + \int_0^t w_s dW_s. \tag{2.15}$$

The proof will be divided in a number of steps.

Step 1. Let $H^n = \langle M(n), M(n)^*\rangle$ and $H = \langle M, M^*\rangle$. Consider the following processes with values in the set of symmetric $(d+q) \times (d+q)$ matrices:

$$K^n = \begin{pmatrix} H^n & G^{n*} \\ G^n & F^n \end{pmatrix}, \qquad K = \begin{pmatrix} H & G^* \\ G & F \end{pmatrix}.$$

By (2.9), (2.10) and (F), we have $K_t^n \to^P K_t$ for all t, while K^n is a nondecreasing process with values in $\mathcal{S}_{d+q}$. So there is a version of K which is also a nondecreasing $\mathcal{S}_{d+q}$-valued process. Further K is continuous in time, so by a classical result we even have

$$\sup_t |K_t^n - K_t| \to^P 0. \tag{2.16}$$

Further we can write $K_t = \int_0^t k_s dA_s$ for some continuous adapted increasing process A and some predictable $\mathcal{S}_{d+q}$-valued process k, and as seen in the proof of Proposition 1-2 we have $k = zz^*$ with z given by (1.6): under the additional assumption of (ii), we can take $A_t = t$, so we have (2.14), and the last claim of (ii) will follow from (i) and from Proposition 1-4.

Step 2. In this step we prove (2.12) can be strenghtened as such:

$$\sup_t |V(N)^n_t| \to^P 0. \tag{2.17}$$

In view of (2.12) it suffices to prove that

$$\forall \varepsilon, \eta > 0,\ \exists \theta > 0,\ \exists n_0 \in \mathbb{N}^*,\ \forall n \geq n_0 \quad \Rightarrow \quad P(w^n(\theta) > \eta) \ \leq\ \varepsilon, \tag{2.18}$$

where $w^n(\theta) = \sup_{0 \leq s \leq \theta, 0 \leq t \leq 1-\theta} |V(N)^n_{t+s} - V(N)^n_t|$ is the θ-modulus of continuity of $V(N)^n$. Denoting by $w'^n(\theta)$ the θ-modulus of continuity of F^n, (2.16) and the continuity of K yield

$$\forall \varepsilon, \eta > 0,\ \exists \theta > 0,\ \exists n_0 \in \mathbb{N}^*,\ \forall n \geq n_0 \quad \Rightarrow \quad P(w'^n(\theta) > \eta) \ \leq\ \varepsilon. \tag{2.19}$$

On the other hand, a classical inequality on quadratic covariations yields that for all $u > 0$ we have $2|V(N)^n_t - V(N)^n_s| \leq |F^n_t - F^n_s|/u + u(\langle N, N\rangle_t - \langle N, N\rangle_s)$ if $s < t$, so that $2w^n(\theta) \leq w'^n(\theta)/u + \langle N, N\rangle_1$, hence

$$P(w^n(\theta) > \eta) \ \leq\ P(w'^n(\theta) > u\eta) + \frac{u}{\eta} E(N(n)^2_1).$$

Then (2.18) readily follows from (2.19), $\sup_n E(N(n)^2_1) < \infty$ and from the arbitraryness of $u > 0$.

Step 3. Here we prove that, instead of proving $Z^n \to^{s\text{-}\mathcal{L}} Z$ with $Z = B + Z'$ as in (i), it is enough to prove that

$$X^n \to^{s\text{-}\mathcal{L}} Z' \tag{2.20}$$

Indeed, set $Z''^n_t = \sum_{s \leq t} h'_q(\Delta Z^n_s)$. By ([5], VI-4.22), (2.11) implies $\sup_t |\Delta Z^n_t| \to^P 0$; since $h'_q(x) = 0$ for $|x|$ small enough, we have $\sup_t |Z''^n_t| \to^P 0$. On the other hand $\Delta B^n_t = \int h_q(x) \nu^n(\{t\}, dx)$, so (2.11) again yields $\sup_t |\Delta B^n_t| \to^P 0$, hence B is continuous by (2.8). Hence the claim follows from (2.3).

Step 4. Here we prove (2.20) under the additional assumption that $\mathcal{F}$ is separable.

a) There is a sequence of bounded variables $(Y_m)_{m \in \mathbb{N}}$ which is dense in $\mathbb{L}^1(\Omega, \mathcal{F}, P)$. We set $N^m_t = E(Y_m | \mathcal{F}_t)$, so $N^m \in \mathcal{M}_b$, and we have two important properties:

(A) Every bounded martingale is the limit in $\mathbb{L}^2$, uniformly in time, of a sequence of sums of stochastic integrals w.r.t. a finite number of N^m's: see (4.15) of [2].

(B) $(\mathcal{F}_t)$ is the smallest filtration, up to P-null sets, w.r.t. which all N^m's are adapted: indeed let $(\mathcal{G}_t)$ be the above-described filtration, and $A \in \mathcal{F}_t$; there is a sequence $Y_{m(n)} \to 1_A$ in $\mathbb{L}^1$, so $N^{m(n)}_t = E(Y_{m(n)} | \mathcal{F}_t)$ is $\mathcal{G}_t$-measurable and converges in $\mathbb{L}^1$ to $E(1_A | \mathcal{F}_t) = 1_A$.

b) Introduce some more notation. First $\mathcal{N} = (N^m)_{m\in\mathbb{N}}$ and $\mathcal{N}(n) = (N^m(n))_{m\in\mathbb{N}}$ (recall Property (F)) can be considered as processes with paths in $\mathbb{D}([0,1],\mathbb{R}^{\mathbb{N}})$. Then (2.6) and (2.16) yield

$$(M(n),\mathcal{N}(n),K^n) \ \to^P \ (M,\mathcal{N},K) \quad \text{in } \mathbb{D}([0,1],\mathbb{R}^d\times\mathbb{R}^{\mathbb{N}}\times\mathbb{R}^{(d+q)^2}). \tag{2.21}$$

On the other hand, VI-4.18 and VI-4.22 in [5] and (2.11) and (2.16) imply that the sequence (X^n) is C-tight. It follows from (2.21) that the sequence $(X^n, M(n),\mathcal{N}(n))$ is tight and that any limiting process $(\hat{X},\hat{M},\hat{\mathcal{N}})$ has $\mathcal{L}(\hat{M},\hat{\mathcal{N}}) = \mathcal{L}(M,\mathcal{N})$.

c) Choose now any subsequence, indexed by n', such that $(X^{n'}, M(n'),\mathcal{N}(n'))$ converges in law. From what precedes one can realize the limit as such: consider the canonical space $(\Omega',\mathcal{F}',\mathbb{F}')$ of all continuous functions from $[0,1]$ into $\mathbb{R}^q$, with the canonical process Z', and define $(\tilde{\Omega},\tilde{\mathcal{F}},(\tilde{\mathcal{F}}_t)_{t\in[0,1]})$ by (1.1); since $\mathcal{F} = \sigma(Y_m : m\in\mathbb{N})$ up to P-null sets, there is a probability measure $\tilde{P}$ on $(\tilde{\Omega},\tilde{\mathcal{F}})$ whose Ω-marginal is P, and such that the laws of $(X^{n'}, M(n'),\mathcal{N}(n'))$ converge to the law of $(X, M,\mathcal{N})$ under $\tilde{P}$.

Therefore we have an extension $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{F}},\tilde{P})$ of $(\Omega,\mathcal{F},\mathbb{F},P)$ (the existence of a disintegration of $\tilde{P}$ as in (1.1) is obvious, due to the definition of $(\Omega',\mathcal{F}')$), and up to $\tilde{P}$-null sets the filtrations $\mathbb{F}$ and $\tilde{\mathbb{F}}$ are generated by $(M,\mathcal{N})$ and $(Z',M,\mathcal{N})$ respectively (use Property (B) of (a)).

Set $Y^n = (M(n),X^n)$ and $Y = (M,Z')$. By contruction, all components of Y^n, $\mathcal{N}(n)$, $Y^nY^{n*} - K^n$ are $\mathbb{F}^n$-local martingales with uniformly bounded jumps. Then IX-1.17 of [5] (applied to processes with countably many components, which does not change the proof) yields that all components of Y, $\mathcal{N}$ and $YY^* - K$ are $\tilde{\mathbb{F}}$-local martingales under $\tilde{P}$. This implies first that on our extension we have

$$F \ = \ \langle Z', Z'^*\rangle, \qquad G \ = \ \langle Z', M^*\rangle \tag{2.22}$$

(since K is continuous increasing in $\mathcal{S}_{d+q}$), and second that all N^m are $\tilde{\mathbb{F}}$-martingales. Then by (9.21) of [2] any stochastic integral $\int_0^\cdot a_s dN^m_s$ with a $\mathbb{F}$-predictable is also an $(\tilde{\mathbb{F}}$-martingale: Property (A) of (a) yields that all elements of $\mathcal{M}_b$ are $\tilde{\mathbb{F}}$-martingales, hence our extension is very good.

d) Let now $N\in\mathcal{M}_b(M^\perp)$. We could have included N in the sequence (N^m): what precedes remains valid, with the same limit, for a suitable subsequence (n'') of (n'). Moreover $X^nN(n) - V(N)^n$ is an $\mathbb{F}^n$-local martingale with bounded jumps, while by (2.17) the sequence $(X^{n''},\mathcal{N}(n''),(n''),V(N)^{n''})$ converges in law to $(Z',\mathcal{N},N,0)$. The same argument as above yields that $Z'N$ is a local martingale on the extension, so Z' is othogonal to all elements of $\mathcal{M}_b(M^\perp)$.

Therefore Z' satisfies (i) of Proposition 1-2: hence Z' is an M-biased continuous $\mathcal{F}$-conditional Gaussian martingale, whose law under Q_ω, which is Q_ω itself, is determined by the processes M, F, G, and in particular it does not depend on the subsequence (n') chosen above.

In other words all convergent subsequence of $(X^n,\mathcal{N}(n))$ have the same limit $(Z',\mathcal{N})$ in law, with the same measure $\tilde{P}$, and thus the original sequence $(X^n,\mathcal{N}(n))$ converges in law to $(Z',\mathcal{N})$. In particular if f is a bounded continuous function on

$\mathbb{D}([0,1],\mathbb{R}^q)$ and since $N(n)^m$ is a component of $\mathcal{N}(n)$ bounded uniformly in n, we get

$$E(f(X^n)N(n)_1^m) \rightarrow \tilde{E}(f(Z')N_1^m).$$

Now (2.4) and (2.6) yield that $N(n)_1^m \rightarrow N_1^m$ in $\mathbb{L}^1$, hence

$$E(f(X^n)N_1^m) \rightarrow \tilde{E}(f(Z')N_1^m).$$

Since $\tilde{E}(UN_1^m) = \tilde{E}(UY_m)$ for any bounded $\tilde{\mathcal{F}}$-measurable variable U, we deduce

$$E(f(X^n)Y_m) \rightarrow \tilde{E}(f(Z')Y_m).$$

Finally any bounded $\mathcal{F}$-measurable variable Y is the $\mathbb{L}^1$-limit of a subsequence of (Y_m), hence one readily deduces that

$$E(f(X^n)Y) \rightarrow \tilde{E}(f(Z')Y), \tag{2.23}$$

which is (2.20).

Step 5. It remains to remove the separability assumption on $\mathcal{F}$. Denote by $\mathcal{H}$ the σ-field generated by the random variables $(M_t, K_t, B_t, X_t^n : t \in [0,1], n \geq 1)$, and let $\mathcal{G}$ be any separable σ-field containing $\mathcal{H}$. Let $(Y_m)_{m\in\mathbb{N}}$ be a dense sequence of bounded variables in $\mathbb{L}^1(\Omega,\mathcal{G},P)$, and $N_t^m = E(Y_m|\mathcal{F}_t)$, and set $\mathbb{G} = (\mathcal{G}_t)_{y\in[0,1]}$ for the filtration generated by the processes $(N^m)_{m\in\mathbb{N}}$.

We have $E(Y_m|\mathcal{F}_t) = E(Y_m|\mathcal{G}_t)$ for all m, so by a density argument $E(Y|\mathcal{F}_t) = E(Y|\mathcal{G}_t)$ for all $Y \in \mathbb{L}^1(\Omega,\mathcal{G},P)$: this implies that any $\mathbb{G}$-martingale is an $\mathbb{F}$-martingale, and in particular each N^m is in $\mathcal{M}_b$, and also that every $\mathbb{F}$-adapted and $\mathcal{G}$-measurable process (like K, B and M) is $\mathbb{G}$-adapted. Thus M is a $\mathbb{G}$-local martingale. Finally, any bounded $\mathbb{G}$-martingale which is orthogonal w.r.t. $\mathbb{G}$ to M is also orthogonal to M w.r.t. $\mathbb{F}$.

In other words, Property (F) is satisfied by $\mathbb{G}$ and the same filtration $\mathbb{F}^n$ and processes $M(n)$, $N(n)$, and (2.8)-(2.12) are satisfied as well with $\mathbb{G}$ instead of $\mathbb{F}$. We can thus apply Step 4 with the same space $(\Omega',\mathcal{F}',\mathbb{F}')$ and process Z', and $\tilde{\Omega} = \Omega\times\Omega'$, $\tilde{\mathcal{G}} = \mathcal{G}\otimes\mathcal{F}'$, $\tilde{\mathcal{G}}_t = \cap_{s>t}\mathcal{G}_s\otimes\mathcal{F}'_s$. We have a transition probability $Q_{\mathcal{G},\omega}(d\omega')$ from $(\Omega,\mathcal{G})$ into $(\Omega',\mathcal{F}')$, such that if $\tilde{P}_\mathcal{G}(d\omega,d\omega') = P_\mathcal{G}(d\omega)Q_{\mathcal{G},\omega}(d\omega')$ (where $P_\mathcal{G}$ is the restriction of P to $\mathcal{G}$), then

$$E_\mathcal{G}(f(X^n)Y) \rightarrow \tilde{E}_\mathcal{G}(f(Z')Y) \tag{2.24}$$

for all bounded continuous function f on $\mathbb{D}([0,1],\mathbb{R}^q)$ and all bounded $\mathcal{G}$-measurable variable Y.

Further, $Q_{\mathcal{G},\omega}$ only depends on M, F, G and so is indeed a transition from $(\Omega,\mathcal{H})$ into $(\Omega',\mathcal{F}')$ not depending on $\mathcal{G}$ and written Q_ω.

It remains to define $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{F}},\tilde{P})$ by (1.1): since $\omega \rightsquigarrow Q_\omega(A)$ is $\mathcal{F}_t$-measurable for $A \in \mathcal{F}'_t$ it is a very good extension of $(\Omega,\mathcal{F},\mathbb{F},P)$. Furthermore $E_\mathcal{G}(f(X^n)Y) = E(f(X^n)Y)$ and $\tilde{E}_\mathcal{G}(f(Z')Y) = \tilde{E}(f(Z')Y)$ for all bounded $\mathcal{G}$-measurable Y: hence (2.24) yields (2.23) for all such Y. Since any $\mathcal{F}$-measurable variable Y is also $\mathcal{G}$-measurable for some separable σ-field $\mathcal{G}$ containing $\mathcal{H}$, we deduce that (2.23) holds for all bounded $\mathcal{F}$-measurable Y, and we are finished. □

2-4. When each Z^n is $\mathbb{F}^n$-locally square integrable, i.e. when we can write

$$Z^n \;=\; B^n + X^n, \tag{2.25}$$

with B^n a $\mathbb{F}^n$-predictable with finite variation and X^n a $\mathbb{F}^n$-locally square-integrable martingale, we have another version, involving a Lindeberg-type condition instead of (2.11), namely:

Theorem 2-2: *Assume Property (F). Assume also that Z^n is as in (2.25), and that there are two continuous processes F and G and a continuous process B of bounded variation on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ satisfying (2.8), (2.9), (2.10), (2.12) and*

$$W(\varepsilon)^n \;:=\; \int_{|x|>\varepsilon} |x|^2 \nu^n([0,1]\times dx) \;\to^P\; 0, \quad \forall \varepsilon > 0. \tag{2.26}$$

Then all results of Theorem 2-1 hold true.

Proof. We have (2.25), and also the decomposition (2.7), i.e.:

$$Z^n_t \;=\; B'^n_t + X'^n_t + \sum_{s\leq t} h'_q(\Delta Z^n_s) \tag{2.27}$$

We will denote by F'^n_t, G'^n_t and $V'(N)^n_t$ the quantities defined in (2.9), (2.10) and (2.12) with X'^n instead of X^n. We will prove that the assumptions of Theorem 2-1 are met, i.e. we have (2.11) and

$$\sup_t |B'^n_t - B_t| \;\to^P\; 0, \tag{2.28}$$

$$F'^n_t \;\to^P\; F_t, \quad \forall t \in [0,1], \tag{2.29}$$

$$G'^n_t \;\to^P\; G_t, \quad \forall t \in [0,1], \tag{2.30}$$

$$V'(N)^n_t \;\to^P\; 0, \quad \forall t \in [0,1], \quad \forall N \in \mathcal{M}_b \text{ orthogonal to } M. \tag{2.31}$$

First (2.11) readily follows from (2.26). Next, comparing (2.25) and (2.27), and if μ^n denotes the jump measure of Z^n, we get

$$B'^n_t \;=\; B^n_t + \int h'_q(x)\nu^n([0,t]\times dx), \qquad X''^n \;:=\; X^n - X'^n \;=\; h'_q \star (\mu^n - \nu^n).$$

We have $|h'_q(x)| \leq C|x| 1_{\{|x|>\theta\}}$ for some constants $\theta > 0$ and C. This implies first that (2.28) follows from (2.8) and (2.26). It also implies

$$\sum_{i=1}^q \langle X''^{i,n}, X''^{i,n}\rangle_t \;\leq\; \int |h'_q(x)|^2 \nu^n((0,t]\times dx) \;\leq\; C^2 W^n(\theta). \tag{2.32}$$

We have

$$|F^n_t - F'^n_t| \;\leq\; |\langle X''^n, X''^{n*}\rangle_t| + \sqrt{|\langle X^n, X^{n*}\rangle_t||\langle X''^n, X''^{n*}\rangle_t|},$$

so (2.9), (2.26) and (2.32) yield (2.29). Similarly, (2.30) follows from (2.5), (2.10), (2.26), (2.32) and from the following inequality:

$$|G^n_t - G'^n_t| \;\leq\; \sqrt{|\langle M(n), M(n)^*\rangle_t||\langle X''^n, X''^{n*}\rangle_t|}.$$

Finally we have

$$|V(N)_t^n - V'(N)_t^n| \leq \sqrt{\langle N(n), N(n)\rangle_t |\langle X''^n, X''^{n*}\rangle_t|},$$

while $E(\langle N(n), N(n)\rangle_t^2) \leq E(N(n)_1^2)$, which is bounded by a constant by (2.4): hence (2.31) follows as above. □

3 Convergence of discretized processes

In this section we specialize the previous results to the case when the filtration $\mathbb{F}^n$ is the "discretized" filtration defined by $\mathcal{F}_t^n = \mathcal{F}_{[nt]/n}$. For every càdlàg process Y write

$$Y_t^n = Y_{[nt]/n}, \qquad \Delta_i^n Y = Y_{i/n} - Y_{(i-1)/n}. \tag{3.1}$$

Here again we have a continuous d-dimensional local martingale M on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$. We denote by h_d a continuous truncation function on $\mathbb{R}^d$. We also consider for each n an $\mathbb{F}^n$-semimartingale, i.e. a process of the form

$$Z_t^n = \sum_{i=1}^{[nt]} \chi_i^n \tag{3.2}$$

where each χ_i^n is $\mathcal{F}_{i/n}$-measurable. We then have:

Theorem 3-1: *Assume that there are two continuous processes F and G and a continuous process B of bounded variation on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ such that*

$$\sup_t |\sum_{i=1}^{[nt]} E(h_q(\chi_i^n)|\mathcal{F}_{\frac{i-1}{n}}) - B_t| \to^P 0, \tag{3.3}$$

$$\sum_{i=1}^{[nt]} \left(E(h_q(\chi_i^n)h_q(\chi_i^n)^*|\mathcal{F}_{\frac{i-1}{n}}) - E(h_q(\chi_i^n)|\mathcal{F}_{\frac{i-1}{n}})E(h_q(\chi_i^n)^*|\mathcal{F}_{\frac{i-1}{n}})\right) \to^P F_t, \ \forall t \in [0,1], \tag{3.4}$$

$$\sum_{i=1}^{[nt]} \left(E(h_q(\chi_i^n)h_d(\Delta_i^n M)^*|\mathcal{F}_{\frac{i-1}{n}}) - E(h_q(\chi_i^n)|\mathcal{F}_{\frac{i-1}{n}})E(h_d(\Delta_i^n M)^*|\mathcal{F}_{\frac{i-1}{n}})\right) \to^P G_t, \quad \forall t \in [0,1], \tag{3.5}$$

$$\sum_{i=1}^{n} P(|\chi_i^n| > \varepsilon|\mathcal{F}_{\frac{i-1}{n}}) \to^P 0, \quad \forall \varepsilon > 0, \tag{3.6}$$

$$\sum_{i=1}^{[nt]} E(h_q(\chi_i^n)\Delta_i^n N|\mathcal{F}_{\frac{i-1}{n}}) \to^P 0, \quad \forall t \in [0,1], \quad \forall N \in \mathcal{M}_b(M^\perp). \tag{3.7}$$

Then all results of Theorem 2-1 hold true.

Proof. We will prove that the assumptions of Theorem 2-1 are in force.

a) First we check Property (F). We will take $N(n) = N^n$, as defined in (3.1), for all $N \in \mathcal{M}_b$, so (2.4) is obvious. Note also that that if $N^1, .., N^m$ are in $\mathcal{M}_b$, then

$$(M^n, N(n)^1, .., N(n)^m) \to^P (M, N^1, .., N^m) \quad \text{in } \mathbb{D}([0,1], \mathbb{R}^{d+m}). \tag{3.8}$$

Next, $M(n)$ is:

$$M(n)_t = \sum_{i=1}^{[nt]} \left(h_d(\Delta_i^n M) - E(h_d(\Delta_i^n M)|\mathcal{F}_{\frac{i-1}{n}})\right), \tag{3.9}$$

so $M^n - M(n) = A^n + A'^n$, where we have put $A_t^n = \sum_{i=1}^{[nt]} E(h_d(\Delta_i^n M)|\mathcal{F}_{\frac{i-1}{n}})$ and $A_t'^n = \sum_{i=1}^{[nt]} h_d'(\Delta_i^n M)$ (with $h_d'(x) = x - h_d(x)$). Then (2.5) follows from combining the results (1.15) and (2.12) in [4] (since M is continuous). These results also yield $\sup_t |A_t^n| \to^P 0$, and for all $\varepsilon > 0$:

$$\sum_{i=1}^{n} P(|\Delta_i^n M| > \varepsilon|\mathcal{F}_{\frac{i-1}{n}}) \to^P 0.$$

This and VI-4.22 of [5], together with the fact that $h_d'(x) = 0$ for $|x|$ small enough, imply that $\sup_t |A_t'^n| \to^P 0$, so finally $\sup_t |M_t^n - M(n)_t| \to^P 0$ and (2.6) follows from (3.9): we thus have (F).

b) The decomposition (2.7) of Z^n has $B_t^n = \sum_{i=1}^{[nt]} E(h_q(\chi_i^n)|\mathcal{F}_{\frac{i-1}{n}})$ and $X_t^n = \sum_{i=1}^{[nt]} (h_q(\chi_i^n) - E(h_q(\chi_i^n)|\mathcal{F}_{\frac{i-1}{n}}))$. Hence (3.3) is (2.8), and the left-hand sides of (3.4), (3.5) and (3.7) are those of (2.9), (2.10) and (2.12). Finally the left-hand sides of (3.6) and of (2.11) are also the same, so we are finished. □

Finally, we could state the "discrete" version of Theorem 2-2. We will rather specialize a little bit more, by supposing that M is square-integrable and that each χ_i^n is square-integrable. This reads as:

Theorem 3-2: *Assume that M is a square-integrable continuous martingale, and that each χ_i^n is square-integrable. Assume also that there are two continuous processes F and G and a continuous process B of bounded variation on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ such that*

$$\sup_t |\sum_{i=1}^{[nt]} E(\chi_i^n|\mathcal{F}_{\frac{i-1}{n}}) - B_t| \to^P 0, \tag{3.10}$$

$$\sum_{i=1}^{[nt]} \left(E(\chi_i^n \chi_i^{n*}|\mathcal{F}_{\frac{i-1}{n}}) - E(\chi_i^n|\mathcal{F}_{\frac{i-1}{n}})E(\chi_i^{n*}|\mathcal{F}_{\frac{i-1}{n}})\right) \to^P F_t, \quad \forall t \in [0,1]; \tag{3.11}$$

$$\sum_{i=1}^{[nt]} E(\chi_i^n \Delta_i^n M^*|\mathcal{F}_{\frac{i-1}{n}}) \to^P G_t, \quad \forall t \in [0,1]; \tag{3.12}$$

$$\sum_{i=1}^{n} E(|\chi_i^n|^2 1_{\{|\chi_i^n| > \varepsilon\}}|\mathcal{F}_{\frac{i-1}{n}}) \to^P 0, \quad \forall \varepsilon > 0, \tag{3.13}$$

$$\sum_{i=1}^{[nt]} E(\chi_i^n \Delta_i^n N | \mathcal{F}_{\frac{i-1}{n}}) \rightarrow^P 0, \quad \forall t \in [0,1], \quad \forall N \in \mathcal{M}_b(M^\perp). \tag{3.14}$$

Then all results of Theorem 2-1 hold true.

Proof. If we write the decomposition (2.26) for Z^n, the left-hand sides of (3.10), (3.11), (3.12), (3.13) and (3.14) are the left-hand sides of (2.8), (2.9), (2.10) with M^n instead of $M(n)$, (2.26) and (2.12). By Theorem 2-2 it thus suffices to prove that (F) is satisfied if $N(n) = N^n$ and $M(n) = M^n$. We have seen (2.4) and (2.6) in the proof of Theorem 3-1, so it remains to prove that $\langle M^n, M^{n*}\rangle_t \rightarrow^P \langle M, M^*\rangle_t$ for all t.

Let us consider $M(n)$ as in (3.9): we have seen that it has (2.5), so it is enough to prove that if $Y^n = M^n - M(n)$, then

$$\langle Y^n, Y^{n*}\rangle_1 \rightarrow^P 0. \tag{3.15}$$

The process $\langle Y^n, Y^{n*}\rangle_t$ is L-dominated by $D_t^n = \sup_{s\leq t} |Y_s^n|$, and $W = \sup_{n,t} |\Delta D_t^n|$ satisfies $W \leq 2C + 2\sup_t |M_t|$ where $C = \sup |h_d|$: hence $E(W) < \infty$. We have seen in the proof of Theorem 3-1 that $D_1^n \rightarrow^P 0$, so the "optional" Lenglart inequality I-3.32 of [5] yields (3.15), and the proof is finished. □

4 Convergence of conditionally Gaussian martingales

Here we still have our basic continuous d-dimensional local martingale M on the basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$, and a sequence Z^n of M-biased continuous $\mathcal{F}$-conditional Gaussian martingales: each one is defined on its own very good extension $(\tilde{\Omega}^n, \tilde{\mathcal{F}}^n, \tilde{\mathbb{F}}^n, \tilde{P}^n)$. Note that $\mathcal{F}$ can be considered as a sub σ-field of $\tilde{\mathcal{F}}^n$ for each n.

Theorem 4-1: *Assume that there are two continuous processes F and G on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ such that*

$$F_t^n := \langle Z^n, Z^{n*}\rangle_t \rightarrow^P F_t, \quad \forall t \in [0,1], \tag{4.1}$$

$$G_t^n := \langle Z^n, M(n)^*\rangle_t \rightarrow^P G_t, \quad \forall t \in [0,1], \tag{4.2}$$

Then there is a very good extension of $(\Omega, \mathcal{F}, \mathbb{F}, P)$ and an M-biased $\mathcal{F}$-conditional Gaussian martingale Z on this extension with

$$\langle Z, Z^*\rangle = F, \quad \langle Z, M^*\rangle = G, \tag{4.3}$$

such that $Z^n \rightarrow^{\mathcal{F}\text{-}\mathcal{L}} Z$.

Proof. Set $H^n = H = \langle M, M^*\rangle$, and define K^n and K as in Step 1 of the proof of Theorem 2-1. (4.1) and (4.2) imply that $K_t^n \rightarrow^P K_t$ for all t, and since K^n is continuous in time the same holds for K, and we have (2.16). Further, if $V(N)^n = \langle Z^n, N\rangle$, by assumption on Z^n we know that $V(N)^n = 0$ for all $N \in \mathcal{M}_b(M^\perp)$.

We can then reproduce Step 4 of the proof of Theorem 2-1, with $M(n) = M$ and $N^m(n) = N^m$ and Z^n and Z instead of X^n and Z'. In place of (2.23), we get

$$\tilde{E}^n(f(Z^n)Y) \rightarrow \tilde{E}(f(Z)Y)$$

for all bounded $\mathcal{F}$-measurable variables Y and all bounded continuous functions f on $I\!D([0,1], I\!R^q)$: this is the desired convergence result when $\mathcal{F}$ is separable. Finally, Step 5 of the same proof may be reproduced here, to relax the separability assumption on $\mathcal{F}$, and the proof is complete. □

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