SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

LEONID I. GALTCHOUK
ALEXANDRE A. NOVIKOV

# On Wald's equation. Discrete time case

*Séminaire de probabilités (Strasbourg)*, tome 31 (1997), p. 126-135
<http://www.numdam.org/item?id=SPS_1997__31__126_0>

# ON WALD'S EQUATION. DISCRETE TIME CASE

Leonid I. GALTCHOUK† and Alexandre A. NOVIKOV‡

† Institut de Recherche Mathématique Avancée
Université Louis Pasteur et C.N.R.S., 7 rue René-Descartes
67084 Strasbourg Cedex, France.
e-mail:galtchou@math.u-strasbg.fr.

‡ Steklov Mathematical Institute
42, Vavilova, 117333, Moscow, Russia.
e-mail: alex@novikov.mian.su.

## 0. Summary

Let $m_t$ be a square integrable martingale, $m_0 = 0$, such that there exists $\lim_{t\to\infty} m_t = m_\infty$ a.s. We study a minimal possible sufficient condition for the validity of Wald's equation $Em_\infty = 0$ in terms of the tail behavior of a square characteristic $S(m)_\infty$ of $m_t$.

## 1. The background for the problem and the main result

Let $(\Omega, \mathbf{F}, \mathbf{F}_t, P), t \in Z^+ = \{0, 1, \ldots\}$ be a stochastic basis with the filtration $\mathbf{F}_t$. We *always* consider in this paper the process $m_t$ as a square integrable martingale, $m_0 = 0$, that is a square integrable process such that $Em_\tau = Em_0 = 0$ for any bounded (by a constant) stopping time $\tau$ (of course, with respect to $\mathbf{F}_t$). Denote a square characteristic of $m_t$ by $S(m)_t := \sum_{k=1}^{t} E_k X_k^2$, where we used notations for the martingale-difference $X_k := m_k - m_{k-1}$ and for conditional expectations $E_t(\,) := E\{(\,)|\mathbf{F}_{t-1}\}$.

It is well known (see Meyer (1972,Theorem 64), Liptser and Shiryaev (1991)) that if $S(m) = S(m)_\infty < \infty$ a.s. then there exists $\lim_t m_t = m_\infty$ a.s. (all limits over $t$ are considered as $t \to \infty$ ). For many applications , for example in sequential analysis, it is of interest to know under what minimal conditions the equation

$$Em_\infty = 0$$

still holds.

The prehistory of this question goes back to classical Wald monograph (1945) in which this equality was used to establish some general properties of sequential tests. The first results obtained by Wald (in modern form) is the following:

---

AMS 1991 classification : 60G42, 60G40.

Key words, phrases :local martingale, Wald's equation, uniform integrability, tauberian theorem.

$$ES(m) < \infty \implies Em_\infty = 0.$$

Later Burkholder and Gundy (1970) proved the maximal inequality

$$E \sup_t |m_t| \le CE(S(m))^{1/2}$$

which implies uniforme integrability of $m$ if

$$E(S(m))^{1/2} < \infty,$$

and, of course, Wald's equation:

$$Em_\infty = 0. \tag{1}$$

(we denote all constants whose values are not important for this exposition by C ).

Note (1) is valid also for continuous time martingale and it seems the first result in this direction was obtained by Novikov (1971) for stochastic integrals with respect to a brownian motion (the paper of Novikov (1971) was presented for publishing at the same time as Burkholder and Gundy (1970)).

Azema, Gundy and Yor (1979) discussed a problem of uniform integrability (U.I.) of continuous martingales ($m_t \in \mathbf{M}^c$) and, particularly, they showed that if $m_t \in \mathbf{M}^c$ and $\sup_t E|m_t| < \infty$ then

$$\lim_t P\{S(m) > t\} t^{1/2} = 0 \iff m_t \text{ is U.I.} \implies Em_\infty = 0. \tag{2}$$

The similar result as in (2) for discrete time case was obtained by Gundy (1981) but for a special case of martingales satisfying the following conditions:

$$\sup_t E|m_t| < \infty\,,\ \lim_t P\{S(m) > t\} t^{1/2} = 0,$$

$$X_t = V_t D_t, V_t \, is \, \mathbf{F}_{t-1}\text{-measurable}, E_t X_t = 0, E_t|X_t| > C > 0, E_t X_t^2 = 1 \tag{3}$$

(all inequalities for random variables in our paper hold with probability one).

In the present paper we prove that under some different bounds for conditional moments of the martingale-difference $X_t$ a weaker condition on $S(m)$ instead of that one in (3) may be used and ever more detailed information concerning the asymptotic behaviour of $P\{S(m) > t\}$ may be obtained (see Lemma 1 and Remark 1 below).

To formulate the basic result introduce the following class of deterministic functions

$$G = \{g(x) > 0,\ g(x) \uparrow,\ \int_1^\infty x^{-3/2} g(x) dx = \infty\}.$$

**Theorem 1.** *Let* $S(m) < \infty, E|m_\infty| < \infty,\ |X_t| < C$. *Then*

$$(\textit{there exists } g(x) \in G : Eg(S(m)) < \infty) \implies Em_\infty = 0. \tag{4}$$

To see that condition of (4) is a less restrictive that the condition

$$\lim_t P\{S(m)\ > t\}t^{1/2} = 0$$

one may take the function $g(x)$ with the step-wise derivative

$$g'(x) = \sum_{k=1}^{\infty}(klogk)^{-1}(x_{k+1}^{1/2} - x_k^{1/2})I\{x_k \leq x < x_{k+1}\}$$

where $I\{\ \}$ is an indicator function, $x_1 = 1, x_{k+1} > x_k+1$ . As for any nondecreasing positive function $f(x)$

$$\int_1^{\infty} x^{-3/2}f(x)dx = \infty \iff \int_1^{\infty} x^{-1/2}df(x) = \infty$$

then $g(x) \in G$.

Take now

$$x_{k+1} = \inf\{x \geq x_k + 1 : \sup_{t\geq x} P\{S(m)\ > t\}t^{1/2} \leq 1/k\}\ .$$

As

$$Eg(S(m))\ < \infty \iff \int_1^{\infty} P(S(m) > x)\sqrt{x}\frac{dg(x)}{\sqrt{x}} < \infty\,,$$

then it is easy to see that $Eg(S(m))\ < \infty$. Note that, of course,

$$(\text{there exists } g(x) \in G : Eg(S(m))\ < \infty) \implies \liminf_t P\{S(m)\ > t\}t^{1/2} = 0.$$

It should be noted that unlike the case of nonnegative martingales the validity of Wald's equation $Em_\infty = 0$ , generally speaking, does not imply Wald's identity , that is the equality $Em_\tau\ = 0$ for any stopping time $\tau$ : consider for example, sums of Rademacher' variables (with jumps 1 and -1) stopped at moment of the first hitting zero after first passaging of the level +1. But if one assumes that $m_t^+$ is U.I. then (as remarked by Vallois (1991)) Wald's equation is equivalent to U.I. of $m_t$ .

The technique used in the present paper is based on exponential martingales and tauberian theorem (see Feller (1966)) and it is very different from one used in Burkholder and Gundy (1970), Azema, Gundy and Yor (1979), Gundy (1981) and related papers of Kinderman (1980), Klass (1988), de la Pena (1993) (all these papers exploited so-called "good-lambda" inequality first appeared in Burkholder and Gundy (1970)).

We note that the idea of using exponential supermartingales was used earlier by Meyer (1972, th. 71) for obtaining some asymptotic results for martingales.

Our method can be easily extended to the case of continuous time martingale (results for quasi left-continuous martingales was reported by the authors to Probability seminar at Strasbourg university, February , 1994) but the authors plan to consider in a separate paper a more general case of so-called optional martingales (that is, without standard condition on right-continuity of $\mathbf{F}_t$ (see Galtchouk (1980)).

Note that the result of Theorem 1 for a special case of stopped processes with independent increments was proved in Novikov (1981a,1982).

## 2. Two lemmas

**Lemma 1.** *Suppose $S(m) < \infty$ and there exists $\lambda_+ > 0$ such that for all $\lambda \in [0, \lambda_+)$*

$$E_t|X_t|^3 exp(\lambda X_t) \le C E_t|X_t|^2 \tag{5}$$

*and*

$$\sup_t Eexp(\lambda m_t) < \infty . \tag{6}$$

*Then*

$$0 \le Em_\infty < \infty, \tag{7}$$

*and*

$$\lim_t P\{S(m) > t\} t^{1/2} = (2/\pi)^{1/2} Em_\infty . \tag{8}$$

Proof. By condition (5) the following predictable function

$$\psi_t(\lambda) = log E_t exp(\lambda X_t),\ 0 \le \lambda < \lambda_+,$$

is finite and it is non negative due to Jensen's inequality and the condition $E_t X_t = 0$.

Below we exploit the following well-known facts: the process

$$Z_t(\lambda) = exp\{\lambda m_t - \sum_1^t \psi_k(\lambda)\},\ 0 \le \lambda < \lambda_+,$$

is a non negative martingale and there exists

$$\lim_t Z_t(\lambda) = Z_\infty(\lambda).$$

The limit $m_\infty$ exists thanks the condition $S(m) < \infty$. By (6) and Fatou's lemma we have $Eexp(\lambda m_\infty) < \infty$ and by the dominated convergence theorem the following equality (Wald's exponential identity) holds

$$EZ_\infty(\lambda) = 1,\ \ 0 \le \lambda < \lambda_+ . \tag{9}$$

Assumption (6) implies uniform integrability of $m_t^+ = max(m_t, 0)$ by Vallée-Poussin's theorem.

As $Em_t^+ = Em_t^-, (m_t^- = max(-m_t, 0))$ then by Fatou's lemma

$$Em_\infty^+ = lim_t Em_t^+ = lim_t Em_t^- \ge Em_\infty^- .$$

So we have $0 \le Em_\infty < \infty$ (this type of arguments was used by Novikov (1981) and a recent paper of Vallois (1991)).

The condition (6) implies

$$E|m_\infty|exp(\lambda m_\infty) < \infty. \tag{10}$$

Indeed

$$E|m_\infty|exp(\lambda m_\infty) = -Em_\infty exp(\lambda m_\infty)I_{m_\infty<0} + Em_\infty exp(\lambda m_\infty)I_{m_\infty\geq 0}. \tag{11}$$

The first right hand term is finite by (7).

Further, for all $\lambda \in [0, \lambda_+)$ there exists a such $\epsilon > 0$ that $\lambda + \epsilon < \lambda_+$. For all $\epsilon > 0$ there exists a such constant $K = K_\epsilon$ that

$$exp((\lambda + \epsilon)m_\infty) > m_\infty exp(\lambda m_\infty)$$

if $m_\infty > K_\epsilon$.

Then for second right hand term of (11) we have

$$Em_\infty exp(\lambda m_\infty)I_{m_\infty\geq 0} = Em_\infty exp(\lambda m_\infty)I_{0\leq m_\infty\leq K_\epsilon} + Em_\infty exp(\lambda m_\infty)I_{m_\infty > K_\epsilon}$$

$$\leq K_\epsilon e^{\lambda K_\epsilon} + Ee^{(\lambda+\epsilon)m_\infty} < \infty.$$

This inequality and (11) imply (10).

Since $E|m_\infty|exp(\lambda m_\infty) < \infty$ then by the dominated convergence theorem from (9) it follows:

$$1 - Eexp\{-\sum_1^\infty \psi_t(\lambda)\} = EZ_\infty(\lambda) - Eexp\{-\sum_1^\infty \psi_t(\lambda)\} =$$

$$\lambda Em_\infty + o(\lambda), \ \ \lambda \to 0.$$

Below, in Lemma 2, we shall prove the following relation

$$Eexp\{-1/2\ \lambda^2 S(m)\} - Eexp\{-\sum_1^\infty \psi_t(\lambda)\} = o(\lambda), \ \ \lambda \to 0, \tag{12}$$

That gives us

$$1 - Eexp\{-\lambda^2/2S(m)\} = \lambda Em_\infty + o(\lambda), \ \ \lambda \to 0.$$

This equation is equivalent to (8) by following tauberian theorem.

**Theorem 2 (Feller W.(1971) Ch.XIII,Example (c)).***Let $Y$ be an $R_+-$ valued random variable, and $L : R_+ \to R_+$ be a slowly varying function at $\infty$. Let $0 \leq \rho < \infty$.*

*Then following relations are equivalent:*

$$i)\ (1 - E\exp(-\nu Y)) \sim \nu^{1-\rho}L(\frac{1}{\nu}), \ \ \nu \to 0,$$

$$ii)\ x^{1-\rho}P(Y \geq x) \sim \frac{1}{\Gamma(\rho)}L(x),\ x \to \infty.$$

This theorem applies with $Y = S(m), L(x) = Em_\infty, \nu = \frac{\lambda^2}{2}$, or $\lambda = \sqrt{2\nu}$.

So to finish the proof of Lemma 1 we need only to prove equation (12). We shall use the following

**Lemma 2.** *Under conditions of Lemma 1*

$$(\lambda^2/2 - \lambda^3 C)S(m) \leq \sum_1^\infty \psi_k(\lambda) \leq (\lambda^2/2 + \lambda^3 C)S(m), 0 \leq \lambda < \lambda_+ \ .$$

Proof of Lemma 2. From the definition of $\psi_t(\lambda)$ it follows that for for all $\lambda \in [0, \lambda_+)$

$$\frac{\partial \psi_t(\lambda)}{\partial \lambda} := \psi_t(\lambda)' = E_t(X_t exp\{\lambda X_t - \psi_t(\lambda)\}),$$

$$0 \leq \psi_t(\lambda)'' = E_t(X_t - \psi_t(\lambda)')^2 exp\{\lambda X_t - \psi_t(\lambda)\}) =$$
$$E_t(X_t^2 exp\{\lambda X_t - \psi_t(\lambda)\}) - (\psi_t(\lambda)')^2.$$

Integrating the last inequality with respect to $\lambda$ and applying the inequality $exp(x) - 1 \leq x^+ exp(x)$ we get

$$\psi_t(\lambda)' \leq \lambda E_t X_t^2 + \frac{\lambda^2}{2} E_t (X_t^+)^3 exp\{\lambda X_t\}. \tag{13}$$

As by (5) $E_t(X_t^+)^3 exp\{\lambda X_t\} \leq C E_t X_t^2$ we get the upper bound in Lemma 2. To prove the lower bound, let us note that the same arguments give

$$\psi_t(\lambda)' \geq \int_0^\lambda E_t(X_t^2 exp\{uX_t - \psi_t(u)\}du - \lambda(\psi_t(\lambda)')^2 \geq$$

$$\lambda(E_t(X_t^2)exp\{-\psi_t(\lambda)\} - \int_0^\lambda E_t X_t^2(1 - exp\{-uX_t^-)du - (\psi_t(\lambda)')^2 \geq$$
$$\lambda exp\{-\psi_t(\lambda)\}E_t X_t^2 - \lambda^2 E_t(X_t^-)^3/2 - \lambda(\psi_k(\lambda)')^2 .$$

Finally, integrating again, by (5) and by the bound (13) we have

$$\psi_t(\lambda) \geq \lambda^2/2\, E_t X_t^2 - \lambda^3 C E_t X_t^2.$$

The proof of Lemma 2 is completed.

Now to complete the proof of Lemma 1 let us note that due to the upper bound from Lemma 2 and the inequality $1 - exp(-x) \le x^+$

$$Eexp\{-\lambda^2/2\ S(m)\} - Eexp\{-\sum_1^\infty \psi_k(\lambda)\} \le$$

$$Eexp\{-\frac{\lambda^2}{2}S(m)\}(1 - exp\{-\lambda^3 CS(m)\}) \le$$

$$C\lambda E(exp\{-\lambda^2/2S(m)\}\lambda^2 S(m)) = o(\lambda)\ \lambda \to 0\,,$$

(by the dominated convergence theorem).

On the other side, by the lower bound from Lemma 2 and the same as above arguments

$$Eexp\{-\lambda^2/2\ S(m)\} - Eexp\{-\sum_1^\infty \psi_k(\lambda)\} \ge$$

$$C\lambda E(exp\{-\lambda^2/2\ (1 - \lambda C)S(m)\}\lambda^2 S(m)) = o(\lambda),\ \ \lambda \to 0$$

That completes the proof of Lemma 1.

### 3. Proof of Theorem 1.

Introduce the stopping time

$$\tau(A) = inf\{t : m_t > A - g(S(m)_t)\},\ \ (inf\{\phi\} = \infty\})$$

where a parameter A is positive, $g \in G$, and consider the stopped martingale

$$m_t^A := m_{t\wedge\tau(A)}.$$

It is easy to see that all conditions of Lemma 1 are fulfilled for $m_t^A$ (condition (5) is fulfilled by the boundness of jumps of $m_t$ and (6) by the definition of $\tau(A)$). So $0 \le Em_\infty^A < \infty$ and

$$\lim_t P\{S(m^A) > \ t\}t^{1/2} = (2/\pi)^{1/2} Em_\infty^A\,. \tag{14}$$

Now show that

$$Em_\infty^A = 0\,.$$

Indeed, due to the relation $E|m_\infty^A| < \infty$ and by $|X_t| < C$ it follows that $Eg(S(m^A)) < \infty$ or, equivalently (integrating by parts),

$$\int_1^\infty P\{S(m^A)\ > t\}dg(t) < \infty\,.$$

But $g \in G$ and so by (14) we have now $Em_\infty^A = 0$ or, equivalently ,

$$EI\{\tau(A)=\infty\}m_\infty + EI\{\tau(A)<\infty\}m_{\tau(A)} = 0$$

Finally, note that since $\tau(A)\to\infty,\ as\, A\to\infty$, then by the assumption of (4)

$$EI\{\tau(A)<\infty\}m_{\tau(A)} \geq -EI\{\tau(A)<\infty\}g(S(m)) \ \to 0, \quad A\to\infty\ .$$

As

$$EI\{\tau(A)=\infty\}m_\infty \to Em_\infty$$

we get the lower bound

$$Em_\infty \geq 0\,.$$

Repeating the same arguments for the martingale $(-m_t)$ we obtain the upper bound $Em_\infty \leq 0$ .

Proof of Theorem 1 is completed.

## 4. Remarks

1. The arguments used in proof of Lemma 1 entail the following result which may be known but we have no references.

**Proposition.** *Let $(m_t)$ be a local martingale, $m_0=0$, such that $m_t > -Z$ for any $t$, where $Z$ is a positive integrable r.v.*

*Then $(m_t)$ is a martingale.*

Proof.

Let $(\tau_l)$ be a localization sequence of stopping times for the local martingale $(m_t)$.

Then by the martingale property

$$Em_{\tau_l\wedge\tau} \ = 0,$$

where $\tau$ is an arbitrary stopping time less than $T$ =const. Hence

$$Em^+_{\tau_l\wedge\tau} = Em^-_{\tau_l\wedge\tau}.$$

Taking a limit as $l$ tends to infinity we get by Fatou's lemma

$$Em^+_\tau = Em^-_\tau < \infty.$$

So $m_\tau$ is an integrable r.v. and the sequence $(m^+_{\tau_l\wedge\tau})$, $l=1,2,\ldots$ is uniformly integrable. Hence we have for any bounded $\tau$

$$Em^+_\tau = Em^-_\tau\,.$$

This fact means that $m_t$ is a martingale.

2. It seems the conditions of Theorem 1 and Lemma 1 concerning boundness of jumps of a martingale $m_t$ may be essentially weekned. That is true , at any rate,

for the case when $m_t = Y_{t\wedge\tau}, Y_t = \sum_1^t X_k, X_k$ are iid, $EX_k = 0, EX_k^2 = \sigma^2 > 0$ . In this case $S(m) = \sigma^2\tau$ and so by Lemma 1 under additional conditions (5) and (6) we have

$$\lim_t P\{\sigma^2\tau > t\}t^{1/2} = (2/\pi)^{1/2} EY_\tau .$$

For the special case of stopping time $\tau = inf\{t : Y_t > f(t)\})$, $f(1) > 0$, which was studied by Novikov (1981b), more stronger results can be obtained. In particular, Novikov (1981b) proved the following result: if a function $f(t)$ is increasing and convex, or $f(t)$ is decreasing, concave and additionally, $E\exp(\lambda X_1) < \infty$ for some $\lambda > 0$ then

$$0 < EY_\tau < \infty \Longleftrightarrow \int_1^\infty \mid f(t) \mid t^{-3/2} dt < \infty.$$

The authors express their gratitude to M.Emery, M.Lifshits, J.Memin, P.-A.Meyer for stimulating conversations about the results.

The paper was completed while the second author was visiting Department of Mathematics of Strasbourg University and he thanks all staff for hospitality.

## References

Azema(J.), Gundy(R.F.), Yor(M.) (1979). Sur l'intégrabilitée uniforme des martingales continues. Séminaire de Probabilités XIV, Lecture Notes in Mathematics, 784, 53-61, Springer-Verlag, Berlin.

Burkholder (D.L.), Gundy(R.F.) (1970). Extrapolation and interpolation of quasilinear operators on martingales. Acta Math., 124, 249-304.

Galtchouk (L.I.) (1980). Optional martingales. Mathematical Sbornik, 112 (154), N 4 (8), 483 - 521 (English translation :(1981) Vol.40, N4, 435-468).

Gundy (R.F.) (1981). On a theorem of F. and M.Riesz and an equation of A.Wald. Indiana Univ.Math.Journal, 30(4), 589-605.

Feller, W. (1971). An Introduction to Probability Theory and Its Applications, vol. 2, Wiley, New York.

Kinderman (R.P.) (1980). Asymptotic comparisons of functionals of Brownian Motion and Random Walk. Ann. Prob., 8,N6)), 1135-1147.

Klass (M.J.) (1988). A best possible improvement of Wald's equation. Ann.prob., 16, N2, 840-853.

Liptser (R.Sh.), Shiryaev (A.N.) (1986). Theory of Martingales, Kluwer Academic Publ.

Meyer (P.-A.) (1972). Martingales and Stochastic Integrals I. Lecture Notes in Mathematics, 284, Springer-Verlag.

Novikov (A.A.) (1971). On the moment of stopping of a Wiener process. Teor. Veroythn. Primen., 16, N3, 458-465 (English translation : pp.449- 456).

Novikov (A.A.) (1981a). A martingale approach to first passage problems and a new condition for Wald's identity. Proc.of the 3rd IFIP-WG 7/1 Working Conf.,Visegrad 1980,Lecture Notes in Control and Inf.Sci. 36, 146-156.

Novikov (A.A.) (1981b). Martingale approach to first passage problems of nonlinear boundaries. Proc.Steklov Inst. 158, 130-158.

Novikov (A.A.) (1982). On the time of crossing of a one-sided nonlinear boundary. Theor.Prob. Appl., 27, N4, 668 - 702 (English translation).

de la Pena (V.H.) (1993). Inequalities for tails of adapted processes with an application to Wald's lemma. J. of Theoretic Prob., 6, N2, 285-302.

Vallois (P.) (1991). Sur la loi du maximum et du temps local d'une martingale continue uniformément intégrable. Preprint, Université de Paris VI.

Wald (A.) (1947). Sequential Analysis, Wiley, New York;