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The Multiplicity of Stochastic Processes

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This paper studies the multiplicity of non-Gaussian, non-infinitely divisible and non-stationary processes associated with the "chaos" space of N. Wiener [12], and for each positive integer N and for $N = \infty$, constructs a process of multiplicity N. The examination of multiplicity of a process has been of interest to many authors such as H. Cramér [2,3,4], T. Hida [5,6], K. Itô [7] and G. Kallianpur and V. Mandrekar [10].

Our approach here begins with a classical, well known theorem on a separable Hilbert space.

Let U_t, $(t \in R)$ be a one parameter group of unitary operators acting on a separable Hilbert space $\mathbf{H}$, and let E_λ be its spectral measure, i.e.,

$$U_t = \int_{-\infty}^{\infty} e^{it\lambda} dE_\lambda.$$

Then there exists a sequence $\{f_n\}$ of elements in $\mathbf{H}$, which will be referred to as cyclic vectors, such that the Hilbert space $\mathbf{H}$ can be Hellinger-Hahn [9] decomposed into a direct sum

$$\mathbf{H} = \sum_{n \geq 1} \bigoplus \mathbf{H}_n,$$

where

$$\begin{aligned} \mathbf{H}_n &= \left\{ \int_{-\infty}^{\infty} g(\lambda) dE_\lambda f_n;\ g \in L^2(R, \mu_n) \right\} \\ &= \text{linear span of } \{U_t f_n;\ -\infty < t < \infty\}, \end{aligned}$$

which will be referred to as a cyclic subspace of $\mathbf{H}$ with f_n, with the notation

$$d\mu_n(\lambda) = ||dE_\lambda f_n||^2,$$

we further have

$$d\mu \overset{\text{def}}{:=} d\mu_1 \gg d\mu_2 \gg \cdots,$$

where $d\mu \gg d\nu$ means that the measure $d\mu$ is absolutely continuous with respect to the measure $d\nu$. The type of the measure sequence $\{d\mu_n\}$ is invariant with respect to the choice of $\{f_n\}'s$. This is to say that if $\mathbf{H} = \sum_{n \geq 1} \oplus \mathbf{H}'_n$ is another decomposition with $\mathbf{H}'_n$, a cyclic subspace with cyclic vector f'_n, then

$$d\mu_n \sim d\mu'_n(\text{ equivalence }),\ n = 1, 2, \cdots,$$

where $d\mu'_n(\lambda) = ||dE_\lambda f'_n||^2$.

Denote the support for $d\mu_n$ by Λ_n. The integer $m(\lambda) = max\{n;\ \lambda \in \Lambda_n\}$ is referred to as the multiplicity of λ, and the pair $\{d\mu, m\}$, the spectral type of U_t. The spectral type of U_t is said to be $\sigma-$Lebesgue if $d\mu$ is equivalent to Lebesgue measure and if $m(\lambda) \equiv \infty$; and that of U_t is said to be simple Lebesgue if $d\mu$ is equivalent to Lebesgue measure and if $m(\lambda) \equiv 1$.

As a further consequence of Hellinger-Hahn decomposition, we have that if U_t and U_t' are one parameter groups of unitary operators acting on $\mathbf{H}$ and $\mathbf{H}'$ respectively, and if they are unitary equivalent, i.e., if there exists an isometry V of $\mathbf{H}$ onto $\mathbf{H}'$ such that $U_t' = VU_tV^{-1}$, then the associated measure sequences $\{d\mu_n\}$ and $\{d\mu_n'\}$ are of the same type. Conversely, if these two sequences are of the same type, then we can construct an isometry between $\mathbf{H}$ and $\mathbf{H}'$ such that $\{U_t\}$ and $\{U_t'\}$ are unitary equivalent. In other words, the sequence $\{d\mu_n\}$ is unitary invariant.

Example Define θ_t to be the transform of $L^2(R)$

$$\theta_t : \begin{array}{ccc} L^2(R) & \rightarrow & L^2(R) \\ F(\cdot) & \rightarrow & F(\cdot - t). \end{array}$$

Then θ_t consists of a one parameter group of unitary operators on $L^2(R)$, and its spectral type is simple Lebesgue.

To see this, let us write

$$\tau(u) = \begin{cases} e^u, & u < 0 \\ 0, & u \geq 0 \end{cases}.$$

Then

$$\hat{\tau}(\lambda) \stackrel{\text{def}}{:=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^u e^{i\lambda u} du = \frac{1}{\sqrt{2\pi}} \frac{1}{1+i\lambda}.$$

Since the Fourier transform is topologically isomorphic on $L^2(R)$ by Plancherel's theorem [13], it follows that

$$\text{linear span } \{e^{i\lambda t}\hat{\tau}(\lambda),\, t \in R\} = \text{ linear span } \{\theta_t \tau(\cdot),\, t \in R\} = L^2(R).$$

Thus $L^2(R)$ itself turns out to be a cyclic space with cyclic vector τ. Now

$$\Big(\int_{-\infty}^{\infty} e^{it\lambda} dE_\lambda \tau\Big)(u) = \tau(u-t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\lambda(u-t)} \hat{\tau}(\lambda) d\lambda,$$

$$\int_{-\infty}^{\infty} e^{it\lambda} ||dE_\lambda \tau||^2 = \int_{-\infty}^{\infty} e^{it\lambda} |\hat{\tau}(\lambda)|^2 d\lambda.$$

Hence

$$d\mu(\lambda) = ||dE_\lambda \tau)||^2 = \frac{1}{2\pi} \frac{1}{1+\lambda^2} d\lambda.$$

This shows that the measure $d\mu$ is equivalent to Lebesgue measure and $m(\lambda) \equiv 1$, and consequently simple Lebesgue.

In the sequel, let U_t be the one parameter group of unitary operators induced by Brownian motion flow T_t on $L^2(\mathbf{B})$ [5], i.e., the collection of all variables measurable with respect to the σ-field generated by Brownian motion $\mathbf{B}$ with finite variances. First, we look at the spectral type of U_t.

To begin with, let $L^2(\mathbf{B}) = \sum_{n=0}^{\infty} \oplus \mathcal{H}_n$ be the Wiener-Itô decomposition [8, 12] of $L^2(\mathbf{B})$. It is well known that each $\mathcal{H}_n$, which consists of an U_t-invariant subspace, is topologically isomorphic to $\sqrt{n!}\widehat{L}^2(R^n)$ (via $\mathcal{J}$-transformation [5]), where $\widehat{L}^2(R^n)$ denotes all the symmetric functions of $L^2(R^n)$, and that each element in $\mathcal{H}_n$ can be expressed as an n-multiple Wiener integral. Without ambiguity, we still write U_t to be the restriction of U_t on $\mathcal{H}_n$. We then have

Theorem For each $n \geq 2$, the spectral type of U_t on $\mathcal{H}_n$ is σ-Lebesgue.

To prove this, we introduce a unitary isometry V_t of U_t. Since spectral type is unitary invariant, the investigation of spectral type of U_t may be reduced to a search for that of V_t. Now let us put

$$L^2_{nc} = L^2((u_1, u_2, \dots u_n) \in R^n;\ u_1 \leq u_2 \leq \dots \leq u_n)$$

and define $\mathcal{C}$:

$$\mathcal{C}: \quad \begin{array}{ccc} \widehat{L}^2(R^n) & \rightarrow & \sqrt{n!}L^2_{nc} \\ F(u_1, u_2, \dots, u_n) & \rightarrow & F(u_{\pi(1)}, u_{\pi(2)}, \dots, u_{\pi(n)}), \end{array}$$

where π is a permutation of $\{1, 2, \cdots, n\}$ such that $u_{\pi(1)} \leq u_{\pi(2)} \leq \dots, \leq u_{\pi(n)}$. Obviously $\mathcal{C}$ defines an isometric mapping from $\widehat{L}^2(R^n)$ to $\sqrt{n!}L^2_{nc}$. Further let

$$A_n = \begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} \\ -1 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & -1 & 1 \end{pmatrix}$$

and define $\mathcal{E}$:

$$\mathcal{E}: \quad \begin{array}{ccc} L^2_{nc} & \rightarrow & L^2(R \times R_+^{n-1}) \\ F(u_1, u_2, \dots u_n) & \rightarrow & G(v_1, v_2, \dots, v_n), \end{array}$$

where $R_+ = [0, \infty)$ and

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = A_n \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}.$$

Then again we verify that $\mathcal{E}$ defines an isometric mapping from ${L^2}_{nc}$ to $L_2(R \times R_+^{n-1})$. Hence if

$$V_t \stackrel{\text{def}}{:=} (\mathcal{E} \cdot \mathcal{C} \cdot \mathcal{J})^{-1} U_t\, (\mathcal{E} \cdot \mathcal{C} \cdot \mathcal{J}),$$

then $\{V_t,\ t \in R\}$ consists of a one parameter group of unitary operators on $L^2(R \times R_+^{n-1})$. As a matter of fact, with the diagram

$$\mathcal{E} \cdot \mathcal{C} \cdot \mathcal{J}: \quad \begin{array}{ccc} \mathcal{H}_n & \xrightarrow{\mathcal{J}} & \sqrt{n!}\ \widehat{L}^2(R^n) \\ \uparrow & & \downarrow \mathcal{C} \\ n!\ L^2(R \times R_+^{n-1}) & \xleftarrow[\mathcal{E}]{} & n!\ {L^2}_{nc} \end{array}$$

in mind, we see that if

$$\mathcal{E} \cdot \mathcal{C} \cdot \mathcal{J}: \quad \begin{array}{ccc} \mathcal{H}_n & \rightarrow & n!\ L^2(R \times R_+^{n-1}) \\ \varphi & \rightarrow & n!\ G(v_1, v_2, \dots, v_n), \end{array}$$

then

$$U_t\varphi \rightarrow n!\ G(v_1 - t, v_2, \dots, v_n) \tag{1}$$

$$= n!\ (V_t G)(v_1, v_2, \dots, v_n). \tag{2}$$

To see the spectral type of V_t on $L^2(R \times R_+^{n-1})$, we decompose $L^2(R \times R_+^{n-1})$ into a direct sum by means of a complete orthonormal basis $\{\eta_n;\, n \geq 0\}$ of $L^2(R_+)$:

(3) $$L^2(R \times R_+^{n-1}) = \sum_{k_2,\cdots,k_n \geq 0} \oplus L_{k_2,\ldots,k_n},$$

where

$$L_{k_2,\cdots,k_n} = \{f(v_1) \otimes \eta_{k_2}(v_2) \otimes \cdots \otimes \eta_{k_n}(v_n);\, f \in L^2(R)\},$$

and $\otimes$ means tensor product. Such η_n's may be taken, for example as the Laguerre functions. Apparently, the subspace $L_{k_2,\cdots,k_n}$ of $L^2(R \times R_+^{n-1})$ by (1) and (2) is V_t invariant, and the spectral type of V_t on each $L_{k_2,\cdots,k_n}$, as seen in the example, is simple Lebesgue. Combining this with (3), we have proven that the spectral type of V_t on $L^2(R \times R_+^{n-1})$ is σ-Lebesgue.

Here, let us note that if we put

$$X_{k_2,\cdots,k_n}(t) = (\mathcal{E} \cdot \mathcal{C} \cdot \mathcal{J})^{-1}(\tau(v_1 - t)\eta_{k_2}(v_2) \cdots \eta_{k_n}(v_n))$$

then $X_{k_2,\cdots,k_n}(t)$ may be expressed as a stochastic integral

$$X_{k_2,\ldots,k_n}(t) = \int_{-\infty}^{t} dB(u_n) \int_{-\infty}^{u_n} \eta_{k_n}(u_n - u_{n-1}) dB(u_{n-1}) \times \ldots \times$$

$$\int_{-\infty}^{u_3} \eta_{k_3}(u_3 - u_2) dB(u_2) \times \int_{-\infty}^{u_2} \tau(\frac{u_1 + \ldots + u_n}{n} - t)\eta_{k_2}(u_2 - u_1) dB(u_1).$$

Hence if we put

$$\mathcal{H}_n(X_{k_2,\cdots,k_n}) = (\mathcal{E} \cdot \mathcal{C} \cdot \mathcal{J})^{-1} L_{k_2,\ldots,k_n},$$

then

$$\mathcal{H}_n = \sum_{k_2,\cdots,k_n \geq 0} \oplus \mathcal{H}_n(X_{k_2,\cdots,k_n}).$$

This is the decomposition of $\mathcal{H}_n$ corresponding to that of $L^2(R \times R_+^{n-1})$.

Further, if we notice that the expectations in $\mathcal{H}_n$ correspond to the multiple integrations in $L^2(R \times R_+^{n-1})$, then we can immediately compute, for example

$$E[(X_{k_2,\cdots,k_n}(t) - X_{k_2,\cdots,k_n}(s))^2] = \int_{-\infty}^{\max\{t,s\}} (\tau(u - t) - \tau(u - s))^2 du,$$

and

(4) $$E[X_{k_2,\cdots,k_n}(t) X_{k_2,\cdots,k_n}(s)] = \frac{1}{2} e^{-|t-s|}.$$

In the case where $n = 2$, which is of particular interest, we will write

$$X_n(t) = \int_{-\infty}^{t} dB(u_2) \int_{-\infty}^{u_2} \tau(\frac{u_1 + u_2}{2} - t)\eta_n(u_2 - u_1) dB(u_1).$$

We now focus on the multiplicity of a process $X(t) \in \mathcal{H}_2$:

$$X(t) \stackrel{\text{def}}{:=} \sum_{n=0}^{\infty} F(t)^n X_n(t),$$

where $F(t)$ on R is an absolutely continuous function with (i) $0 < F(t) \leq \delta < 1$.

Theorem If $F(t)$ further satisfies the conditions (ii) the derivative F' of F is in $L^1(R)$; (iii) for any open interval (a,b),

$$\int_a^b F'^2 dt = +\infty,$$

then the multiplicity of $X(t)$ is infinity.

The proof will be done by constructing another process $Y(t)$ which is both canonically represented by Brownian motion and has the same reproducing kernel Hilbert space as that of $X(t)$. Consequently, the determination of the multiplicity for process $X(t)$ may be reduced to that for $Y(t)$.

Before constructing $Y(t)$, let us first find a process $T(t)$ such that $T(t)$ can be canonically represented by Brownian motion, and that $T(t)$ shares the same covariance with $X_n(t)$. Since the covariance of $X_n(t)$ is given by (4), it follows from N. Wiener [11] that such a process must be Ornstein-Uhlenbeck process

$$T(t) = \int_{-\infty}^t e^{-(t-u)} dB(u).$$

Let us prepare a sequence of independent Brownian motions on R: $B_0, B_1, B_2, \cdots$, and let

$$Y_n(t) = \int_{-\infty}^t e^{-(t-u)} dB_n(u).$$

Then a process $Y(t)$ defined as

$$Y(t) = \sum_{n=0}^{\infty} F(t)^n Y_n(t)$$

shares the same reproducing kernel Hilbert space as that of $X(t)$. Hence the multiplicity of $Y(t)$ equals that of $X(t)$.

To say that the multiplicity of $Y(t)$ is infinity, it suffices to show by T. Hida [5,6] that the representation of $Y(t)$ is canonical, i.e., fix $T \in R$, for $n = 0, 1, 2, \cdots$, take $f_n \in L^2((-\infty, T])$ such that

$$\sum_{n=0}^{\infty} \int_{-\infty}^T |f_n(t)|^2 dt < \infty$$

and let

$$g_n(t) = \int_{-\infty}^{\min(t,T)} e^{-(t-u)} f_n(u) du.$$

We then have to show that if

$$h_0(t) := \sum_{n=0}^{\infty} F(t)^n g_n(t) = 0,$$

then $f_n = 0$ in $L^2((-\infty, T])$, $n = 0, 1, 2, \cdots$. For this purpose, let

$$\begin{aligned} h_k(t) &= \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) F(t)^{n-k} g_n(t), \; k \geq 1 \\ l_k(t) &= \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) F(t)^{n-k} g_n'(t), \; k \geq 1 \\ l_0(t) &= \sum_{n=0}^{\infty} F(t)^n g_n'(t). \end{aligned}$$

It is clear that for all k,

$$l_k(t) \in L^2_{loc}(R), \quad h_k(t) \in C(R),$$

where $L^2_{loc}(R)$ and $C(R)$ denote all the locally L^2 integrable functions and all the continuous functions on R respectively. It then follows, by mathematical induction and hypotheses on F that

$$\begin{array}{rcl} h'_0(t) = l_0(t) + F'(t)h_1(t) = 0 & \Longrightarrow & h_1(t) = 0 \\ h'_1(t) = l_1(t) + F'(t)h_2(t) = 0 & \Longrightarrow & h_2(t) = 0 \\ \cdots & \cdots & \cdots \\ h'_k(t) = l_k(t) + F'(t)h_{k+1}(t) = 0 & \Longrightarrow & h_{k+1}(t) = 0 \\ \cdots & \cdots & \cdots \end{array}$$

In matrix form,

$$A_t \cdot \begin{pmatrix} g_0(t) \\ g_1(t) \\ \vdots \\ g_n(t) \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix},$$

where

$$A_t = \begin{pmatrix} 1 & F(t) & F(t)^2 & F(t)^3 & \cdots & \binom{n}{0}F(t)^n & \cdots & \cdots \\ 0 & 1 & 2F(t) & 3F(t)^2 & \cdots & \binom{n}{1}F(t)^{n-1} & \cdots & \cdots \\ 0 & 0 & 1 & 3F(t) & \cdots & \binom{n}{2}F(t)^{n-2} & \cdots & \cdots \\ 0 & 0 & 0 & 1 & \cdots & \binom{n}{3}F(t)^{n-3} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \binom{n}{n-1}F(t) & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

On the other hand, if we let

$$B_t = \begin{pmatrix} 1 & -F(t) & F(t)^2 & -F(t)^3 & \cdots & (-1)^n\binom{n}{0}F(t)^n & \cdots & \cdots \\ 0 & 1 & -2F(t) & 3F(t)^2 & \cdots & (-1)^{n-1}\binom{n}{1}F(t)^{n-1} & \cdots & \cdots \\ 0 & 0 & 1 & -3F(t) & \cdots & (-1)^{n-2}\binom{n}{2}F(t)^{n-2} & \cdots & \cdots \\ 0 & 0 & 0 & 1 & \cdots & (-1)^{n-3}\binom{n}{3}F(t)^{n-3} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & -\binom{n}{n-1}F(t) & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

then $B_tA_t = A_tB_t$ turns out to be an infinite unit matrix. This results in $g_n(t) = 0$ and hence $f_n = 0, n = 0, 1, 2, \cdots$. The proof of the theorem is thus completed.

As a consequence of the approach, we may easily prove that for each positive integer N, the multiplicity of a process defined as

$$X(t) = \sum_{n=0}^{N-1} F(t)^n X_n(t)$$

is exactly N.

The argument for this follows if, in the proof, we define $Y(t)$ as

$$Y(t) = \sum_{n=0}^{N-1} F(t)^n B_n(t)$$

and A_t as

$$A_t = \begin{pmatrix} 1 & F(t) & F(t)^2 & F(t)^3 & \cdots & \binom{N-1}{0}F(t)^{N-1} \\ 0 & 1 & 2F(t) & 3F(t)^2 & \cdots & \binom{N-1}{1}F(t)^{N-2} \\ 0 & 0 & 1 & 3F(t) & \cdots & \binom{N-1}{2}F(t)^{N-3} \\ 0 & 0 & 0 & 1 & \cdots & \binom{N-1}{3}F(t)^{N-4} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \binom{N-1}{N-2}F(t) \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

and B_t as

$$B_t = \begin{pmatrix} 1 & -F(t) & F(t)^2 & -F(t)^3 & \cdots & (-1)^{N-1}\binom{N-1}{0}F(t)^{N-1} \\ 0 & 1 & -2F(t) & 3F(t)^2 & \cdots & (-1)^{N-2}\binom{N-1}{1}F(t)^{N-2} \\ 0 & 0 & 1 & -3F(t) & \cdots & (-1)^{N-3}\binom{N-1}{2}F(t)^{N-3} \\ 0 & 0 & 0 & 1 & \cdots & (-1)^{N-4}\binom{N-1}{3}F(t)^{N-4} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & -\binom{N-1}{N-2}F(t) \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Finally, we need to demonstrate the existence of the function F. The construction will be done by using the Monotone Convergence Theorem.

Notation: Let $f(t)$ be a function locally symmetric at $t = x$ and let $N(x)$ denote the local support of f at x and $|N(x)|$ denote the Lebesgue measure of the support $N(x)$.

We first proceed to construct a sequence of functions $s_n(t), n = 1, 2, \cdots$ as follows. $s_1(t)$: (i) symmetric about y-axis, (ii) locally symmetric at $t = \frac{n}{2}$, $n = 1, 2, \cdots$ and $|N(\frac{n}{2})| \leq \frac{1}{2^2}$, and (iii) $0 < \int_{N(\frac{n}{2})} s_1(t)dt \leq \frac{1}{2}\frac{\delta}{2^{2+n}}$ and $\int_{N(\frac{n}{2})} s_1^2(t)dt = +\infty, n = 1, 2, \cdots$; $s_2(t)$: (i) symmetric about y-axis, (ii) locally symmetric at $t = \frac{n}{2^2}$, $n = 1, 3, 5, \cdots$, and $|N(\frac{n}{2^2})| \leq \frac{1}{2^3}$, and (iii) $0 < \int_{N(\frac{n}{2^2})} s_2(t)dt \leq \frac{1}{2}\frac{\delta}{2^{3+n}}$ and $\int_{N(\frac{n}{2^2})} s_2^2(t)dt = +\infty, n = 1, 2, \cdots$. In general, for $k \geq 3$, we similarly construct $s_k(t)$ as $s_k(t)$: (i) symmetric about y-axis, (ii) locally symmetric at $t = \frac{n}{2^k}$, $n = 1, 3, 5, \cdots$, and $|N(\frac{n}{2^k})| \leq \frac{1}{2^{k+1}}$, and (iii) $0 < \int_{N(\frac{n}{2^k})} s_k(t)dt \leq \frac{1}{2}\frac{\delta}{2^{k+1+n}}$ and $\int_{N(\frac{n}{2^k})} s_k^2(t)dt = +\infty, n = 1, 2, \cdots$.

Now, let us consider the sum

$$S_n(t) \stackrel{\text{def}}{:=} \sum_{k=1}^{n} s_k(t).$$

Since we obviously have

$$0 \leq S_1(t) \leq S_2(t) \leq \cdots,$$

and

$$0 < \lim_{n\to\infty} \int_{-\infty}^{\infty} S_n(t)dt \leq \sum_{k=1}^{\infty} 2^{-k}\delta = \delta,$$

it follows from the Monotone Convergence Theorem that

$$S(t) \stackrel{\text{def}}{:=} \lim_{n\to\infty} S_n(t)$$

exists for almost all t. Now define function F as

$$F(t) = \int_{-\infty}^{t} S(u)du.$$

We may easily verify that the function F satisfies the conditions as in the theorem.

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