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SOME POLAR SETS FOR THE BROWNIAN SHEET

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§1. Introduction. Let $W \triangleq (W(s); s \in \mathbb{R}_+^N)$ denote d -dimensional N -parameter Brownian sheet. That is, W is a centered Gaussian process on $\mathbb{R}^d$ indexed by $\mathbb{R}_+^N$ such that

$$\mathbb{E}W_i(s)W_j(t) = \begin{cases} \prod_{k=1}^N (s_k \wedge t_k), & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}.$$

We will write V_i for the i -th coordinate of the k -dimensional vector V and the norm of $V \in \mathbb{R}^k$ is $\|V\| \triangleq (\sum_{j=1}^k V_j^2)^{1/2}$.

In this article, we are concerned with some interesting sets which are avoided by the path of W . In the language of Markov processes, such sets are said to be *polar*. Let us begin with a result of OREY AND PRUITT [OP] on when singletons are polar.

(1.1) **Theorem.** ([OP, Theorems 3.3, 3.4]) *For any $a \in \mathbb{R}^d$,*

$$\mathbb{P}(W(t) = a, \text{ for some } t \in \mathbb{R}_+^N) = \begin{cases} 1, & \text{if } d < 2N \\ 0, & \text{if } d \geq 2N \end{cases}.$$

(1.2) **Remark.** When the Brownian sheet is non-critical, i.e., $d \neq 2N$, we provide an elementary proof which can be easily extended to show the following: suppose $E \subset \mathbb{R}^d$ is compact and $\liminf_{h \rightarrow 0} h \ln(1/h) N_E^{d/2}(h) = 0$ where $N_E(h)$ is the upper (or lower) Kolmogorov entropy of E . Then $\mathbb{P}(W(t) \in E, \text{ for some } t \in \mathbb{R}_+^N) = 0$. See TAYLOR [T1] for definitions and properties.

The next result concerns k -multiple points. We say that W has k -multiple points, if there exists k distinct times $t^1, \dots, t^k$, such that $W(t^1) = \dots = W(t^k)$.

(1.3) **Theorem.** *The probability that W has k -multiple points is 1 or 0 according as whether $(d - 2N)k < d$ or $(d - 2N)k > d$.*

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Clearly, the above leaves out the critical case, $(d - 2N)k = d$. There does not seem to be an elementary way to resolve this problem when $(d - 2N)k = d$. However, the problem can be solved. See the forthcoming paper of SALISBURY AND FITZSIMMONS [FS-2]

In Section 2, we prove Theorem (1.1) in the non-critical case, i.e., when $d \neq 2N$. Theorem (1.3) is proved in Section 3.

A historical account of these problems is in order. When $N = 1$, W is d -dimensional Brownian motion and the above are amongst the results of DVORETSKY, ERDŐS AND KAKUTANI [DEK1, DEK2] and DVORETSKY, ERDŐS, KAKUTANI AND TAYLOR [DEKT]; see TAYLOR [T1] for a detailed account of this celebrated problem (as well as many other related developments). In this case, (i.e., when $N = 1$), much more can be done due to the Markovian structure of the underlying process. For further advances in this area see, for example, BASS, BURDZY AND KHOSHNEVISAN [BBK], BASS AND KHOSHNEVISAN [BK], DYNKIN [D1, D2], FITZSIMMONS AND SALISBURY [FS-1], HAWKES AND PRUITT [HaP], HENDRICKS [He], LE GALL [LG], PERES [P], ROSEN [R1-R3], SALISBURY [S], SHIEH [Sh], TAYLOR [T1-T3], VARADHAN [V], WERNER [W] and YOR [Y], to cite a small sample. When $N > 1$ and $k < 4N$, the existence of 2-multiple points was discovered simultaneously and independently by EHM [E] and ROSEN [R2]; see ADLER [A1] and DYNKIN [D1, D2] for improvements and other works. Similar methods to the ones mentioned above (i.e., local time techniques) can be used to show the existence of k -multiple points for any $k \geq 2$ satisfying $(d - 2N)k \leq d$; cf. CHEN [C]. (In light of Theorem (1.1) above, the condition $d \geq 2N$ in [C] is superfluous for non-polarity.) For our purposes, the crux of the argument is the proof of the non-existence of k -multiple points. The need to solve this problem was brought to our attention by the review of FRISTEDT [F].

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§2. The Proof of Theorem (1.1) in the non-critical case. Without loss of much generality, let us only consider the case $a = 0$. When $d < 2N$, there exists a non-trivial measure which lives on $\{s \in \mathbb{R}_+^N : W(s) = 0\}$; see ADLER [A1] and EHM [E]. Consequently, $\mathbb{P}(\exists s \in \mathbb{R}_+^N : W(s) = 0) = 1$. For the sake of completion, we will give a simple Fourier analytic proof of this fact (when $N = 1$, this method appears in KAHANE [K], Chapters 16 and 18). Fix a closed cube $I \subset (0, \infty)^N$ and consider the occupation measure, $\nu(A) \triangleq \int_I \mathbf{1}\{W(s) \in A\} ds$. The Fourier transform $\hat{\nu}$ of ν is $\hat{\nu}(\xi) = \int_I \exp(i\xi \cdot W(s)) ds$, where $\xi \in \mathbb{R}^d$ and $\cdot$ denotes the Euclidean dot

product. Note that

$$\begin{aligned}\mathbb{E}|\widehat{\nu}(\xi)|^2 &= \mathbb{E} \int_I \int_I \exp(i\xi \cdot (W(s) - W(t))) ds dt \\ &= \int_I \int_I \exp\left(-\frac{\|\xi\|^2}{2} \sigma^2(s, t)\right) ds dt,\end{aligned}$$

where $\sigma^2(s, t) \triangleq \prod_{j=1}^N s_j + \prod_{j=1}^N t_j - 2 \prod_{j=1}^N (s_j \wedge t_j)$ for $s, t \in \mathbb{R}_+^N$. Define, $\sigma^2 \circ \pi(u, v) = \exp(\sum_j u_j) + \exp(\sum_j v_j) - 2 \exp(\sum_j (u_j \wedge v_j))$. Then by a change of variables,

$$\mathbb{E}|\widehat{\nu}(\xi)|^2 = \int_{\ln(I)} \int_{\ln(I)} \exp\left(-\|\xi\|^2 \sigma^2 \circ \pi(u, v)/2\right) \exp \sum_j (u_j + v_j) du dv.$$

For $u, v \in \ln(I)$, let $\mathcal{S} = \{1 \leq j \leq N : u_j \leq v_j\}$. Recalling that $I \subset (0, \infty)^N$ is a fixed closed cube, consider,

$$\begin{aligned}\sigma^2 \circ \pi(u, v) &= \exp\left(\sum_{j \in \mathcal{S}} u_j\right) \left[\exp \sum_{j \in \mathcal{S}^c} u_j - \exp \sum_{j \in \mathcal{S}^c} v_j\right] + \\ &\quad + \exp\left(\sum_{j \in \mathcal{S}^c} v_j\right) \left[\exp \sum_{j \in \mathcal{S}} v_j - \exp \sum_{j \in \mathcal{S}} u_j\right] \\ &= e^{\sum u_j} \left[1 - \exp \sum_{j \in \mathcal{S}^c} |u_j - v_j|\right] + e^{\sum v_j} \left[1 - \exp \sum_{j \in \mathcal{S}} |u_j - v_j|\right] \\ &\geq c_0 \sum_{j=1}^N |u_j - v_j|,\end{aligned}$$

where c_0 depends only on d, N and the size of I . Therefore, for some c_1 depending on d, N and the size of I ,

$$\begin{aligned}\mathbb{E}|\widehat{\nu}(\xi)|^2 &\leq \int_{\ln(I)} \int_{\ln(I)} \exp\left(-\frac{c_0 \|\xi\|^2 \sum_j |u_j - v_j|}{2}\right) e^{\sum_j (u_j + v_j)} du dv \\ &\leq c_1 \int_{\ln(I) \ominus \ln(I)} \exp\left(-c_0 \|\xi\|^2 \sum_j |w_j|/2\right) dw,\end{aligned}$$

where $A \ominus B \triangleq \{x - y : x \in A, y \in B\}$. By scaling, it follows that for some c_2 (which depends only on d, N and the size of I),

$$\mathbb{E}|\widehat{\nu}(\xi)|^2 \leq c_2 (\|\xi\|^{-2N} + 1).$$

Since $d < 2N$, this implies that $\mathbb{E} \int_{\mathbb{R}^d} |\widehat{\nu}(\xi)|^2 d\xi < \infty$. In particular, with probability one, $\widehat{\nu} \in L^2(\mathbb{R}^d, d\xi)$. By Parseval's identity, almost surely, $\nu(d\xi) \ll d\xi$ and the density is a.s. in $L^2(\mathbb{R}^d, d\xi)$. Writing the density as $\ell_I^\sharp$, it follows that $\nu(A) = \int_A \ell_I^\sharp dx$. Note that $\mathbb{E} \ell_I^0 = \int_I (2\pi \prod_{j=1}^N s_j)^{-d/2} ds > 0$. Therefore, $\ell_I^0 > 0$ with

positive probability. Since the "measure" $I \mapsto \ell_I^0$ is supported in $W^{-1}(\{0\})$, with positive probability, $I \cap W^{-1}(\{0\}) \neq \emptyset$. An application of Kolmogorov's 0-1 law shows that $W^{-1}(\{0\}) \neq \emptyset$. a.s. .

It remains to investigate the case $d > 2N$: our proof is motivated by the work of KAUFMAN [Ka].

By taking $\eta \rightarrow 0$, we see that it suffices to show that for any $\eta \in (0, 1)$.

$$(2.1) \quad \mathbb{P}(\exists t \in [\eta, \eta^{-1}]^N : W(t) = 0) = 0.$$

For any $\varepsilon > 0$ cover $[\eta, \eta^{-1}]^N$ by closed non-overlapping boxes, $B_j(\varepsilon)$, $1 \leq j \leq n(\varepsilon)$, of side ε . It is easy to see that there exist suitable constants $K_i = K_i(\eta, N)$, $i = 1, 2$, such that

$$(2.2) \quad K_1 \varepsilon^{-N} \leq n(\varepsilon) \leq K_2 \varepsilon^{-N}.$$

Define the random process N by

$$N(\varepsilon) \triangleq \sum_{j=1}^{n(\varepsilon)} \mathbf{I}\{\exists s \in B_j(\varepsilon) : W(s) = 0\},$$

where $\mathbf{I}\{\dots\}$ is 1 or 0 according to whether or not the event between the braces occurs. Recall the uniform modulus of continuity of W (cf. OREY AND PRUITT [OP] or the proof of ADLER [A2, p.8], for example):

$$(2.3) \quad \limsup_{\varepsilon \rightarrow 0} \max_{1 \leq j \leq n(\varepsilon)} \sup_{s, t \in B_j(\varepsilon)} \frac{\|W(s) - W(t)\|}{\sqrt{\varepsilon \ln(1/\varepsilon)}} \leq K_3,$$

where $K_3 = K_3(\eta, d, N) \in (0, \infty)$. It follows that for all ε small enough, $N(\varepsilon) \leq M(\varepsilon)$, where M is defined by the following:

$$M(\varepsilon) \triangleq \sum_{j=1}^{n(\varepsilon)} \mathbf{I}\{\forall s \in B_j(\varepsilon) : \|W(s)\| \leq 2K_3 \sqrt{\varepsilon \ln(1/\varepsilon)}\}.$$

To finish the proof of the theorem, it suffices to show that with probability one,

$$\liminf_{\varepsilon \rightarrow 0} M(\varepsilon) = 0.$$

We will achieve this by proving that

$$(2.4) \quad \lim_{\varepsilon \rightarrow 0} \mathbb{E} M(\varepsilon) = 0.$$

Note that

$$\mathbf{I}\{\forall s \in B_j(\varepsilon) : \|W(s)\| \leq 2K_2 \sqrt{\varepsilon \ln(1/\varepsilon)}\} \leq \mathbf{I}\{\|W(b_j(\varepsilon))\| \leq 2K_3 \sqrt{\varepsilon \ln(1/\varepsilon)}\},$$

where $b_j(\varepsilon)$ is the center of $B_j(\varepsilon)$, say. Hence,

$$\mathbb{E}M(\varepsilon) \leq \sum_{1 \leq j \leq n(\varepsilon)} \mathbb{P}(\|W(b_j(\varepsilon))\| \leq 2K_3\sqrt{\varepsilon \ln(1/\varepsilon)}).$$

For $s \in \mathbb{R}_+^N$ and $a \in \mathbb{R}^d$, let $\varphi_s(a)$ denote the Gaussian density of $W(s)$ at a . From the properties of Gaussian densities, there exist some $K_4 = K_4(\eta, N, d)$ so that

$$\sup_{a \in \mathbb{R}^d} \sup_{s \in [\eta, \eta^{-1}]^N} \varphi_s(a) \leq K_4.$$

Hence, using (2.2), we see that there exists some $K_5 = K_5(\eta, d, N)$ such that

$$\mathbb{E}M(\varepsilon) \leq K_5 \varepsilon^{-N+(d/2)} (\ln(1/\varepsilon))^{d/2}.$$

Since $d > 2N$, (2.4) and hence the result follow. $\square$

§3. The Proof of Theorem (1.3). When $d < 2N$, Theorem (1.3) follows from Theorem (1.1). Suppose $d \geq 2N$. When $(d - 2N)k < d$, the existence of k -multiple points follows immediately from CHEN [C]. Equivalently, one can show (as we did for Theorem 1.1) that uniformly in $\varepsilon > 0$, $\varepsilon^{d(1-k)} \hat{\mu}_\varepsilon \in L^2(\mathbb{R}^d, d\xi)$, where $\mu_\varepsilon(A)$ is given by,

$$\int_{I_1} \cdots \int_{I_k} \mathbf{I}\{W(s^1) \in A\} \prod_{j=2}^k \mathbf{I}\{\|W(s^1) - W(s^j)\| \leq \varepsilon\} ds^1 \cdots ds^k,$$

and I_j is the box $[2j, 2j+1]^N$, $1 \leq j \leq k$. We will omit the details.

Suppose, next, that $(d - 2N)k > d$. Let $\eta \in (0, 1)$ be very small and fixed; also fix disjoint boxes $C_1, \dots, C_k$ such that $C_i \subset [\eta, \eta^{-1}]^N$, $1 \leq i \leq k$ and that if $i \neq j$, $d(C_i, C_j) \geq \eta$, where d denotes the usual Euclidean (that is, ℓ^2) distance on $\mathbb{R}^N$. It suffices to show the following:

$$(3.1) \quad \mathbb{P}(\forall 1 \leq j \leq k, \exists t^j \in C_j : W(t^1) = \cdots = W(t^k)) = 0.$$

Fix any such $\eta \in (0, 1)$ and $C_1, \dots, C_k \subset [\eta, \eta^{-1}]^N$. For any $\varepsilon > 0$ and $j \in \{1, \dots, k\}$, cover C_j with disjoint boxes $B_{i,j}(\varepsilon)$ of side ε , $1 \leq i \leq n_j(\varepsilon)$. Note that there exists some $K_6 = K_6(\eta, N)$ such that

$$(3.2) \quad \max_{j \leq k} n_j(\varepsilon) \leq K_6 \varepsilon^{-N}.$$

Define,

$$N_k(\varepsilon) \triangleq \sum_{i_1=1}^{n_1(\varepsilon)} \sum_{i_2=1}^{n_2(\varepsilon)} \cdots \sum_{i_k=1}^{n_k(\varepsilon)} \mathbf{I}\{\forall 1 \leq p \leq k, \exists t^p \in B_{i_p,p}(\varepsilon) : W(t^1) = \cdots = W(t^k)\}.$$

From (2.3), a little thought shows that for all ε small enough, $N_k(\varepsilon) \leq M_k(\varepsilon)$, where $M_k(\varepsilon)$ is given by

$$M_k(\varepsilon) \triangleq \sum_{i_1=1}^{n_1(\varepsilon)} \cdots \sum_{i_k=1}^{n_k(\varepsilon)} \mathbf{I} \{ \forall 1 \leq p \leq k : \sup_{\substack{s \in B_{i_1,1}(\varepsilon) \\ t \in B_{i_p,p}(\varepsilon)}} \|W(s) - W(t)\| \leq 2K_3 \sqrt{\varepsilon \ln(1/\varepsilon)} \}.$$

As in §2, Theorem (1.3) follows once we show the following:

$$(3.3) \quad \lim_{\varepsilon \rightarrow 0} \mathbb{E} M_k(\varepsilon) = 0.$$

Let $b_{i,j}(\varepsilon)$ denote the center of $B_{i,j}(\varepsilon)$, say. Note that $\mathbb{E} M_k(\varepsilon)$ is bounded above by

$$\sum_{i_1=1}^{n_1(\varepsilon)} \cdots \sum_{i_k=1}^{n_k(\varepsilon)} \mathbb{P} \left(\forall 1 \leq p \leq k : \|W(b_{i_1,1}(\varepsilon)) - W(b_{i_p,p}(\varepsilon))\| \leq 2K_3 \sqrt{2\varepsilon \ln(1/\varepsilon)} \right).$$

However, by the construction of $C_1, \dots, C_k$, we see that for any $1 < j \leq k$, conditional on $\{W(b_{i_{j-1},j-1}(\varepsilon)), \dots, W(b_{i_1,1}(\varepsilon))\}$, $W(b_{i_j,j}(\varepsilon))$ is a vector of independent normal random variables. Moreover, the (conditional) variance of any of the components of $W(b_{i_j,j}(\varepsilon))$ is bounded below by $K_7\eta$, for some $K_7 = K_7(N)$. By iteration, and since normal distributions are unimodal, the mode being at the mean, we see that

$$\begin{aligned} \mathbb{E} M_k(\varepsilon) &\leq K_8 \prod_{j=1}^k n_j(\varepsilon) \cdot \left(\varepsilon \ln(1/\varepsilon) \right)^{d(k-1)/2} \\ &\leq K_9 \varepsilon^{-kN+d(k-1)/2} \left(\ln(1/\varepsilon) \right)^{d(k-1)/2}, \end{aligned} \quad (3.4)$$

by (3.2). Here, $K_8 = K_8(\eta, d)$ and $K_9 \triangleq K_8 \cdot K_6^k$. Recall that we have $(d-2N)k > d$. Equivalently, we have $d(k-1) > 2Nk$. From (3.4) we obtain (3.3) and hence the result. $\square$

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