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Branching processes, the Ray-Knight theorem, and sticky Brownian motion

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1 Introduction

Diffusions with boundary conditions were studied by Ikeda and Watanabe [5] by means of associated stochastic differential equations. Here we are interested in a fundamental example. Let θ and x be real constants satisfying $0 < \theta < \infty$ and $0 \leq x < \infty$. Suppose $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a filtered probability space satisfying the usual conditions, and that $(X_t; t \geq 0)$ is a continuous, adapted process taking values in $[0, \infty)$ which satisfies the stochastic differential equation

$$X_t = x + \theta \int_0^t I_{\{X_s=0\}} ds + \int_0^t I_{\{X_s>0\}} dW_s, \tag{1.1}$$

where $(W_t; t \geq 0)$ is a real valued $(\mathcal{F}_t)$-Brownian motion. We say that X_t is sticky Brownian motion with parameter θ, started from x. Sticky Brownian motion has a long history. Arising in the work of Feller [3] on the general strong Markov process on $[0, \infty)$ that behaves like Brownian motion away from 0, it has been considered more recently by several authors, see Yamada [12] and Harrison and Lemoine [4], as the limit of storage processes, and by Amir [1] as the limit of random walks.

Ikeda and Watanabe show that (1.1) admits a weak solution and enjoys the uniqueness-in-law property. In [2], Chitashvili shows that, indeed, the joint law of X and W is unique (modulo the initial value of W), and that X is not measurable with respect to W, so verifying a conjecture of Skorokhod that (1.1) does not have a strong solution. The filtration $(\mathcal{F}_t)$ cannot be the (augmented) natural filtration of W and the process X contains some 'extra randomness'. It is our purpose to identify this extra randomness in terms of killing in a branching process. To this end we will study the squared Bessel process, which can be thought of as a continuous-state branching process, and a simple decomposition of it induced by introducing a killing term. We will then be able to realise this decomposition in terms of the local-time processes of X and W. Finally we will prove the following result which essentially determines the conditional law of sticky Brownian motion given the driving Wiener process.

Theorem 1. *Suppose that X is sticky Brownian motion starting from zero, and that W is the driving Wiener process, also starting from zero. Letting $L_t = \sup_{s\leq t}(-W_s)$, the conditional law of X given W satisfies*

$$\mathbb{P}\big(X_t \leq x | \sigma(W)\big) = \exp\big(-2\theta(W_t + L_t - x)\big) \qquad a.s.$$

for $x \in [0, W_t + L_t]$.

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Note in particular that $X_t \in [0, W_t + L_t]$ a.s.. The proof of this result is given in Section 4, and depends on the construction of the pair (X, W) discussed in Section 3. Section 2 is essentially independent, but helps provide us with the intuitive reason for believing Theorem 1.

We begin with a simple but illuminating lemma on sticky Brownian motion, and fix some notation we will need in the sequel.

We denote:

$$A_t^+ = \int_0^t I_{\{X_s>0\}} ds; \qquad \alpha_t^+ = \inf\{u : A_u^+ > t\}; \tag{1.2}$$

$$A_t^0 = \int_0^t I_{\{X_s=0\}} ds; \qquad \alpha_t^0 = \inf\{u : A_u^0 > t\}. \tag{1.3}$$

Then we have

Lemma 2. *If we time change both sides of (1.1) with α^+, the right-continuous inverse of A^+, we find that $(X_{\alpha_t^+}, t \geq 0)$ solves Skorokhod's reflection equation*

$$X_{\alpha_t^+} = W_t^+ + L_t^+.$$

where $W_t^+ = x + \int_0^{\alpha_t^+} I_{\{X_s>0\}} dW_s$ is a Brownian motion, and $L_t^+ = \sup_{s\leq t}\left((-W_s^+)\vee 0\right)$.

Proof. On time changing we have

$$X_{\alpha_t^+} = W_t^+ + \theta A^0_{\alpha_t^+}.$$

Observe that W^+ is a Brownian motion by Lévy's characterization. Now, A_t^0 is a continuous and increasing function of t, A_t^+ is a continuous and strictly increasing function of t, and so $L_t^+ = \theta A^0_{\alpha_t^+}$ is also a continuous and increasing function of t. Furthermore it is constant on the set $\{t : X_{\alpha_t^+} > 0\}$. The criteria of Skorokhod's lemma, see [9], are thus satisfied and $L_t^+ = \sup_{s\leq t}(-W_s^+)$ as claimed. □

This lemma shows us that sticky Brownian motion is just the time change of a reflecting Brownian motion so that the process is held momentarily each time it visits the origin. In this way it spends a real amount of time at the origin, proportional to the amount of local time the reflecting Brownian motion has spent there, in fact,

$$\theta A^0_{\alpha_t^+} = L_t^+. \tag{1.4}$$

The laws of A_t^+,A_t^0 and other quantities can be obtained directly from this, as has been accomplished by Chitashvili and Yor [13].

2 A decomposition of the squared Bessel process

We consider two processes $(R_t, t \geq 0)$ and $(Y_t, t \geq 0)$ satisfying

$$dR_t = 2\sqrt{R_t}\, dB_t - 2\theta R_t\, dt, \qquad R_0 = x, \tag{2.1a}$$

$$dY_t = 2\sqrt{Y_t}\, d\tilde{B}_t + 2\theta R_t\, dt, \qquad Y_0 = 0, \tag{2.1b}$$

where B and $\tilde{B}$ are independent Brownian motions.

Proposition 3. *$V_t = R_t + Y_t$ is a squared Bessel process of dimension 0 started from x.*

Proof. One need only make a simple application of Pythagoras's theorem, following Shiga and Watanabe [11]. We sum the two equations of (2.1) and note that

$$\int_0^t \frac{\sqrt{R_s}\, dB_s + \sqrt{Y_s}\, d\tilde{B}_s}{\sqrt{R_s + Y_s}}$$

is a Brownian motion. □

This simple decomposition can be thought of in the following manner. V_t is the total-mass process of a continuous-state critical branching process and R_t that of a subcritical process. But a subcritical process can be obtained from a critical process by introducing killing at some fixed rate into the latter. Y_t represents the mass of that part of the critical process descended from killed particles. The idea that 'R_t is V_t with killing at rate 2θ' will pervade this paper.

V_t has some finite extinction time $\tau = \inf\{t : V_t = 0\}$, see for example Revuz and Yor [9], and the same is true of R_t, its extinction time being denoted by σ. It is clear that $\tau \geq \sigma$; perhaps surprisingly τ can equal σ, and we will calculate the probability of this. This will be accomplished first via the Lévy-Khintchine formula and then extended using martingale techniques.

Lemma 4. *The laws of the extinction times τ and σ are given by*

$$\mathbb{P}(\tau \in dt) = \frac{x}{2t^2} \exp(-x/2t)\, dt,$$

and

$$\mathbb{P}(\sigma \in dt) = \tfrac{1}{2}x \left[\frac{\theta}{\sinh(t\theta)}\right]^2 \exp\left[\tfrac{1}{2}x\theta\big(1 - \coth(t\theta)\big)\right]\, dt.$$

Proof. From Pitman and Yor [8],

$$\mathbb{P}(V_t = 0) = \exp(-x/2t),$$

and

$$\mathbb{P}(R_t = 0) = \lim_{\lambda \to \infty} \mathbb{E} \exp(-\lambda R_t) = \exp\left[\tfrac{1}{2}x\theta\big(1 - \coth(t\theta)\big)\right].$$

The lemma follows on differentiating. □

We wish to prove the following.

Proposition 5. *The conditional law of the extinction time of the subcritical process given the extinction time of the critical process satisfies*

$$\mathbb{P}(\sigma = \tau | \tau) = \exp(-2\theta\tau) \qquad a.s..$$

This can be loosely interpreted as the probability that the last surviving particle of the critical process also belongs to the subcritical process, an event that depends on whether there has been any killing along its line of ancestry.

Let us denote the law of a process satisfying

$$dZ_t = 2\sqrt{Z_t}\,dB_t + 2(\beta Z_t + \delta)\,dt, \qquad\qquad Z_0 = y,$$

by ${}^{\beta}\mathbb{Q}^{\delta}_{y}$, and the law of the Z-process conditioned to be at x at time t by ${}^{\beta}\mathbb{Q}^{\delta,t}_{y\to x}$. Now the following Lévy-Khintchine formula comes from Yor [14],

$$\mathbb{E}[\exp(-\lambda Y_t)|\sigma(R)] = \exp\left\{-\int n^+(d\epsilon)\int_0^t ds\,2\theta R_s\Big(1-\exp\big(-\lambda l_{t-s}(\epsilon)\big)\Big)\right\},$$

where n^+ is the restriction of Itô excursion measure for Brownian motion to positive excursions and $l_t(\epsilon)$ the local time at height t of the excursion ϵ. Letting $\lambda\uparrow\infty$, we have

$$\exp\big(-\lambda l_{t-s}(\epsilon)\big) \to \begin{cases} 0 & \text{if } \sup\epsilon > t-s, \\ 1 & \text{otherwise.}\end{cases}$$

Hence, since $n^+(\sup\epsilon > t-s) = 1/2(t-s)$ we obtain

$$\mathbb{P}\big(Y_t = 0|\sigma(R)\big) = \exp\left\{-\int_0^t ds\,\theta R_s/(t-s)\right\}. \tag{2.2}$$

From this it follows that

$$\mathbb{P}(Y_t = 0|\sigma = t) = {}^{-\theta}\mathbb{Q}^{4,t}_{x\to 0}\exp\left\{-\theta\int_0^t Z_s/(t-s)\,ds\right\}. \tag{2.3}$$

Note that, because we are conditioning to hit 0 at time t and not before, we obtain ${}^{-\theta}\mathbb{Q}^{4,t}_{x\to 0}$, and not ${}^{-\theta}\mathbb{Q}^{0,t}_{x\to 0}$ as one might expect, see [8] for a full discussion. To evaluate this we begin by observing that by the change of measure given in Pitman and Yor [8],

$$\begin{aligned}&{}^{-\theta}\mathbb{Q}^{4,t}_{0\to 0}\exp\left\{-\theta\int_0^t Z_s/(t-s)\,ds\right\} = \\ &\qquad\qquad \frac{{}^{0}\mathbb{Q}^{4,t}_{0\to 0}\exp\left\{-\theta\int_0^t Z_s/(t-s)\,ds \;-\frac{1}{2}\theta^2\int_0^t Z_s\,ds\right\}}{{}^{0}\mathbb{Q}^{4,t}_{0\to 0}\exp\left\{-\frac{1}{2}\theta^2\int_0^t Z_s\,ds\right\}}.\end{aligned} \tag{2.4}$$

Now from [9], under ${}^{0}\mathbb{Q}^{4,t}_{0\to 0}$, Z_t solves, for $u\le t$,

$$Z_u = 2\int_0^u \sqrt{Z_s}\,dB_s + 2\int_0^u [2 - Z_s/(t-s)]\,ds,$$

where B is a Brownian motion. Hence,

$$\theta\int_0^t Z_s/(t-s)\,ds = 2t\theta + \theta\int_0^t \sqrt{Z_s}\,dB_s,$$

but, of course, $\int_0^u \sqrt{Z_s}\,dB_s$ is a martingale with quadratic variation $\int_0^u Z_s\,ds$, so

$$\exp\left\{-\theta\int_0^u \sqrt{Z_s}\,dB_s - \tfrac{1}{2}\theta^2\int_0^u Z_s\,ds\right\}$$

is a martingale too (it's bounded above by $\exp(2\theta t)$!!). We take expectations and have succeeded in evaluating the numerator of (2.4),

$$\text{(2.5)}\qquad {}^0\mathbb{Q}^{4,t}_{0\to 0}\exp\left\{-\theta\int_0^t Z_s/(t-s)\,ds - \tfrac{1}{2}\theta^2\int_0^t Z_s\,ds\right\} = \exp(-2t\theta).$$

We find directly from Pitman and Yor [8] that the denominator satisfies

$$\text{(2.6)}\qquad {}^0\mathbb{Q}^{4,t}_{0\to 0}\exp\left\{-\tfrac{1}{2}\theta^2\int_0^t Z_s\,ds\right\} = \left[\frac{t\theta}{\sinh(t\theta)}\right]^2.$$

Next we observe, recalling (2.2),

$$\text{(2.7)}\qquad \begin{aligned} {}^{-\theta}\mathbb{Q}^{0,t}_{x\to 0}\exp\left\{-\theta\int_0^t Z_s/(t-s)\,ds\right\} &= \frac{{}^{-\theta}\mathbb{Q}^0_x\exp\left\{-\theta\int_0^t Z_s/(t-s)\,ds\right\}}{{}^{-\theta}\mathbb{Q}^0_x I_{\{Z_t=0\}}} \\ &= \frac{{}^0\mathbb{Q}^0_x I_{\{Z_t=0\}}}{{}^{-\theta}\mathbb{Q}^0_x I_{\{Z_t=0\}}} = \frac{\mathbb{P}(\tau\ge t)}{\mathbb{P}(\sigma\ge t)}. \end{aligned}$$

We can now proceed to

Proof of proposition 5. The Pitman-Yor decomposition, [8],

$${}^{-\theta}\mathbb{Q}^{4,t}_{x\to 0} = {}^{-\theta}\mathbb{Q}^{4,t}_{0\to 0}\oplus {}^{-\theta}\mathbb{Q}^{0,t}_{x\to 0},$$

allows us, combining (2.5),(2.6) and (2.7), to compute $\mathbb{P}(\tau = t|\sigma = t)$. Then we have

$$\mathbb{P}(\sigma = t|\tau = t) = \mathbb{P}(\tau = t|\sigma = t)\frac{\mathbb{P}(\sigma\in dt)}{\mathbb{P}(\tau\in dt)},$$

and substituting from the lemma we are done. □

We will now extend this result by conditioning on the whole of V, instead of just its extinction time. We will need the following lemma, which is perhaps of some independent interest.

Lemma 6. *Suppose M and N are continuous, orthogonal martingales with respect to a filtration $(\mathcal{F}_t; t\ge 0)$, and suppose that M has the following representation property. Any bounded, $\sigma(M)$-measurable variable Φ is of the form*

$$\Phi = c + \int_0^\infty H_t\,dM_t,$$

where H_t is $\mathcal{F}_t$-previsible, and $c\in\sigma(M_0)$. Let $\mathcal{G}_t = \mathcal{F}_t\vee\sigma(M)$, then N is a $\mathcal{G}_t$-martingale.

Proof. By an application of the monotone-class lemma, it suffices to show that for bounded $\sigma(M)$-measurable variables Φ,

$$\mathbb{E}[\Phi(N_t - N_s)|\mathcal{F}_s] = 0.$$

But, by the representation property,

$$\begin{aligned}\mathbb{E}[\Phi(N_t - N_s)|\mathcal{F}_s] &= \mathbb{E}\left[\left\{\int_0^t H_u\,dM_u\right\}(N_t - N_s)\middle|\mathcal{F}_s\right]\\ &= \mathbb{E}[(H\cdot M)_t N_t - (H\cdot M)_s N_s|\mathcal{F}_s]\\ &= 0,\end{aligned}$$

since $(H\cdot M)$ and N are orthogonal. □

Now on the stochastic interval $[0,\tau)$ we define

$$\Theta_t = \frac{R_t}{V_t}\exp(2\theta t).$$

Applying Itô's formula gives

$$d\Theta_t = \left\{2\frac{\sqrt{R_t}}{V_t}\,dB_t - \frac{R_t}{V_t^2}\,dV_t\right\}\exp(2\theta t),$$

which shows Θ_t to be a local martingale on $[0,\tau)$. Moreover, since $\Theta_t < \exp(2\theta t)$, Θ_t tends to a finite limit as $t\uparrow\tau$, and if we define $\Theta_t = \Theta_{\tau-}$ for $t \ge \tau$, then Θ_t is a martingale for $0 \le t < \infty$.

If we continue to calculate with Itô's formula, we find that, for $t < \tau$,

$$d\Theta_t dV_t = 0 \tag{2.8a}$$

$$d\Theta_t d\Theta_t = 4\frac{\Theta_t}{V_t}\big(\exp(2\theta t) - \Theta_t\big). \tag{2.8b}$$

Thus we have proved

Lemma 7. *Θ_t is a $\mathcal{F}_t$-martingale with quadratic variation*

$$[\Theta]_t = \int_0^{t\wedge\tau} ds\,4\frac{\Theta_s}{V_s}\big(\exp(2\theta s) - \Theta_s\big),$$

and furthermore Θ is orthogonal to V.

So if we put $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(V)$, we can apply Lemma 6 to deduce that Θ_t is a $\mathcal{G}_t$-martingale. Moreover, τ is $\mathcal{G}_0$- measurable, and so for any positive constant K,

$$\mathbb{E}\Theta_\infty I_{\{\tau<K\}} = \mathbb{E}\Theta_0 I_{\{\tau<K\}},$$

since $\Theta_t I_{\{\tau<K\}}$ is a bounded $\mathcal{G}_t$-martingale. But as $K\uparrow\infty$ we obtain

$$\mathbb{E}\Theta_\infty = \mathbb{E}\Theta_0 = 1,$$

whence Θ is uniformly integrable. Now we are able to prove

Proposition 8. *The conditional law of the extinction time of the subcritical process given $\sigma(V_t; 0 \le t < \infty)$ satisfies*

$$\mathbb{P}\big(\sigma = \tau | \sigma(V)\big) = \exp(-2\theta\tau) \qquad a.s..$$

Proof. We have already remarked that $\Theta_{\tau-}$ exists and hence $[\Theta]_\tau$ is finite almost surely. It is easy to confirm, for example by time inversion, that $\int_0^\tau V_s^{-1}\, ds = \infty$, and thus we deduce from the formula for the quadratic variation process of Θ, given in Lemma 7, that $\Theta_s(\exp(2\theta s) - \Theta_s) \to 0$ as $s \uparrow \tau$. Hence Θ_τ is either 0 or $\exp(2\theta\tau)$. Furthermore,

$$\mathbb{E}[\Theta_\tau | \mathcal{G}_0] = \mathbb{E}[\Theta_0 | \mathcal{G}_0] = 1,$$

and so,

$$\mathbb{P}\big(\Theta_\tau = \exp(2\theta\tau) | \sigma(V)\big) = \exp(-2\theta\tau).$$

Now observe that $\tau > \sigma$ implies that $\Theta_\tau = 0$ (but the converse isn't so evident!), whence

$$\mathbb{P}\big(\tau = \sigma | \sigma(V)\big) \ge \exp(-2\theta\tau).$$

But

$$\mathbb{P}(\tau = \sigma | \tau) = \mathbb{E}\big[\mathbb{P}\big(\tau = \sigma | \sigma(V)\big) \big| \tau\big] = \exp(-2\theta\tau),$$

implying the desired equality. □

3 A decomposition of Brownian motion

It is now well known, as excellently described by Le Gall [7], that if we interpret the squared Bessel process of dimension zero as a continuous-state branching process then the associated genealogical structure is carried by Brownian excursions. In this section we will give a decomposition of Brownian motion that corresponds to the decomposition of the squared Bessel process induced by the killing considered previously. By looking at local times we will be able to recover Proposition 3.

To begin we recall:

Theorem 9 (Ray-Knight). *If $\bar{W}_t$ is reflecting Brownian motion, starting from zero, with l_t^y its local time at level y, then, letting $\tau_x = \inf\{t : l_t^0 \ge x\}$, we have $(l_{\tau_x}^y, y \ge 0)$ is a squared Bessel process of dimension 0 started from x.*

If we introduce drift we can obtain the subcritical process of the previous section in a similar manner.

Theorem 10. *If $\bar{S}_t$ is reflecting Brownian motion with drift θ towards the origin, starting from zero, and if l_t^y is its local time at level y, then letting $\tau_x = \inf\{t : l_t^0 \ge x\}$, we have the law of the process $(l_{\tau_x}^y, y \ge 0)$ is ${}^{-\theta}\mathbb{Q}_x^0$.*

Proof. We follow Yor [14]. Let ${}^{\theta}\mathbb{W}$ denote the law of reflecting Brownian motion with drift θ towards the origin, with similar notation for the corresponding expectation. Then the Girsanov theorem gives us

$$\left.\frac{d\,{}^{\theta}\mathbb{W}}{d\,{}^{0}\mathbb{W}}\right|_{\mathcal{F}_t} = \exp\bigl(\theta(X_t - \tfrac{1}{2}l_t^0) - \tfrac{1}{2}\theta^2 t\bigr).$$

Hence for a positive measurable functional F, we have using the Ray-Knight theorem,

$$\begin{aligned}{}^{\theta}\mathbb{W}[F(l^y_{\tau_x}; y \geq 0)] &= {}^{0}\mathbb{W}[F(l^y_{\tau_x}; y \geq 0)\exp(-\tfrac{1}{2}\theta x - \tfrac{1}{2}\theta^2\tau_x)] \\ &= {}^{0}\mathbb{Q}^0_x\left[F(Z_y; y \geq 0)\exp\left(-\tfrac{1}{2}\theta x - \tfrac{1}{2}\theta^2\int_0^\infty Z_y\,dy\right)\right] \\ &= {}^{-\theta}\mathbb{Q}^0_x[F(Z_y; y \geq 0)],\end{aligned}$$

the last line following from the change of measure given in [8]. □

Now we give the fundamental results of this section, recalling the notation of Section 1.

Theorem 11. *Suppose that X is a sticky Brownian motion starting from x, and W is a Wiener process, with $W_0 \geq x$, so that equation (1.1) is satisfied. Define, for $t \geq 0$,*

$$\bar{W}_t = W_t + L_t,$$

where $L_t = \sup_{s\leq t}\bigl((-W_s) \vee 0\bigr)$, so $\bar{W}$ is a reflecting Brownian motion. Then

$$\bar{W}_t = \bar{S}_{A_t^0} + X_t,$$

where $\bar{S}_t$ is a reflecting Brownian motion, with drift θ towards the origin, independent of X.

Proof. Take (X, W) solving (1.1) with $W_0 \geq x$. Then

$$S_t = W_0 - x + \int_0^{\alpha_t^0} I_{\{X_s=0\}}dW_s - \theta t$$

defines a Brownian motion with drift $(-\theta)$, independent by Knight's theorem from X_t. It is easy to check that $W_t = S_{A_t^0} + X_t$. Let

$$L_t = \sup_{s\leq t}\bigl((-W_s) \vee 0\bigr) \qquad \text{and} \qquad K_t = \sup_{s\leq t}\bigl((-S_s) \vee 0\bigr).$$

Now $\bar{W}_t = W_t + L_t$ is reflecting Brownian motion, and $\bar{S}_t = S_t + K_t$ is a reflecting Brownian motion with drift θ towards the origin, independent of X. Moreover, if we can show $K_{A_t^0} = L_t$ then we will have $\bar{W}_t = \bar{S}_{A_t^0} + X_t$ as required. But $W_t \geq S_{A_t^0}$, whence

$$\sup_{s\leq t}(-W_s) \leq \sup_{s\leq t}(-S_{A_s^0}),$$

and so $L_t \leq K_{A_t^0}$. If there exists an $s \leq t$ so that $X_s = 0$ then, putting

$$t^0 = \alpha^0_{A_t^0} = \sup\{s \leq t : X_s = 0\},$$

so $X_{t^0} = 0$ and $S_{A_t^0} = S_{A_{t^0}^0}$, we have $W_{t^0} = S_{A_t^0}$, and hence

$$\sup_{s\leq t}(-W_s) \geq \sup_{s\leq t}(-S_{A_s^0}).$$

If no such s exists then $A_t^0 = 0$, and $W_s \geq 0$, for all $s \leq t$. In either case $L_t \geq K_{A_t^0}$. $\square$

For the rest of the section we assume that X_0 and $\bar{W}_0$ are both 0, and we are able to interpret the above result in terms of branching processes. A point $(t, \bar{W}_t)$ represents part of the subcritical process if $X_t = 0$; otherwise it is part of an excursion of the X process away from 0, and such an excursion represents mass descended from a single killed ancestor. Letting l_t^y be the local time of $\bar{W}$ and τ_x be as before, we have,

$$l^y_{\tau_x} = \int_0^{\tau_x} I_{\{X_t=0\}} dl_t^y + \int_0^{\tau_x} I_{\{X_t>0\}} dl_t^y. \tag{3.1}$$

The Ray-Knight theorem applies to the left-hand side and the following applies to the right-hand side.

Proposition 12. *For $y \geq 0$ define*

$$R_y = \int_0^{\tau_x} I_{\{X_t=0\}} dl_t^y,$$

and

$$Y_y = \int_0^{\tau_x} I_{\{X_t>0\}} dl_t^y.$$

Then R and Y satisfy the stochastic differential equations (2.1).

Proof. From the occupation-time formula, we have, for any positive Borel measurable function f,

$$\begin{aligned}
\int_0^\infty f(y) R_y\, dy &= \int_0^\infty f(y) \int_0^{\tau_x} I_{\{X_t=0\}} dl_t^y dy \\
&= \int_0^{\tau_x} f(\bar{W}_t) I_{\{X_t=0\}} dt \\
&= \int_0^{\tau_x} f(\bar{W}_t)\, dA_t^0 \\
&= \int_0^{A^0_{\tau_x}} f(\bar{S}_t)\, dt.
\end{aligned}$$

Next observe, since we demonstrated in the proof of the previous theorem $K_{A_t^0} = L_t$, that $A^0_{\tau_x}$ is the first time that the local time of $\bar{S}$ at 0 reaches x. Thus $(R_y; y \geq 0)$ is the family of local times of $\bar{S}$ stopped after it has spent local time x at the origin, and, appealing to Theorem 10, the first part of the result follows.

Similarly, for any positive Borel measurable function f,

$$\begin{aligned}\int_0^\infty f(y)Y_y dy &= \int_0^\infty f(y)\int_0^{\tau_x} I_{\{X_t>0\}}dl_t^y dy \\ &= \int_0^{\tau_x} f(\bar{W}_t)I_{\{X_t>0\}}dt \\ &= \int_0^{\tau_x} f(\bar{S}_{A_t^0}+X_t)\,dA_t^+ \\ &= \int_0^{A_{\tau_x}^+} f(\bar{S}_{\theta^{-1}L_t^+}+X_{\alpha_t^+})\,dt.\end{aligned}$$

Recall that $(X_{\alpha_t^+}; t\geq 0)$ is a reflecting Brownian motion and that L^+ half its local time at the origin. Note that $A_{\tau_x}^+$is the first time that the local time of X_{α^+} at the origin reaches $2\theta A_{\tau_x}^0$. Now put

$$M_t^f = \exp\left\{-\int_0^t f(\bar{S}_{\theta^{-1}L_t^+}+X_{\alpha_t^+})\,dt\right\}.$$

Since, conditional on $\bar{S}$, M_t^f is a skew multiplicative functional of $(X_{\alpha_t^+}; t\geq 0)$, it is a consequence of excursion theory, see [9] and [14], that

$$\mathbb{E}\Big[M_{A_{\tau_x}^+}^f\Big|\sigma(\bar{S})\Big] =$$
$$\exp\left\{-\int_0^{2\theta A_{\tau_x}^0} ds\int n^+(d\epsilon)\left[1-\exp\left(-\int_0^{T(\epsilon)} du\, f\big(\epsilon(u)+\bar{S}_{(2\theta)^{-1}s}\big)\right)\right]\right\},$$

where $T(\epsilon)$ denotes the lifetime of the excursion ϵ. This, by the occupation-time formula, remembering $A_{\tau_x}^0$ is the first time that the local time of $\bar{S}$ at 0 reaches x, equals

$$\exp\left\{-\int n^+(d\epsilon)\int_0^\infty 2\theta dy\, R_y\left[1-\exp\left(-\int_0^{T(\epsilon)} du\, f\big(\epsilon(u)+y\big)\right)\right]\right\}.$$

Thus we have

$$\mathbb{E}\left[\exp\left\{-\int_0^\infty f(y)Y_y\,dy\right\}\middle|\sigma(R)\right] =$$
$$\exp\left\{-2\theta\int_0^\infty dy\, R_y\int n^+(d\epsilon)\left[1-\exp\left(-\int_0^{T(\epsilon)} du\, f\big(\epsilon(u)+y\big)\right)\right]\right\},$$

and this characterises the solution to (2.1), see [14]. □

4 The conditional law of sticky Brownian motion

In this section we will prove Theorem 1; however we do not work directly with the pair of processes (X, W). Instead, motivated by the previous section, we consider a Markov process $(X, \bar{W})$ on the state space $E = \{(x,a)\in\mathbb{R}^2 : x\geq 0, a\geq x\}$, defined by taking X to be a sticky Brownian motion and $\bar{W}_t = \bar{S}_{A_t^0} + X_t$ where $\bar{S}$ is a reflecting Brownian motion, independent of X, with drift θ towards the origin. We denote by $\mathbb{P}^{(x,a)}$ the law of $(X, \bar{W})$ started from $X_0 = x$ and $\bar{W}_0 = a$, with similar notation for expectations. We will prove that

$$\mathbb{P}^{(0,0)}\big(X_t \le x | \sigma(\bar{W})\big) = \exp\big(-2\theta(\bar{W}_t - x)\big) \qquad \text{a.s.} \tag{4.1}$$

for $x \in [0, \bar{W}_t]$. This has a clear interpretation in terms of our branching process with killing. We can think of the value of X_t as depending on whether, and if so where, killing occurs along a line of ancestry of length $\bar{W}_t$. Theorem 1 follows from (4.1) and Theorem 11, noting that $\sigma(\bar{W}) = \sigma(W)$.

We proceed by computing some resolvents. We need, first, to convince ourselves that $(X, \bar{W})$ has the strong Markov property; but this follows, conditioning on X, from the simple Markov property of S and the strong Markov property of X. The first part of the following result, the calculation of the resolvent of sticky Brownian motion, has been obtained previously by several authors, see for example Knight [6].

Proposition 13. *The resolvent operators $(\mathcal{U}_\lambda, \lambda > 0)$ and $(\mathcal{V}_\lambda, \lambda > 0)$ of X and $(X, \bar{W})$ respectively are given, letting $\gamma^2 = 2\lambda$, by*

$$\mathcal{U}_\lambda f(x) = (2\theta)^{-1} u_\lambda(x,0) f(0) + \int_0^\infty u_\lambda(x,y) f(y)\, dy,$$

where

$$u_\lambda(x,y) = \gamma^{-1}\left[e^{-\gamma|y-x|} + \frac{\theta\gamma - \lambda}{\theta\gamma + \lambda} e^{-\gamma|y+x|}\right],$$

and

$$\begin{aligned}\mathcal{V}_\lambda f(x,a) = &\int_0^\infty \int_y^\infty f(y,b) v_\lambda(x,a,y,b)\, db dy \\ &+ (2\theta)^{-1}\int_0^\infty f(0,b) v_\lambda(x,a,0,b)\, db \\ &+ \int_{\{a+y=b+x\}} f(y,b) r_\lambda^-(x,y)\, dy,\end{aligned}$$

where

$$v_\lambda(x,a,y,b) = \frac{2\theta}{\gamma+\theta} e^{\theta(a-b+y-x)-\gamma(y+x)}\left[e^{-(\theta+\gamma)|b-a+x-y|} + \frac{\gamma+2\theta}{\gamma} e^{-(\theta+\gamma)|a+b-y-x|}\right],$$

$$\textit{and} \qquad r_\lambda^-(x,y) = \gamma^{-1}\left[e^{-\gamma|y-x|} - e^{-\gamma|y+x|}\right].$$

Proof. We are guided (as always!) by Rogers and Williams [10].

We begin by supposing that $X_0 = 0$ and $\bar{W}_0 = a$, where $a \ge 0$. Take two independent exponential random variables, T_1 and T_2, both independent of X and $\bar{W}$, and both with mean λ^{-1}. Let

$$T = \alpha_{T_1}^0 \wedge \alpha_{T_2}^+,$$

this also being exponentially distributed with mean λ^{-1}. Now $X_T = 0$ precisely if $\alpha^0_{T_1} < \alpha^+_{T_2}$, or equivalently if $T_1 < A^0_{\alpha^+_{T_2}}$. But recall $\theta A^0_{\alpha^+_t}$ equals L^+_t, which is exponentially distributed with mean γ^{-1}, where $\gamma^2 = 2\lambda$, and so,

$$\mathbb{P}^{(0,a)}(X_T = 0) = \frac{\lambda}{\lambda + \theta\gamma}.$$

For $y > 0$, since $X_{\alpha^+_t}$ is reflecting Brownian motion, and hence $X_{\alpha^+_{T_2}}$ is independent of $L^+_{T_2}$,

$$\begin{aligned}\mathbb{P}^{(0,a)}(X_T \in dy) &= \mathbb{P}^{(0,a)}(X_{\alpha^+_{T_2}} \in dy)\mathbb{P}^{(0,a)}(\alpha^0_{T_1} > \alpha^+_{T_2}) \\ &= \frac{2\theta\lambda}{\lambda + \theta\gamma}\exp(-\gamma y)\, dy.\end{aligned}$$

Now let us note that the resolvent of reflecting Brownian motion with drift θ towards the origin has density

$$^\theta r_\lambda(x,y) = \alpha^{-1}e^{\theta(x-y)}\left[e^{-\alpha|y-x|} + \frac{\alpha+\theta}{\alpha-\theta}e^{-\alpha|x+y|}\right],$$

with respect to Lebesgue measure, where $\alpha^2 = 2\lambda + \theta^2$.

We have that A^0_T equals $T_1 \wedge \theta^{-1}L^+_{T_2}$, and hence is exponentially distributed at rate $\lambda + \theta\gamma$. Thus,

$$\begin{aligned}\mathbb{P}^{(0,a)}(X_T = 0, \bar{W}_T \in db) &= \mathbb{P}^{(0,a)}(\alpha^0_{T_1} < \alpha^+_{T_2} \text{ and } \bar{S}_{A^0_T} \in db) \\ &= \lambda\, {}^\theta r_{\lambda+\theta\gamma}(a,b)db.\end{aligned}$$

Similarly, and again crucially using the independence of $X_{\alpha^+_{T_2}}$ and $L^+_{T_2}$,

$$\begin{aligned}\mathbb{P}^{(0,a)}(X_T \in dy, \bar{W}_T \in db) &= \mathbb{P}^{(0,a)}(X_T \in dy)\mathbb{P}^{(0,a)}(\bar{S}_{A^0_T} \in d(b-y)) \\ &= 2\theta\lambda\exp(-\gamma y)\, {}^\theta r_{\lambda+\theta\gamma}(a, b-y)\, dbdy.\end{aligned}$$

The above arguments have determined $\mathcal{U}_\lambda f(0)$ and $\mathcal{V}_\lambda f(0,a)$. If we now consider the process $(X, \bar{W})$ started from an arbitrary point $(x,a) \in E$, we may apply the strong Markov property at the time H_0, the first time that X_t is zero. We obtain

$$\mathcal{U}_\lambda f(x) = \mathcal{R}^-_\lambda f(x) + \psi_\lambda(x)\mathcal{U}_\lambda f(0),$$

and, defining the the function $f_{x,a}$ by $f_{x,a}(y) = f(y, a+y-x)$ for $y \geq 0$,

$$\mathcal{V}_\lambda f(x,a) = \mathcal{R}^-_\lambda f_{x,a}(x) + \psi_\lambda(x)\mathcal{V}_\lambda f(0, a-x),$$

where $\mathcal{R}^-_\lambda$ is the resolvent of Brownian motion killed at 0, which has density $r^-_\lambda(x,y)$ with respect to Lebesgue measure, and

$$\psi_\lambda(x) = \mathbb{E}^{(x,a)}[\exp(-\lambda H_0)] = \exp(-\gamma x).$$

This completes the proof. $\square$

Let $T_1, T_2, \dots, T_n$ be independent exponential times with means $\lambda_1^{-1}, \lambda_2^{-1}, \dots, \lambda_n^{-1}$. We will show, for arbitrary bounded, measurable functions $f_1, \dots, f_n$ on E, that

$$\text{(4.2)}\quad \mathbb{E}^{(0,0)}\left[\mathcal{I}_{\{X_{T_1+\dots+T_n}\le x\}} f_1(\bar{W}_{T_1})\dots f_n(\bar{W}_{T_1+\dots+T_n})\right] = \mathbb{E}^{(0,0)}\left[\exp\left(-2\theta(\bar{W}_{T_1+\dots T_n} - x)\wedge 0\right) f_1(\bar{W}_{T_1})\dots f_n(\bar{W}_{T_1+\dots+T_n})\right].$$

Our argument will essentially depend on time reversal and the fact that $\bar{W}_t = 0$ implies that $X_t = 0$. We begin by making some remarks concerning the resolvent of $(X, \bar{W})$ that follow from the preceding proposition.

Define the measure m on Borel subsets of the state space E by

$$\text{(4.3)}\qquad mA = \int_A e^{2\theta(x-a)}\, dx da + \int_{\{a:(0,a)\in A\}} e^{-2\theta a}/2\theta\, da,$$

and then $\mathcal{V}_\lambda$ is self-adjoint with respect to m in the sense that for any bounded, measurable functions f and g on the state space

$$\text{(4.4)}\qquad \int_E dm(y,a)\, f(y,a)[\mathcal{V}_\lambda g](y,a) = \int_E dm(y,a)\, [\mathcal{V}_\lambda f](y,a) g(y,a).$$

We also have

$$\text{(4.5)}\qquad \begin{aligned} \mathcal{V}_\lambda f(0,0) &= 2\theta \int_E dm(y,a)\, f(y,a) v_\lambda(y,a,0,0) \\ &= 2\theta \int_E dm(y,a)\, f(y,a) r_\lambda(0,a). \end{aligned}$$

where $r_\lambda(\cdot,\cdot)$ is the density, with respect to Lebesgue measure, of the resolvent $\mathcal{R}_\lambda$ of reflecting, driftless Brownian motion. Slightly abusing notation we will write r_λ for the function $r_\lambda(0,\cdot)$. If f is a bounded measurable function on $[0,\infty)$ let us define $f^\star$ to be the function on the state space satisfying $f^\star(x,a) = f(a)$ for all $x \in [0,a]$. For such f observe that

$$\text{(4.6)}\qquad \mathcal{V}_\lambda f^\star = (\mathcal{R}_\lambda f)^\star;$$

this being nothing more than the statement that $\bar{W}$ is a reflecting Brownian motion.

We define the functions $e_x : \mathbb{R}^+ \to [0,1]$ and $i_x : E \to \{0,1\}$ by

$$e_x(a) = \begin{cases} e^{-2\theta(a-x)} & \text{if } a > x, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$i_x(y,a) = \begin{cases} 1 & \text{if } y \le x, \\ 0 & \text{otherwise.} \end{cases}$$

Now, using the above observations, and that $\mathcal{R}_\lambda$ is self-adjoint with respect to Lebesgue measure, we have

$$\begin{aligned}
[\mathcal{V}_{\lambda_1} f_1^\star \mathcal{V}_{\lambda_2} f_2^\star \dots \mathcal{V}_{\lambda_n} i_x f_n^\star](0,0) &= 2\theta \int_E dm(y,a) r_{\lambda_1}(0,a)[f_1^\star \mathcal{V}_{\lambda_2} f_2^\star \dots \mathcal{V}_{\lambda_n} i_x f_n^\star](y,a) \\
&= 2\theta \int_E dm(y,a) i_x(y,a)[f_n^\star \mathcal{V}_{\lambda_n} f_{n-1}^\star \dots \mathcal{V}_{\lambda_2} f_1^\star r_{\lambda_1}^\star](y,a) \\
&= 2\theta \int_{\{(y,a)\in E: y\le x\}} dm(y,a)\, [f_n \mathcal{R}_{\lambda_n} f_{n-1} \dots \mathcal{R}_{\lambda_2} f_1 r_{\lambda_1}](a) \\
&= \int_0^\infty da\, e_x(a)[f_n \mathcal{R}_{\lambda_n} f_{n-1} \dots \mathcal{R}_{\lambda_2} f_1 r_{\lambda_1}](a) \\
&= \int_0^\infty da\, r_{\lambda_1}(0,a)[f_1 \mathcal{R}_{\lambda_n} f_2 \dots \mathcal{R}_{\lambda_n} e_x f_n](a) \\
&= [\mathcal{V}_{\lambda_1} f_1^\star \mathcal{V}_{\lambda_2} f_2^\star \dots \mathcal{V}_{\lambda_n} e_x^\star f_n^\star](0,0).
\end{aligned}$$

This proves equation (4.2). Moreover it follows simply from (4.6), that given further, independent, exponential times $T_{n+1}, \dots, T_{n+m}$, and bounded, measurable functions $f_{n+1}, \dots, f_{n+m}$, the stronger statement

$$\begin{aligned}
(4.7)\quad & \mathbb{E}^{(0,0)}\left[\mathcal{I}_{\{X_{T_1+\dots+T_n}\le x\}} f_1(\bar W_{T_1}) \dots f_{n+m}(\bar W_{T_1+\dots+T_{n+m}})\right] = \\
&\quad \mathbb{E}^{(0,0)}\left[\exp\big(-2\theta(\bar W_{T_1+\dots+T_n} - x)\wedge 0\big) f_1(\bar W_{T_1}) \dots f_{n+m}(\bar W_{T_1+\dots+T_{n+m}})\right],
\end{aligned}$$

holds. Now, by the uniqueness of Laplace transforms,

$$\begin{aligned}
(4.8)\quad & \mathbb{E}^{(0,0)}\left[\mathcal{I}_{\{X_{t_1+\dots+t_n}\le x\}} f_1(\bar W_{t_1}) \dots f_{n+m}(\bar W_{t_1+\dots t_{n+m}})\right] = \\
&\quad \mathbb{E}^{(0,0)}\left[\exp\big(-2\theta(\bar W_{t_1+\dots+t_n} - x)\wedge 0\big) f_1(\bar W_{t_1}) \dots f_{n+m}(\bar W_{t_1+\dots+t_{n+m}})\right],
\end{aligned}$$

for almost all $t_1, t_2, \dots, t_{n+m} \ge 0$. In order to extend this equality, so that it holds for all $t_1, t_2, \dots, t_{n+m} \ge 0$, we first assume that $f_1, \dots, f_{n+m}$ are continuous, and subsequently apply the monotone-class lemma. Now observe that

$$t_1, t_2, \dots, t_{n+m} \longmapsto \exp\big(-2\theta(\bar W_{t_1+\dots+t_n} - x)\wedge 0\big) f_1(\bar W_{t_1}) \dots f_{n+m}(\bar W_{t_1+\dots+t_{n+m}})$$

is continuous and bounded, so the bounded convergence theorem implies that the right-hand side of (4.8) is a continuous function of $t_1, t_2, \dots, t_{n+m}$ too. Note that, as we can check from Lemma 2,

$$t_1 + t_2 + \dots + t_n \longmapsto \mathbb{E}^{(0,0)}\left[\mathcal{I}_{\{X_{t_1+\dots+t_n}\le x\}}\right]$$

is continuous, and so by adding large constant multiples of $\mathcal{I}_{\{X_{t_1+\dots+t_n}\le x\}}$ we can assume that $f_1, f_2, \dots, f_{n+m}$ are all positive functions. Now

$$t_1, t_2, \dots, t_{n+m} \longmapsto \mathcal{I}_{\{X_{t_1+\dots+t_n}\le x\}} f_1(\bar W_{t_1}) \dots f_{n+m}(\bar W_{t_1+\dots+t_{n+m}})$$

is upper semi-continuous, and Fatou's lemma thus implies that the left-hand side of (4.8) is an upper semi-continuous function of $t_1, t_2, \dots, t_{n+m}$. But we can argue the same way with $f_1, f_2, \dots, f_{n+m}$ replaced by $-f_1, -f_2, \dots, -f_{n+m}$, and so the left-hand side of (4.8) must in fact be continuous, and hence equality holds for all $t_1, t_2, \dots, t_{n+m} \ge 0$. All that remains is, on applying the monotone-class lemma, to deduce (4.1), and the proof of Theorem 1 is finally complete.

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