Séminaire de probabilités (Strasbourg)

Michel Émery

Closed sets supporting a continuous divergent martingale

Séminaire de probabilités (Strasbourg), tome 31 (1997), p. 176-189

<http://www.numdam.org/item?id=SPS_1997__31__176_0>

CLOSED SETS SUPPORTING A CONTINUOUS DIVERGENT MARTINGALE

by M. Émery[1]

Let $(X_t)_{t\geqslant 0}$ be a continuous martingale with values in a finite-dimensional affine space E. We shall call X *divergent* if almost surely $\lim_{t\to\infty} X_t$ does not exist in E. For which subsets F of E does there exist in E a divergent, continuous martingale with values in F? Unable to answer this question in general, we shall restrict ourselves to the case when F is closed; this note is devoted to giving a non-probabilistic characterization of the closed subsets of E that contain a divergent, continuous martingale.

When $\dim E = 1$, no strict subset of E can contain a divergent, continuous martingale, but E itself does; this case is trivial and the problem is interesting for $\dim E \geqslant 2$ only (if at all!).

As we are interested in continuous martingales only, the adjective 'continuous' will be omitted and all martingales will be implicitely assumed continuous. By time-change, considering continuous local martingales instead of martingales would make no difference.

Our statements will involve only the affine structure on E (and the associated topology); but in some proofs, E will be endowed with an additional Euclidean structure: the distance will be denoted by d, the open balls will be called $B(x,r)$, the closed ones $\overline{B}(x,r)$, the spheres $S(x,r)$, orthogonality will be used, etc.

1. Prominent points and humpless kernel of a closed set

If X is a topological space and if $A \subset B \subset X$, $\mathrm{i}_B A$ and $\partial_B A$ will respectively denote the interior and the boundary of A in the topological space B (endowed with the topology inherited from X, of course). One always has $(\mathring{A} =)\, \mathrm{i}_X A \subset \mathrm{i}_B A$, for $\mathrm{i}_X A$ is an open subset of X included in A, hence also an open subset of B included in A. The reverse inclusion may fail (for instance when $A = B$ and A is not open).

LEMMA 1. — *Let A, B and C be three subsets of a topological space.*

a) *If $A \subset B$, one has $\partial_A(A\cap C) \subset \partial_B(B\cap C)$.*

b) *If $A \subset B\cap C$ and if A is both open and closed in $B\cap C$, then $\partial_B A \subset \partial C$.*

PROOF. — a) Let $x \in \partial_A(A\cap C)$. If V is a neighbourhood of x in B, $V\cap A$ is a neighbourhood of x in A and must meet $A\cap C$ and $A\cap C^{\mathrm{c}}$; a fortiori, V itself meets $B\cap C$ and $B\cap C^{\mathrm{c}}$. As V is arbitrary, x is in $\partial_B(B\cap C)$.

---

1. This note originates from enjoyable conversations with Chris Burdzy.

b) Calling X the ambient topological space, a) yields

$$\partial_B(B \cap C) \subset \partial_X(X \cap C) = \partial C \,;$$

so it suffices to verify that $\partial_B A \subset \partial_B(B \cap C)$. Setting $D = B \cap C$ and taking B as the new ambient topological space, it now suffices to verify that *if A is a closed and open subset of D, then $\partial A \subset \partial D$.*

Let $x \in \partial A$. We must show that any neighbourhood V of x meets both D and D^c. We already know that it meets A, hence also D; it remains to see that it meets D^c. Consider first the case when $x \in A$. Write $A = D \cap O$ with O open; $V \cap O$ is a neighbourhood of x, so it meets $A^c = D^c \cup O^c$, hence also D^c, and we are done. Now the other case: $x \notin A$. Write A as $D \cap F$ with F closed; from $A \subset F$ one gets $\partial A \subset F$ and $x \in F$; since x does not belong to $A = D \cap F$, it must be in D^c, and V meets D^c at point x. ∎

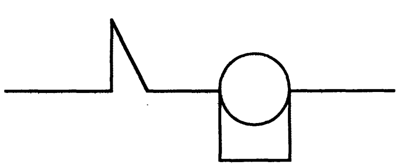
DEFINITIONS. — *Let F be a closed subset of the affine space E and x a point of F. One says that x is* a prominent point of F *if there exist an affine hyperplane H in E not containing x and a compact K included in F, containing x and with boundary $\partial_F K$ included in H.*

The set of all points of F that are not prominent points of F will be called the non-prominence *of F and abbreviated* $\mathrm{np}(F)$.

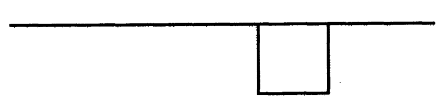
Very roughly, x is a prominent point of F if a plane blade can cut off a bounded part of F containing x.

We shall see below (Lemma 5) a seemingly stronger but equivalent definition of the prominent points of a closed set: generality is not restricted by demanding, in the above definition, that the compact K be also open in the closed set $F \cap D$, where D is the closed half-space with boundary H and containing x.

The figure below shows a closed set F_1 in the plane (in gray; it consists of a half-plane, minus a square, plus a disk and a triangle) and its non-prominence $\mathrm{np}(F_1)$. The prominent points of F_1 are the points of the triangle minus its base and the points of the disk minus its horizontal diameter. This can be checked directly from the definitions, or, more easily, by using Proposition 4 a) below. Notice on this exemple that the requirement $H \not\ni x$ in the definition of prominent points cannot be weakened to $x \notin \partial_F K$: the points of the horizontal diameter of the disk are not prominent, but would become so after this modification (take $K =$ the disk, so that $\partial_{F_1} K$ consists of two points).



F_1



$\mathrm{np}(F_1)$

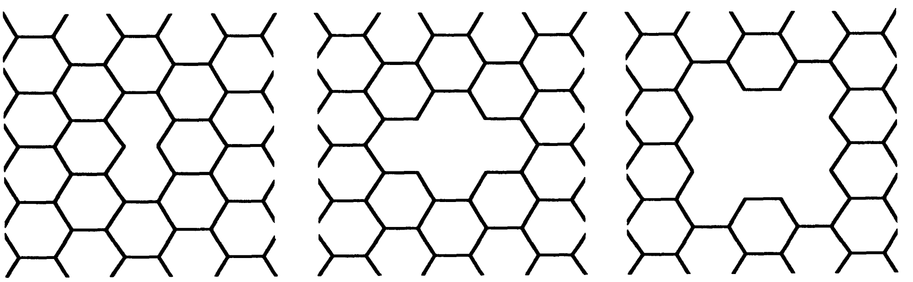
If F is compact, all its points are prominent and $\mathrm{np}(F)$ is empty (take $K = F$, so the boundary $\partial_F K$ is empty, and call H any hyperplane not meeting F). More generally, for the same reason, every compact open in F (for instance, every connected component of F that is isolated in F and bounded) consists of prominent points of F.

REMARK. — The definition of prominent points seems to involve the reference space E, via the constraint that H must be a hyperplane. But actually it does not: if E' is an affine sub-space of E and F a closed subset of E', the prominent points of F are the same, whether defined with respect to E or E'. Indeed, every hyperplane H of E' is the trace on E' of some hyperplane H of E; conversely, if a hyperplane H of E does not contain a given point of E' (here, the prominent point), $H \cap E'$ is either empty or a hyperplane of E'.

LEMMA 2. — *Let F be a closed subset of E. The set* $\mathrm{np}(F)$ *is closed too.*

PROOF. — Let x be a prominent point of F; there exist a hyperplane H and a compact K such that $x \notin H$, $x \in K \subset F$ and $\partial_F K \subset H$. One has $x \notin \partial_F K$, whence $x \in \mathrm{i}_F K$. All points of F close enough to x are also in the open subset $H^{\mathrm{c}} \cap \mathrm{i}_F K$ of F, hence they are also prominent. So the set of all prominent points of F is open in F, and $\mathrm{np}(F)$ is closed in F, hence also closed in E. ∎

F_1 in the preceding picture is such that $\mathrm{np}(F_1)$ has no prominent point. This is not a general rule: the figure below shows a closed set F_2 made of all the edges of an infinite hexagonal lattice except one (call this one e); $\mathrm{np}(F_2)$ consists of all the edges except the neighbours of e, and the set of all prominent points of $\mathrm{np}(F_2)$ is the union of all the edges that are second-order neighbours of e.



F_2 $\mathrm{np}(F_2)$ $\mathrm{np}(\mathrm{np}(F_2))$

LEMMA 3. — *Let F and G be two closed subsets of E such that $F \subset G$. Every point of F prominent for G is also prominent for F, and* $\mathrm{np}(F) \subset \mathrm{np}(G)$.

PROOF. — Let x be a point of F prominent for G. There exist a compact K such that $x \in K \subset G$ and a hyperplane H of E such that $x \notin H$ and $\partial_G K \subset H$. The set $F \cap K$ is a compact of F containing x; according to Lemma 1 a), its boundary verifies $\partial_F(F \cap K) \subset \partial_G(G \cap K) = \partial_G K \subset H$. Hence x is a prominent point of F. The first statement is proved, the inclusion follows. ∎

Definition. — *A closed subset of E is* humpless *if it has no prominent point.*

Lemme 4. — *Let $\mathcal{F}$ be a set of humpless closed subsets of E and $G = \bigcup_{F \in \mathcal{F}} F$ the union of this set. The closure $\overline{G}$ of G is humpless.*

Proof. — The prominent points of $\overline{G}$ form an open subset of $\overline{G}$ (Lemma 2). If this open set were not empty, it would meet G, for G is dense in $\overline{G}$; hence there would exist some $F \in \mathcal{F}$ containing some prominent point x of $\overline{G}$; x would also be a prominent point of F (Lemma 3); this would contradict the humplessness of F. So $\overline{G}$ has no prominent point. ∎

Proposition 1 and definition. — *Let F be a closed subset of E. All humpless closed subsets of F are included in one of them.*

This biggest humpless closed subset of F will be called the humpless kernel *of F and denoted by $\check{F}$.*

Proof. — Apply Lemma 4 to the closure of the union of all humpless closed subsets of F. ∎

Proposition 1 can also be proved by transfinite induction, using only the fact that the mapping np from the set of all closed subsets of E to itself is a *derivation* (that is, it is increasing, and it verifies $\mathrm{np}(F) \subset F$ for all F). This makes it easy to construct, for every ordinal α, the transfinite iterate $\mathrm{np}^{\alpha}(F)$; the so-obtained transfinite sequence is decreasing, hence stationary, and it is not difficult to see that its limit is the biggest fixed point of np included in F.

Such a transfinite induction is not necessary here, and ordinary induction will suffice: Proposition 2 will show that the limit is reached at or before the first infinite ordinal.

Remark. — One has always $\check{F} \subset \mathrm{np}(F)$. In other words, no prominent point of F can belong to the humpless kernel $\check{F}$; indeed, according to Lemma 3, such a point should be prominent for $\check{F}$ too, but this is impossible since $\check{F}$ is humpless. Hence the humpless kernel is made of non prominent points only.

But the reverse inclusion is false: F may have points that are neither prominent, nor in the humpless kernel $\check{F}$. Consider for instance F_2 drawn on the preceding page; the iterate $\mathrm{np}^n(F_2)$ is obtained by deleting from F_2 all the edges that are neighbours of order $\leqslant n$ of the edge e; this can be checked by induction using Proposition 4 b). Consequently, the humpless kernel $\check{F}_2$, included in each iterate $\mathrm{np}^n(F_2)$, is empty, though $\mathrm{np}(F_2)$ is not.

Lemma 5. — *Let x be a prominent point of a closed set F. There exist a closed half-space D such that $x \in \mathring{D}$, and a compact open subset K of $F \cap D$, such that $x \in K$ and $\partial_F K \subset \partial D$.*

Proof. — By hypothesis there exist a hyperplane $H \not\ni x$ and a compact L such that $x \in L \subset F$ and $\partial_F L \subset H$. Let H' be a hyperplane parallel to H and separating x and H; call D the closed half-space with boundary H' and containing x. $K = L \cap D$ is a compact containing x. Lemma 1 a) gives $\partial_{F\cap D} K = \partial_{F\cap D}\big((F\cap D)\cap L\big) \subset \partial_F(F\cap L) = \partial_F L \subset H$; but $\partial_{F\cap D} K$ is a subset of $F\cap D$, hence also of D, which does not meet H. Consequently, $\partial_{F\cap D} K = \varnothing$, and the compact K of $F\cap D$ is also open in $F\cap D$. Last, inclusion $\partial_F K \subset \partial D$ follows immediately from Lemma 1 b). ∎

LEMMA 6. — *Let* $(F_n)_{n\in\mathbb{N}}$ *be a decreasing sequence of closed subsets of* E*; call* F_∞ *the limit of this sequence.*

a) *If* K_∞ *is a compact open subset of* F_∞*, then for every* n *large enough there exists a compact open subset* K_n *of* F_n *such that* $K_n \supset K_\infty$.

b) *Each prominent point of* F_∞ *is a prominent point of all* F_n*'s but finitely many.*

c) *The closed sets* $\mathrm{np}(F_n)$ *form a decreasing sequence with limit* $\mathrm{np}(F_\infty)$.

PROOF. — a) There exists an open subset U of E such that $K_\infty = F_\infty \cap U$; there exists a compact L such that $K_\infty \subset \overset{\circ}{L} \subset L \subset U$. The compacts $F_n \cap \partial L$ are decreasing with limit $F_\infty \cap \partial L = F_\infty \cap (U \cap \partial L) = (F_\infty \cap U) \cap \partial L = K_\infty \cap \partial L = \varnothing$; so, for n large enough, $F_n \cap \partial L$ is empty and $F_n \cap L = F_n \cap \overset{\circ}{L}$ is a compact and open subset of F_n containing K_∞.

b) Let x be a prominent point of F_∞. Lemma 5 gives a closed half-space D such that $x \in \overset{\circ}{D}$ and a compact K open in $F_\infty \cap D$ such that $x \in K$. Apply a) to the decreasing closed sets $F_n \cap D$ with limit $F_\infty \cap D$: for n large enough, there is a compact K_n open in $F_n \cap D$ and containing K, hence also x; Lemma 1 b) yields the inclusion $\partial_{F_n} K_n \subset \partial D$, showing that x is a prominent point of these F_n's.

c) Inclusions $\mathrm{np}(F_{n+1}) \subset \mathrm{np}(F_n)$ and $\mathrm{np}(F_\infty) \subset \bigcap_n \mathrm{np}(F_n)$ are straightforward from Lemma 3. Conversely, a point belonging to all the $\mathrm{np}(F_n)$'s is in each F_n hence in F_∞; but it cannot be prominent in F_∞ because of b); so it belongs to $\mathrm{np}(F_\infty)$. ∎

PROPOSITION 2. — *Let* F *be closed in* E*. The decreasing sequence of iterates* $\mathrm{np}^n(F)$ *of* F *converges to the humpless kernel* $\check{F}$ *of* F.

PROOF. — Decreasingness comes from Lemma 3; call F_∞ the limit $\bigcap_n \mathrm{np}^n(F)$. Applied to $F_n = \mathrm{np}^n(F)$, Lemma 6 c) entails that F_{n+1} tends to $\mathrm{np}(F_\infty)$, whence $\mathrm{np}(F_\infty) = F_\infty$ and F_∞ is a humpless closed set contained in F. If G is any humpless closed set included in F, Lemma 3 implies $\mathrm{np}^n(G) \subset \mathrm{np}^n(F)$, that is, $G \subset \mathrm{np}^n(F)$ since G is a fixed point of np. As a result, $G \subset F_\infty$; this shows that F_∞ is the biggest humpless closed set contained in F, that is, $F_\infty = \check{F}$. ∎

2. The case when E is a plane

The case when $\dim E = 1$ is trivial: the only humpless closed sets are E and $\varnothing$ and one has $\mathrm{np}(F) = \check{F} = \varnothing$ if $F \neq E$ and $\mathrm{np}(E) = \check{E} = E$. The simplest non-trivial examples occur when $\dim E = 2$; this section is devoted to describing the non-prominence and the humpless kernel of a planar closed set. But some statements extend to higher dimensions as well; so when we assume E is a plane, we shall mention it explicitely.

Summarized in Proposition 4, the results are quite intuitive; scribbling a few sketchy pictures will convince you much more pleasantly than reading the pedestrian but tedious proofs given below. Experts have shown me how some homological considerations could have saved paper and ink; but I prefer walking the way rather than taking readers in a jet I can hardly pilot. In any case, this section will not be used in the sequel.

In two dimensions, the mapping $F \mapsto \mathrm{np}(F)$ becomes easier to describe when passing to complementaries and dealing with open sets instead of closed ones. If O is open in E, we shall call $\tilde{\mathrm{np}}(O)$ the complementary of the closed set $\mathrm{np}(O^c)$. Lemmas 2 and 3 and Propositions 1 and 2 say that $\tilde{\mathrm{np}}(O)$ is open and contains O, that $\tilde{\mathrm{np}}$ is an increasing mapping from the set of open subsets of E to itself, and that the sequence of iterates $\tilde{\mathrm{np}}^n(O)$ is increasing, with limit the complementary of the humpless kernel of the closed set O^c.

LEMMA 7. — *Let D be a closed half-plane, K a compact in D, V a neighbourhood of K in D and x a point in K. There exists in $V \setminus K$ a continuous curve with endpoints y and z such that x belongs to the segment $[y, z]$.*

PROOF. — Call D' the closed half-plane included in D and whose boundary $\partial D'$ contains x; by replacing D by D', K by $K \cap D'$ and V by $V \cap D'$, we may suppose that x belongs to the boundary $\Delta = \partial D$ of the half-plane D. Without loss of generality, we shall also suppose V open in D and $V \neq D$.

Let $a > 0$ be the distance from the compact K to the closed set $D \setminus V$. Cover K with a finite family $\big(B(c_i, \frac{1}{2}a), i \in I\big)$ of open disks with the same radius $\frac{1}{2}a$ and with centres c_i in K. Choose a number $b \in (\frac{1}{2}a, a)$ meeting the following requirements: any circle $C_i = \partial B(c_i, b)$ with centre c_i and radius b is not tangent to Δ, any two of these circles are not tangent, any three of them have empty intersection, and no intersection point of two of them is on Δ. (This is possible because only finitely many values of b are forbidden, namely the distances $d(c_i, \Delta)$, the half-distances $\frac{1}{2}\, d(c_i, c_j)$, the outradii of the triangles $c_i c_j c_k$, and the distances from c_i to the intersections of Δ with the perpendicular bissectors of the segments $c_i c_j$.) Call F_i the closed disk $\overline{B}(c_i, b)$; its boundary is C_i. The compact $L = \bigcup_i F_i$ verifies $K \subset \bigcup_i B(c_i, b) \subset \mathring{L}$ and $L \cap D \subset V$, hence also $\partial L \cap D \subset V \setminus K$. To prove the lemma, we shall construct in $\partial L \cap D$ a continuous curve whose endpoints are on Δ and encompass x.

Orient the plane, thus defining a counter-clockwise direction on each circle. Orient also the line Δ, in such a way that if a circle C meets Δ at two points y and z, and if y is before z on Δ, z is before y on the arc $C \cap D$ with endpoints y and z.

The intersection $\Delta \cap L = \bigcup_i \big(\Delta \cap F_i\big)$ is a finite union of segments with strictly positive lengths, hence also a finite union of disjoint segments with strictly positive lengths; call $s_\alpha = [y_\alpha, z_\alpha]$ these disjoint segments, where α ranges over a finite set A and where y_α is before z_α on the oriented line Δ. Let Y be the set $\{y_\alpha, \alpha \in A\}$ of all left-endpoints of these segments and Z the set $\{z_\alpha, \alpha \in A\}$ of all right-endpoints of these segments. Point x belongs to one of the segments s_α; on Δ, there are before x more points of Y than of Z (exactly one more). We shall construct a one-to-one correspondence between Y and Z, such that any two corresponding points can always be linked by a continuous curve lying in $\partial L \cap D$. At least one point of Y before x will be linked to some point of Z after x, thus proving the lemma.

Remark first that each point of ∂L is in L, hence in one of the closed disks F_i; and it belongs to the closure of the exterior of L and a fortiori to the closure of the exterior of the disk F_i; so it must be on the boundary C_i and this gives $\partial L \subset \bigcup_i C_i$.

Start at a point $z \in Z$. It belongs to ∂L, hence to some C_i, unique owing to the conditions on b. Follow this C_i counter-clockwise until meeting the line Δ or another circle C_j. If Δ is met first, stop; if some C_j (unique owing to b) is met first, leave C_i and follow C_j counter-clockwise until meeting Δ or some C_k ($k \neq j$, but k can be equal to i). If on Δ, stop, else switch to the new circle. Keep doing this as long as possible, that is, indefinitely or until meeting Δ.

To conclude, it suffices to show that, *starting from* $z \in Z$, *the line* Δ *is reached after finitely many steps, at some point* $y \in Y$; that *the path followed from* z *to* y *lies in* $\partial L \cap D$; and that *the so-defined mapping from* Z *to* Y *is one-to-one and onto.*

Consider the reverse algorithm, analogously defined, but with 'clockwise' instead of 'counter-clockwise'. Applied after starting with the direct one, the reverse algorithm follows backwards the same path; this shows that two paths obtained with the direct algorithm cannot merge (that is, coincide after some step without having coincided in the past). In particular, starting from $z \in Z$, it is impossible to pass twice the same point, for the past of the second time should be the same as the past of the first time, and the starting point z should have been met between both times; but meeting Δ terminates the algorithm. As the total number of arcs at our disposal is finite, Δ must be met after finitely many steps, and the algorithm eventually stops.

The starting point z belongs to ∂L; it is on C_i but not in any of the closed disks F_j with $j \neq i$ (it is not on the boundary of those disks because of the condition on b; nor in their interior since it is on ∂L). The first step of the path, until another circle is met, remains in the exterior of all the F_j's, hence on the boundary ∂L; it is also in the half-plane D because of the choice of the orientations. It can be checked inductively that, at each step of the algorithm, the arc of some circle C_k used lies in the exterior of all the other circles C_m, $m \neq k$: indeed, when passing from some circle C_k to another circle C_ℓ, since C_ℓ is reached from the exterior and both motions are counter-clockwise, the path will leave C_k outwards. So the algorithm never leaves ∂L. Similarly, when eventually reaching Δ, the path is counter-clockwise following some C_p while remaining in the exterior of all the other circles. Hence, the point where Δ is met is a point of $\partial L \cap \Delta$ having a right-neighbourhood $F_p \cap \Delta$ included in L; so this meeting point must belong to Y.

This defines a mapping from Z to Y. To see that it is one-to-one and onto, it suffices to exhibit its inverse. The latter is obtained by applying the reverse algorithm starting from the points of Y: by the same argument, one eventually reaches Δ after following the same path backwards. ∎

LEMMA 8. — *Let* O *be a connected open subset of* E. *Any two points of* O *can be linked by a simple curve in* O (that is, with no multiple points).

PROOF. — It suffices to verify that if x is any point of O, the set of endpoints of all simple curves in O started at x is both open and closed in O. Since O is locally convex, it suffices to show that if x, y and z are three points, if c is a simple curve from x to y and if s is the segment $[y, z]$, there exists a simple curve from x to z included in $c \cup s$. Calling t the point of $s \cap c$ closest to z, one gets the required curve by chaining together the (unique) part of c linking x to t and the segment (possibly a singleton) $[t, z]$. ∎

DEFINITION. — *If* A *is a subset of* E, *the union of all segments* $[u, v]$, *where* u *and* v *range over* A, *will be called the* segment-span *of* A.

Clearly, the segment-span of A is included in any convex set containing A, in particular in the convex hull of A. The converse is false in general, but holds in dimension 1 (immediate) and, when A has at most 2 connected components, also in dimension 2. This is a theorem of Fenchel; for references and generalizations, see O. Hanner & H. Rådström [4]. We shall need only the particular case when A is connected:

LEMMA 9. — *Let A be a connected subset of a plane. The convex hull of A is equal to the segment-span of A.*

PROOF. — It suffices to see that the segment-span S of A contains the convex hull of A. Taking $x \in S^c$, we have to show that x is not in the convex hull of A.

Remark first that $S \supset A$ (a point is a segment), so x does not belong to A. Let E be the plane, Γ a circle with centre x and f the mapping from $E \setminus \{x\}$ to Γ such that, for every $y \neq x$, the points y and $f(y)$ are on the same ray emanating from x. Since f is continuous and A connected, the range $f(A)$ is a connected subset of Γ, hence an arc $a \subset \Gamma$. Hypothesis $x \notin S$ entails that this arc never contains both endpoints of a diameter of Γ; so it is either an arc with measure less than π, or a non-closed arc with measure π. In either case, the set $f^{-1}(a) \subset E \setminus \{x\}$ is a convex part of E, containing A, but not x. This prevents the convex hull of A from containing x. ∎

PROPOSITION 3. — *Suppose* $\dim E \geqslant 2$; *let O be open and F be closed in E.*

a) *Every prominent point of F is in the segment-span of some connected component of the open set F^c.*

b) *The open set $\widetilde{\text{np}}(O)$ is included in the union of the segment-spans of the connected components of O.*

c) *If each connected component of F^c is convex, F is humpless.*

This proposition holds a fortiori if 'segment-span' is replaced by 'convex hull'.

When $\dim E \geqslant 3$, the converse statements to a), b) and c) are false: consider the case when F is a line in E (it is a humpless closed set) and O the complementary of a line (this connected open set segment-spans the whole space).

PROOF OF PROPOSITION 3. — a) Given a prominent point x of F, we have to find in the same connected component of F^c two points y and z such that the segment $[y, z]$ contains x.

By Lemma 5, there exist a closed half-space D' and a compact and open subset K' of $F \cap D'$ such that $x \in K'$ and $\partial_F K' \subset \partial D'$. There exists a V' open in D' such that $K' = F \cap V'$. Let Δ be a line containing x and parallel to the hyperplane $\partial D'$ (this is where the hypothesis $\dim E \geqslant 2$ comes in) and let P be the 2-plane perpendicular to $\partial D'$ and containing Δ. Call D the half-plane $D' \cap P$, K the compact $K' \cap D$ and V the set $V' \cap D$. Since $x \in K$ and V is open in D and contains K, Lemma 7 applies and gives two points y and z of $V \setminus K$ linked by a continuous curve in $V \setminus K$ and such that the segment $[y, z]$ contains x. But $V \setminus K$ is included in $V' \setminus K'$ and hence in F^c; the points y and z, linked by a continuous curve in F^c, are in the same connected component of F^c, and we are done.

b) Observe that the points of $\widetilde{\text{np}}(O)$ are the points of O and the prominent points of O^c and apply a) to $F = O^c$.

c) If the connected components of F^c are convex, each of these components is its own segment-span, and F is humpless according to a). ∎

In two dimensions, the converse statements to a), b) and c) hold true, and prominency and humplessness can be more expressively rephrased.

In the next statement, 'convex hull' can as well be replaced with 'segment-span' since, according to Lemma 9, they are equivalent for planar connected sets.

PROPOSITION 4. — *Suppose* $\dim E = 2$*; let* O *be open and* F *be closed in* E.

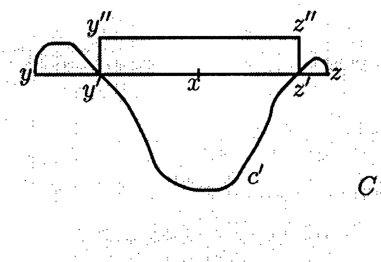
a) *A point* $x \in F$ *is prominent if and only if it belongs to the convex hull of some connected component of the open set* F^{c}.

b) *The open set* $\check{\mathrm{np}}(O)$ *is the union of the convex hulls of the connected components of* O.

c) *The closed set* F *is humpless if and only if each connected component of* F^{c} *is convex.*

PROOF. — a) The necessary condition has been seen in Proposition 3 a); now for the converse. Supposing x in F and in the convex hull of a connected component C of F^{c}, we shall show it is prominent.

Lemma 9 yields two points y and z of C such that $x \in [y,z]$; Lemma 8 gives the existence of a simple curve c linking y and z in C. Let y' (respectively z') the point of $c \cap [x,y]$ (respectively $c \cap [x,z]$) closest to x and c' the piece of c linking y' and z'. The union $\gamma = c' \cup [y',z']$ is a simple closed curve; by Jordan's theorem, γ^{c} has two connected components, one of which (call it J) is bounded and verifies $\partial J = \gamma$. Since x is in F, it is not in C and a fortiori not on c'. In a neighbourhood of x, γ is straight, and the oriented normal line to γ at x can be defined: let $\vec{v}$ be a normal vector to $[y',z']$, going from J towards J^{c}, and of length $r > 0$ small enough for the closed disks $\overline{B}(y',r)$ and $\overline{B}(z',r)$ to be included in C. The translation by $\vec{v}$ transforms y' and z' into y'' and z''; let R be the compact convex rectangle with vertices y', z', z'' and y'', L the compact $\bar{J} \cup R$ and K the compact $L \cap F$.



As $x \in K$, to establish that x is prominent, it suffices to show that the boundary $\partial_F K$ is included in the segment $[y'',z'']$. Lemma 1 a) gives

$$\partial_F K = \partial_F(L \cap F) \subset \partial_E(L \cap E) = \partial L = \partial(R \cup \bar{J})$$
$$\subset \partial R \cup \partial \bar{J} = \partial R \cup \gamma = \partial R \cup c' \cup [y',z'] = \partial R \cup c' \ ;$$

but $\partial_F K$ is also included in F, wherefrom

$$\partial_F K \subset (\partial R \cap F) \cup (c' \cap F) = \partial R \cap F \, .$$

Remarking that ∂R consists of four segments, two of which, $[y',y'']$ and $[z',z'']$, are in C, transforms the above inclusion into

$$\partial_F K \subset (y'',z'') \cup \big((y',z') \cap F\big) \, .$$

Now, for every point t of the open interval (y',z'), the segment $[y',z']$ splits a small disk centred at t into two half-disks, one included in $\bar{J}$ and the other one in R; hence L is a neighbourhood of t. Consequently, K is a neighbourhood in F of each point of $(y',z') \cap F$, and those points cannot be on $\partial_K F$; finally $\partial_F K \subset (y'',z'')$ and x is prominent.

b) Apply a) to $F = O^c$, and notice that the points of $\mathring{\text{np}}(O)$ are the points of O and the prominent points of F.

c) If the connected components of F^c are convex, F is humpless by Proposition 3 c).

If F is humpless, let C be a connected component of F^c. By a), no point of F can belong to the convex hull $\hat{C}$ of C, so $\hat{C}$ is included in F^c. Hence $\hat{C}$ is a connected part of F^c containing C, so $\hat{C} = C$, and C is convex. ∎

3. Martingales, at last!

Prominence and humplessness will be used to describe the closed sets of E that contain a divergent martingale (that is, almost surely not convergent in E when t tends to infinity; recall that we consider only continuous martingales).

The Euclidean structure (balls, distance, etc.) already used several times is also able to measure the length of a curve and its analogue for a martingale: The *Euclidean quadratic variation* of a martingale X in E is the increasing process

$$\langle X, X\rangle_t = \lim_n \textstyle\sum_k d\big(X_{t\wedge k2^{-n}}, X_{t\wedge(k+1)2^{-n}}\big)^2$$

where the limit is in probability; equivalently, it is also the sum $\sum_i \langle X^i, X^i\rangle_t$, where the real martingales X^i are the coordinates of X in an orthonormal affine frame. Recall the equivalence, valid for almost all ω,

$$\lim_{t\to\infty} X_t(\omega) \text{ exists in } E \qquad \Longleftrightarrow \qquad \langle X, X\rangle_\infty(\omega) < \infty\ .$$

LEMMA 10. — *Let x be a prominent point of a closed subset F of E. There exist a number $\alpha > 0$ and a set U open in F such that $x \in U \subset F \setminus \text{np}(F)$ and that, for every F-valued martingale X verifying $X_0 \in U$,*

$$\mathbf{P}\big[\lim_{t\to\infty} X_t \text{ exists in } E\big] \geqslant \alpha\ .$$

PROOF. — There exist a hyperplane H and a compact K such that $x \in K \subset F$, $\partial_F K \subset H$ and $x \notin H$. Let ℓ denote the affine function on E vanishing on H and such that $\ell(x) = 2$; the number $a = \sup_K \ell$ verifies $2 \leqslant a < \infty$. As x is not in $\partial_F K$, it lies in $\text{i}_F K$, and the set $U = \{\ell > 1\} \cap \text{i}_F K$ is open in F and verifies $x \in U \subset K$ and $\ell > 1$ on U. The properties of H and K imply that each point of U is a prominent point of F. If X is an F-valued martingale such that $X_0 \in U$, call T the stopping time $\inf\{t : X_t \in \partial_F K\}$. On $\{T < \infty\}$, $X_T \in \partial_F K \subset H$ and $\ell(X_T) = 0$; on the interval $[\![0, T]\!]$, X is in K for, in F, no continuous curve starting in U can leave K without meeting the boundary $\partial_F K$. Hence, the stopped process $X^{|T}$ is a bounded martingale, and the real process $M = \ell(X^{|T})$ is a bounded real martingale verifying $M_0 > 1$, $M \leqslant a$ and $M_\infty = 0$ on $\{T < \infty\}$. Consequently,

$$a\,\mathbf{P}[T = \infty] \geqslant \mathbf{E}[M_\infty \mathbb{1}_{\{T=\infty\}}] = \mathbf{E}[M_\infty] = \mathbf{E}[M_0] > 1\ ,$$

wherefrom $\mathbf{P}[T = \infty] > 1/a$. But on the event $\{T = \infty\}$ the paths of X are in K, hence bounded, hence convergent; so, $\mathbf{P}[X \text{ converges}] > \alpha = 1/a$. ∎

LEMMA 11. — *Let F be closed in E and X be an F-valued divergent martingale. The subset $\{X \notin \mathrm{np}(F)\}$ of $\mathbf{R}_+ \times \Omega$ is evanescent.*

In other words, an F-valued divergent martingale lives in fact in the smaller closed set $\mathrm{np}(F)$.

PROOF. — As the set $O = F \backslash \mathrm{np}(F)$ is open in F, it is a countable union of compacts, and every open covering of O in the topological space F contains a countable sub-covering. Now Lemma 10, applied to each point of O, gives a covering of O by open sets $U_x \subset O$, each of them associated with a number $\alpha_x > 0$. Hence there exist a sequence $(V_n)_{n\in\mathbf{N}}$ of open sets of F and a sequence $(\beta_n)_{n\in\mathbf{N}}$ in $(0,\infty)$ such that $\bigcup_n V_n = O$ and that, for all n and all F-valued martingale X verifying $X_0 \in V_n$, the minoration $\mathbf{P}[X \text{ converges}] \geqslant \beta_n$ holds.

Let X be a martingale in F such that the optional set $\{X \notin \mathrm{np}(F)\}$ is not evanescent. This set contains the graph of some stopping time (section theorem); so there are a stopping time T and an n such that the event $\Omega' = \{T < \infty\} \cap \{X_T \in V_n\}$ verifies $\mathbf{P}[\Omega'] > 0$. Applying the above minoration to the martingale $X'_t = X_{T+t}$ (defined on Ω' with the filtration $\mathcal{F}'_t = \mathcal{F}_{T+t}$ and the probability $\mathbf{P}'[A] = \mathbf{P}[A \,|\, \Omega']$) yields $\mathbf{P}'[X \text{ converges}] \geqslant \beta_n$, whence $\mathbf{P}[X \text{ converges}] \geqslant \beta_n \mathbf{P}[\Omega'] > 0$, and X is not divergent. ∎

PROPOSITION 5. — *Let F be closed in E. Every F-valued divergent martingale takes its values in the humpless kernel $\check{F}$ of F.*

PROOF. — Let X be such a martingale. By induction on n, Lemma 11 shows that each set $\{X \notin \mathrm{np}^n(F)\}$ is evanescent. So is also the union of these sets; now, according to Proposition 2, this union is nothing but $\{X \notin \check{F}\}$. ∎

PROPOSITION 6. — *Let F be a humpless closed set of E and x a point of F. There exists an F-valued divergent martingale X such that $X_0 = x$.*

PROOF. — *Step one.* Given any $a > 0$, we shall construct a Markov kernel N in F (endowed with its Borel σ-field) such that for every $y \in F$, the probability $\varepsilon_y N$ has mass centre y and is carried by the compact $\{z \in F : d(y,z) = a\} = F \cap S(y,a)$.

For $y \in F$, let L_y denote the compact $F \cap S(y,a)$; we shall first show that y is in the convex hull C_y of L_y. If it were false, y and L_y would be separated by a hyperplane H : there would exist a closed half-space D with boundary H such that $y \in \mathring{D}$ and $L_y \subset D^c$. The intersection $F \cap D \cap S(y,a)$ would be empty and the compact $K = F \cap D \cap \overline{B}(y,a)$ would also be equal to $F \cap D \cap B(y,a)$; it would be both closed and open in $F \cap D$. Lemma 1 b) would give $\partial_F K \subset \partial D = H$ and y would be prominent in F. As F is humpless, this is impossible.

Since $y \in C_y$, y is by Carathéodory's theorem the mass centre of a probability N_y carried by $r+1$ points of L_y, where $r = \dim E$. To conclude step one, it suffices to verify that N_y can be chosen mesurable in y. The set of all systems $(y; z_0, \dots, z_r; \lambda_0, \dots, \lambda_r)$ verifying $z_i \in L_y$, $0 \leqslant \lambda_i \leqslant 1$, $\sum_{i=0}^r \lambda_i = 1$ and $\sum_{i=0}^r \lambda_i z_i = y$ is closed in $F^{r+2} \times [0,1]^{r+1}$, with non-empty y-sections; so it has a Borel section $y \mapsto (z_0(y), \dots, z_r(y); \lambda_0(y), \dots, \lambda_r(y))$ (see for instance Dellacherie [3], page 350). Defining N_y as $\sum_{i=0}^r \lambda_i(y) \varepsilon_{z_i(y)}$ gives the claimed kernel.

Step two. Given $a > 0$, we shall construct a martingale X^a starting at x, with values in the closed set $F^a = \{y \in E : d(y,F) \leqslant a\}$ and with Euclidean quadratic variation $\langle X^a, X^a \rangle_t \equiv t$.

Recall that to each centred probability μ on a vector space V is associated a V-valued *Walsh martingale* (unique in law): denoting by R the absolute value of a real Brownian motion started at the origin, the Walsh martingale is obtained by independently multiplying each excursion of R by a random vector in V with law μ; see for instance [2] for more details. This process is a martingale because μ is centred.

If now y is a point in the affine space E and ν a probability on E centred at y, one can similarly define "the" E-valued Walsh martingale W started at y and associated to ν; if T is the first time when R hits 1, the random variable W_T has law ν.

Take $\nu = \varepsilon_x N$, where x is the given point and N the kernel constructed in step one; since $\varepsilon_x N$ is carried by the sphere $S(x, a)$, the so-obtained Walsh martingale W has Euclidean quadratic variation $\langle W, W \rangle_t = a^2 t$ and its distribution at time $T_1 = \inf\{t \geqslant 0 : W_t \in S(x, a)\}$ is $\varepsilon_x N$. Define a martingale Y^a equal to W on the interval $[\![0, T_1]\!]$; after T_1, start the same construction again independently with x replaced by $Y^a_{T_1}$ (it is in F by construction of N) and $\varepsilon_x N$ by $\varepsilon_{Y^a_{T_1}} N$; and stop at the first hitting time T_2 of $S(Y^a_{T_1}, a)$ after T_1, etc. Since the differences $T_{i+1} - T_i$ are i.i.d. (they are distributed as the time needed by R to reach 1), T_i tend to infinity and this construction can be performed step by step, yielding a process Y^a_t well-defined for every $t \geqslant 0$. Moreover, this process is a martingale, with Euclidean quadratic variation $\langle Y^a, Y^a \rangle_t = a^2 t$: this holds on $[\![0, T_i]\!]$ by induction on i (this is where the measurability of N is used). Last, Y^a is in F at times T_i and in the ball $\overline{B}(Y^a_{T_i}, a)$ during the interval $[\![T_i, T_{i+1}]\!]$. Consequently, its distance to F remains bounded by a and it lives in F^a. To get X^a as claimed, it suffices to time-change Y^a by a constant factor: $X^a_t = Y^a_{t/a^2}$ is a F^a-valued martingale starting at x and its Euclidean quadratic variation is $\langle X^a, X^a \rangle_t = \langle Y^a, Y^a \rangle_{t/a^2} = t$.

Step three. Construction of a F-valued martingale X started at x, with Euclidean quadratic variation $\langle X, X \rangle_t = t$.

Carrying the construction of the previous step for $a = 1/n$ yields a sequence $(Z^n)_{n \in \mathbb{N}}$ of continuous, E-valued martingales started at x and with the same Euclidean quadratic variation $\langle Z^n, Z^n \rangle_t = t$. Such a sequence has a subsequence convergent in law, whose limit is a martingale X in E verifying also $X_0 = x$ and $\langle X, X \rangle_t = t$ (see Rebolledo [5]). Furthermore, since Z^k is $F^{1/n}$-valued for $k \geqslant n$, so is also X, which lives in each $F^{1/n}$, hence in F. Last, X is divergent since $\langle X, X \rangle_\infty = \infty$ a. s. ∎

COROLLARY 1. — *Let F be closed in E and x be a point in F. There exists in F a divergent martingale starting from x if and only if x is in the humpless kernel $\check{F}$.*

PROOF. — If there exists a divergent martingale in F started at x, it lives in $\check{F}$ according to Proposition 5; consequently its starting point x is in $\check{F}$.

Conversely, if $x \in \check{F}$, Proposition 6 applied to the humpless closed set $\check{F}$ gives the existence of a divergent martingale, started at x, living in $\check{F}$ and a fortiori in F. ∎

COROLLARY 2. — *Let F be closed in E. The following three statements are equivalent :*

(i) *there exists an F-valued divergent martingale;*

(ii) *the humpless kernel $\check{F}$ is not empty;*

(iii) *F contains a non-empty humpless closed set.*

PROOF. — Implication (i) $\Rightarrow$ (ii) stems from Proposition 5, its converse (ii) $\Rightarrow$ (i) from Corollary 1, and equivalence (ii) $\Leftrightarrow$ (iii) from the definition of $\check{F}$. ∎

COROLLARY 3. — *Let F be closed in an affine plane E. There exists an F-valued divergent martingale if and only if there exists an open set U, whose connected components are convex, and such that $F^c \subset U \neq E$.*

PROOF. — This is a restatement of the equivalence (i) $\Leftrightarrow$ (iii) in Corollary 2 using Proposition 4 c). ∎

4. Remarks

a) (Remark by P. A. Meyer.) Prominence with respect to some closed F is far from being a local property: proving that x is prominent requires considering only the intersection of F with some ball centred at x, but this ball can be arbitrarily large, and proving that x is *not* prominent is impossible if you know only a bounded part of F. But humplessness is, in some sense, local. Say that F is r-*humpless* if each point x of F belongs to the convex hull of $F \cap S(x,r)$. *The following are equivalent :*

(i) *F is r_n-humpless for some sequence $(r_n)_{n\in\mathbb{N}}$ with $r_n > 0$ and $r_n \to 0$;*

(ii) *F is r-humpless for every $r > 0$;*

(iii) *F is humpless.*

Indeed, the proof of Proposition 6 first establishes that (iii) $\Rightarrow$ (ii), then uses only the (seemingly) weaker statement (i) to construct in F a martingale started at any given point. So (i) implies the existence of such martingales, and (iii) follows by Corollary 1.

b) Replace now E by a C^2-manifold, endowed with an affine connection. Given a closed set $F \subset E$, do there exist divergent martingales in F? No generalization of humplessness to that case seems to exist. But using some complete Riemannian metric on the manifold (not related to the connection; notice that any two such metrics are comparable on compacts), it is still possible to define r-humplessness and the construction in Proposition 6 carries over to this situation: F contains divergent martingales iff it contains non-empty, closed, r_n-humpless subsets for a sequence $r_n > 0$ tending to 0.

c) What happens if F is no longer supposed closed? Prominent points can still be defined: x is prominent if there are an affine hyperplane $H \subset E$ not containing x and a bounded, closed subset K of F containing x and whose boundary $\partial_F K$ is included in H. As in Lemma 10, it is easily seen that no divergent martingale contained in F can start from a prominent point. But I do not know if Lemma 6 generalizes: if one cuts off the prominent points of F, then the prominent points of the remaining set, and so on, does he eventually get a humpless residue? Or is it necessary to transfinitely iterate this cutting off?

These questions are probably uninteresting since, even if a definition of humpless kernels and Proposition 5 extend, one way or another, to a non-closed F, there is no reason to expect the converse, that is, the existence in any humpless set of a divergent martingale (Proposition 6). The proof given above, constructing the martingale in a slightly larger set and passing to the limit, is clearly doomed to failure for non-closed sets.

Another attempt would be to try to reduce the non-closed case to what we already know. For instance, if a set contains a divergent martingale, does there always exist a smaller closed set containing also a divergent martingale? The answer is no, even for a smooth open set; here is a counter-example.

Call A the open planar set $\{(x,y) \in \mathbf{R}^2 : 0 < y < f(x)\}$, where $f : \mathbf{R} \to \mathbf{R}$ is convex, C^2, strictly positive, and has limit 0 when $x \to -\infty$ (for instance $f = \exp$). We shall show that A contains a divergent martingale, but no closed set included in A shares this property.

First, there exists in A a divergent martingale. Let X be a real Brownian motion started at 0 and I be the current infimum of X, given by $I_t = \inf_{0 \leqslant s \leqslant t} X_s$. The process $Y = \frac{1}{2} f{\circ}I + \frac{1}{2}(X-I)\, f'{\circ}I$ is a martingale owing to the change of variable formula

$$dY = \tfrac{1}{2} f'{\circ}I\, dX + \tfrac{1}{2}(X-I)\, f''{\circ}I\, dI = \tfrac{1}{2} f'{\circ}I\, dX\ ,$$

where $(X-I)\, dI = 0$ because I varies when $X = I$ only. (This formula extends to the case when f is not smooth: see Azéma & Yor [1], page 92.) As f is increasing, $Y \geqslant \frac{1}{2} f{\circ}I > 0$; as $\frac{1}{2}f$ is convex, $Y \leqslant \frac{1}{2} f{\circ}X < f{\circ}X$. Thus the planar martingale (X,Y) lives in A. And it is divergent, for so is already its projection X on the x-axis.

And yet, every humpless closed set included in A is empty, so no closed set included in A can contain a divergent martingale. To see this, let F be a humpless closed set included in A. Choose any non-empty open ball centred on the x-axis and having no intersection with F. The union of this ball with the x-axis is connected and does not meet F; as F is humpless, Proposition 4 a) says that the convex hull of this union does not meet F either; so F has no point in some strip $\{|y| < \varepsilon\}$. Choose x_0 such that $f(x_0) < \varepsilon$; F cannot meet the line $\{x = x_0\}$, so it does not meet the union of this line with the x-axis, nor the convex hull of this union (same reason as above). As this convex hull is the whole plane, F is empty.

References

[1] J. Azéma & M. Yor. Une solution simple au problème de Skorokhod. *Séminaire de Probabilités XIII*, Lecture Notes in Mathematics 721, Springer 1979.

[2] M. Barlow, J. Pitman & M. Yor. On Walsh's Brownian Motions. *Séminaire de Probabilités XXIII*, Lecture Notes in Mathematics 1372, Springer 1989.

[3] C. Dellacherie. Ensembles analytiques : théorèmes de séparation et applications. *Séminaire de Probabilités IX*, Lecture Notes in Mathematics 465, Springer 1975.

[4] O. Hanner & H. Rådström. A Generalization of a Theorem of Fenchel. *Proc. Amer. Math. Soc. 2,* 1951.

[5] R. Rebolledo. La méthode des martingales appliquée à l'étude de la convergence en loi des processus. *Bull. Soc. math. France,* Mémoire 62, supplément au numéro d'octobre 1979.

Université Louis Pasteur et C.N.R.S.
I.R.M.A.
7 rue René Descartes
67 084 STRASBOURG Cedex