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On the tails of the supremum and the quadratic variation
of strictly local martingales

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Introduction and main results.

In this paper, we study some properties of continuous strictly local martingales, i.e : local martingales which are not martingales. Our interest for this class of local martingales stems from the fact, under some mild additional conditions on such a process $(M_t, t \geq 0)$, the tails of the distributions of $\sup_{t \geq 0} M_t$ and $\langle M \rangle_\infty^{1/2}$ are equivalent to $\frac{c}{x}$, as $x \rightarrow \infty$, for two related constants c_1 and c_2 (depending on M). Precisely, one of our main results, which has a number of applications, is the

Theorem 1 : Let $(M_t, t \geq 0)$ be a continuous local martingale taking its values in \mathbb{R}_+ , and satisfying $E[M_0] < \infty$.

Then, both :

$$\ell = \lim_{x \rightarrow \infty} \left\{ x P \left(\sup_{t \geq 0} M_t \geq x \right) \right\} \quad \text{and} \quad \sigma = \lim_{y \rightarrow \infty} \left\{ y P \left(\langle M \rangle_\infty^{1/2} \geq y \right) \right\}$$

exist in \mathbb{R}_+ , and satisfy :

$$(1) \quad \ell = \sqrt{\frac{\pi}{2}} \sigma = E[M_0 - M_\infty].$$

It is particularly easy to prove this theorem if $M_\infty = 0$, and $M_0 = c > 0$, for simplicity. In this case, using the Dubins-Schwarz representation of $(M_t, t \geq 0)$ as :

$M_t = \beta_{\langle M \rangle_t}$, where $(\beta_u, u \geq 0)$ denotes a Brownian motion starting from c ,

we obtain :

$$\sup_{t \geq 0} M_t = \sup_{u \leq T_0} \beta_u \quad \text{and} \quad \langle M \rangle_\infty = T_0 \equiv \inf\{u : \beta_u = 0\}.$$

It is now easy to show that :

$$\sup_{t \geq 0} M_t \stackrel{(law)}{=} \frac{c}{U}, \quad \text{and} \quad \langle M \rangle_\infty^{1/2} \stackrel{(law)}{=} \frac{c}{|N|},$$

where U is uniform on $[0,1]$, and N is a standard reduced gaussian r.v. The double equality (1) now follows easily.

In fact, in the first paragraph below, we shall prove a more general result than that of Theorem 1 ; indeed, we shall consider a general \mathbb{R} -valued continuous local martingale M and we shall prove the following

Theorem 1' : Let $(M_t, t \geq 0)$ be a continuous local martingale, with $M_0 = 0$. Assume that :

(i) $\{M_V^- ; V \text{ finite stopping time}\}$ is uniformly integrable.

Then, $\{M_t, t \rightarrow \infty\}$ converges a.s. ; we denote this limit by M_∞ .

Assume furthermore that :

(ii) there exists $\varepsilon > 0$ such that : $E[\exp(\varepsilon M_\infty^-)] < \infty$.

Then both :

$$\ell = \lim_{x \rightarrow \infty} \left\{ x P \left(\sup_{t \geq 0} M_t \geq x \right) \right\} \quad \text{and} \quad \sigma = \lim_{y \rightarrow \infty} \left\{ y P(\langle M \rangle_\infty^{1/2} \geq y) \right\} \quad \text{exist in } \mathbb{R}_+,$$

and satisfy :

$$(2) \quad \ell = \sqrt{\frac{\pi}{2}} \sigma = -E(M_\infty).$$

In our second paragraph, we apply Theorem 1' to transient diffusions, in particular Bessel processes, and we show how the identity (2) translates into some remarkable identities involving Bessel functions.

A more general discussion of strictly local martingales and their relations with strong completeness of stochastic flows is made in [9].

Acknowledgment and priority :

The proof of Theorem 1', concerning σ , uses essentially the Tauberian theorem ; after writing a first draft of this paper in May 1995, we learnt that Galtchouk - Novikov [10] already went through a similar discussion.

Ron Doney (Manchester) and J. Warren (Bath) also convinced us that the argument, if not the result, was "well-known" (to some...).

1. Proof of Theorem 1'.

(1.1) We first show that $\{M_t, t \rightarrow \infty\}$ converges a.s. ; indeed, we remark that (i) implies, from Fatou's lemma, that : $E[L_\infty] < \infty$, where $(L_t, t \geq 0)$ is the local time at 0 of M .

But, it is well-known that the sets : $\{M_t \xrightarrow[t \rightarrow \infty]{} \cdot\}$, $\{L_\infty < \infty\}$, and $\{<M>_\infty < \infty\}$ are all a.s. equal ; in our situation, they all have probability 1.

(1.2) We first show that ℓ exists, and satisfies :

$$(1.a) \quad \ell = E[-M_\infty].$$

To prove this (fairly well-known result), we apply the optional stopping theorem to $\tilde{M} = (M_{t \wedge T_x} ; t \geq 0)$, for $x > 0$; from (i), \tilde{M} is uniformly integrable ; hence, we obtain :

$$0 = E[M_{T_x}] = E[M_\infty 1_{(T_x = \infty)}] + x P(T_x < \infty)$$

Consequently :

$$x P\left(\sup_{t \geq 0} M_t \geq x\right) = E\left[(-M_\infty) 1_{(T_x = \infty)}\right].$$

The right-hand side converges, as $x \rightarrow \infty$, to : $E[-M_\infty]$, thanks to the dominated convergence theorem, since $E[|M_\infty|] < \infty$. This integrability property follows from the equality : $E[M_{T_x}^+] = E[M_{T_x}^-]$, our hypothesis (i), and Fatou's lemma.

(1.3) The proof that σ exists, and satisfies :

$$(1.b) \quad \sqrt{\frac{\pi}{2}} \sigma = E(-M_\infty)$$

hinges essentially on the following variant of the Tauberian theorem.

Lemma 1 (Feller [0], XIII.5 : Tauberian theorems, Example (c)).

Let X be an \mathbb{R}_+ -valued random variable, and $L : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a slowly

varying function at ∞ ; finally, let $0 < \alpha < 1$.

The following properties are equivalent :

$$i) \quad \frac{1}{\lambda} (1 - E[\exp(-\lambda X)]) \xrightarrow{\lambda \rightarrow 0} L\left(\frac{1}{\lambda}\right)$$

$$ii) \quad x^\alpha P(X \geq x) \xrightarrow{x \rightarrow \infty} \frac{1}{\Gamma(1-\alpha)} L(x).$$

Proof of (1.b) : We write :

$$\begin{aligned} & \frac{1}{\nu} E[1 - \exp(-\frac{\nu^2}{2} \langle M \rangle_\infty)] \\ (*) \quad &= \frac{1}{\nu} E[\exp(\nu(-M_\infty) - \frac{\nu^2}{2} \langle M \rangle_\infty) - \exp(-\frac{\nu^2}{2} \langle M \rangle_\infty)] \\ &= E\left[\exp\left(-\frac{\nu^2}{2} \langle M \rangle_\infty\right) \left(\frac{\exp(\nu(-M_\infty)) - 1}{\nu}\right)\right] \end{aligned}$$

It is then easily shown that, thanks to the hypothesis (ii) in Theorem 1', the last written expectation converges towards : $E(-M_\infty)$ (precisely, we use dominated convergence, and the elementary fact :

$$\left| \frac{1}{\nu} (\exp(\nu x) - 1) \right| \leq \begin{cases} \exp(\nu x), & \text{if } x \geq 0 ; \\ x^-, & \text{if } x \leq 0 ; \end{cases} \quad)$$

Thus, we see that Lemma 1 applies with $\lambda = \frac{\nu^2}{2}$, or equivalently : $\nu = \sqrt{2\lambda}$, $X = \langle M \rangle_\infty$, and $L(\mu) \equiv \sqrt{2}E(-M_\infty)$. \square

To be complete, we add the following justification of (*) : we need to show that :

$$(**) \quad E[\exp(\nu(-M_\infty) - \frac{\nu^2}{2} \langle M \rangle_\infty)] = 1,$$

which also follows from the hypotheses (i) and (ii) ; indeed, they imply that :

$$\exp(\nu M_V^-) \leq E[\exp(\nu M_\infty^-) | \mathcal{F}_V], \text{ for } \nu \leq \varepsilon$$

hence the uniform integrability of

$$\{\exp(\nu M_V^-) ; V \text{ finite stopping time, } \nu \leq \varepsilon\}$$

which yields (**).

(1.4) We now make some comments about the hypotheses and the conclusion of Theorem 1' :

- first, remark that (i) and (ii) imply, using both Jensen's and Doob's L^p inequalities, that :

$$\text{for } \varepsilon' < \varepsilon, \quad E[\exp(\varepsilon' \sup_{t \geq 0} M_t^-)] < \infty$$

- consequently, ℓ is also equal to :

$$\ell^* = \lim_{x \rightarrow \infty} \left\{ x P \left(\sup_{t \geq 0} |M_t| \geq x \right) \right\} ;$$

likewise, σ is also equal to

$$\sigma^+ \stackrel{\text{def}}{=} \lim_{y \rightarrow \infty} \{ y P((\langle M^* \rangle_\infty)^{1/2} \geq y) \}$$

$$\left(\text{recall that : } \langle M^* \rangle_t = \int_0^t 1_{(M_s > 0)} d\langle M \rangle_s \right),$$

since, from (ii), $\langle M^* \rangle_\infty$ is integrable, and in fact admits moments of all orders.

To summarize, starting from an asymmetric hypothesis about a local martingale M , the conclusion of Theorem 1' may be presented in "symmetric" terms (i.e. involving only $|M|$).

The following variant of Theorem 1' seems to have a wider domain of applicability.

Theorem 1'' : Let $X_t = M_t + A_t$ be an \mathbb{R}_+ -valued continuous local sub-martingale such that its increasing process $(A_t, t \geq 0)$ satisfies :

$$(+) \quad E[\exp(\varepsilon A_\infty)] < \infty, \text{ for some } \varepsilon > 0.$$

Then, the following limits exist in \mathbb{R}_+ :

$$\ell = \lim_{x \rightarrow \infty} \left\{ x P \left(\sup_{t \geq 0} X_t > x \right) \right\} \quad \text{and} \quad \sigma = \lim_{y \rightarrow \infty} \left\{ y P(\langle X \rangle_\infty^{1/2} \geq y) \right\}$$

and satisfy :

$$(3) \quad \ell = \sqrt{\frac{\pi}{2}} \sigma = E[M_0 - M_\infty].$$

The proof of Theorem 1" is quite similar to that of Theorem 1' ; hence, it is left to the reader.

As an illustration, we remark that Theorem 1" applies to :

$$X_t^{(1)} = B_{t \wedge \tau_\lambda}^*, \text{ and } X_t^{(2)} = |B_{t \wedge \tau_\lambda}|, \text{ for some } \lambda > 0,$$

where : $\tau_\lambda = \inf\{u : \ell_u > \lambda\}$, with (ℓ_u) the local time at 0 of the Brownian motion B ; note that Theorem 1' does not apply to $(B_{t \wedge \tau_\lambda}, t \geq 0)$. For these examples, (3) becomes :

$$\ell^{(1)} = \sqrt{\frac{\pi}{2}} \sigma^{(1)} = \frac{\lambda}{2} \text{ and } \ell^{(2)} = \sqrt{\frac{\pi}{2}} \sigma^{(2)} = \lambda.$$

These results may also be checked directly, since it is well-known that :

$$\sup_{t \leq \tau_1} B_t \stackrel{(law)}{=} 1/2e \text{ and } \sup_{t \leq \tau_1} |B_t| \stackrel{(law)}{=} 1/e,$$

$$\text{whereas : } \int_0^{\tau_1} ds \, 1_{(B_s > 0)} \stackrel{(law)}{=} \frac{1}{4} \tau_1 \stackrel{(law)}{=} \frac{1}{4} T_1, \text{ with}$$

$T_1 = \inf\{t : B_t = 1\}$, and e is an exponential variable with mean 1.

Finally, concerning possible further generalizations, it would be most interesting to know whether one can avoid the Tauberian argument, and weaken the hypothesis (+) in Theorem 1".

2. Strictly local martingales, transient diffusions and some remarkable identities.

(2.1) Our examples will take place in the framework considered by Pitman-Yor ([3], [4] ; (1981, 1982)) and Le Gall ([1] ; (1986)) of a regular diffusion $(R_t, 0 \leq t < \zeta, P_r, 0 < r < \infty)$ on the interval $]0, \infty[$ of \mathbb{R} ; let $s(\cdot)$ denote a scale function for the diffusion, and m the speed measure normalized so that the infinitesimal generator is $\frac{1}{2} \frac{d}{dm} \frac{d}{ds}$. We assume :

- (i) $\zeta = \inf\{t > 0 : X_{t-} = 0 \text{ or } \infty\}$
- (ii) $s(0) = -\infty, s(\infty) < \infty$
- (iii) 0 is an entrance point for the diffusion R .

In the sequel, we shall always take $s(\infty) = 0$.

We still need to introduce, for $0 < \rho < \infty$, the last passage time

$L_\rho = \sup\{t \geq 0 : R_t = \rho\}$, and the expression of the semi-group

$$Q_t(r, d\rho) = p_t^\bullet(r, \rho) m(d\rho).$$

Then, we have the following

Theorem 2 : 1. For $0 \leq r < \rho < \infty$,

$$(4) \quad P_r(L_\rho \in dt) = -\frac{1}{2s(\rho)} p_t^\bullet(r, \rho) dt$$

2. For every $r > 0$, and $t \geq 0$, the limit

$$(5) \quad \lim_{\alpha \rightarrow \infty} \left\{ \alpha P_r \left(\sup_{u \leq t} \left(-s(R_u) \right) \geq \alpha \right) \right\}$$

exists, and is equal to :

$$(6) \quad E_r \left[s(R_t) - s(r) \right] = \frac{1}{2} \int_0^t du p_u^\bullet(0, r).$$

In particular, $(s(R_t), t \geq 0)$ is a strictly local martingale.

Proof : a) For the first statement, see Pitman-Yor (1981).

b) For the second statement, the existence of the limit and its equality to the left-hand side of (6) follows from (1) in Theorem 1, whereas

Le Gall ([1] (1986) ; Theorem 1.1, p. 1222)) expresses the limit (5) as the right-hand side of (6). \square

(2.2) The most standard example of a diffusion which satisfies the above hypothesis is the Bessel process with dimension $d = 2(1+\nu) > 2$, i.e : $\nu > 0$.

We then have : $s(\rho) = -\frac{1}{\rho^{d-2}}$,

and the identity (4), taken for $r = 0$, becomes :

$$(7) \quad P_0(L_\rho \in dt) = \left(\frac{\rho^2}{2}\right)^\nu \frac{dt}{\Gamma(\nu)t^{\nu+1}} \exp\left(-\frac{\rho^2}{2t}\right).$$

Now, the identity (6) may be written as :

$$(6') \quad E_r \left[\frac{s(R_t)}{s(r)} \right] = 1 - \left(\frac{-1}{2s(r)} \right) \int_0^t du p_u^\bullet(0, r) = P_0(L_r \geq t)$$

which, as a consequence of (7), becomes :

$$(8) \quad \int_0^\infty \frac{d\rho}{\rho^{2\nu}} p_t(r, \rho) = \frac{1}{2^\nu \Gamma(\nu)} \int_t^\infty \frac{du}{u^{1+\nu}} \exp\left(-\frac{r^2}{2u}\right),$$

$$\text{where} \quad p_t(r, \rho) = \frac{1}{t} \rho \left(\frac{\rho}{r}\right)^\nu \exp\left(-\frac{r^2 + \rho^2}{2t}\right) I_\nu\left(\frac{r\rho}{t}\right)$$

is the density of the semi-group $Q_t(r, d\rho)$ with respect to $d\rho$.

Easy changes of variables then show that (8) is equivalent to

$$(8') \quad \int_0^\infty \frac{\xi d\xi}{\xi^\nu a^{\nu-1}} \exp\left(-\frac{a^2}{2}(1+\xi^2)\right) I_\nu(a\xi) = \frac{1}{2^\nu \Gamma(\nu)} \int_0^1 \frac{du}{u} u^\nu \exp\left(-\frac{a^2 u}{2}\right)$$

and also to :

$$(8'') \quad \int_0^\infty \eta d\eta \exp\left(-\frac{\eta^2}{2}\right) \frac{I_\nu(\sqrt{a}\eta)}{(\sqrt{a}\eta)^\nu} = \frac{1}{2^\nu \Gamma(\nu)} \int_0^1 dv (1-v)^{\nu-1} \exp\left(\frac{a}{2}v\right).$$

This identity (8'') may be verified by developing both sides as a series expansion in powers of a with the help, for the left-hand side, of the classical formula :

$$(9) \quad I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^\infty \frac{1}{n! \Gamma(\nu+n+1)} \left(\frac{z}{2}\right)^{2n}.$$

In any case, the identity (8'') is a particular case of the Lipschitz-Hankel integrals ; see, e.g., chap. XIII of Watson [6], formula 3, p. 394, which gives a formula for :

$$\int_0^\infty dt t^{\mu-1} e^{-p^2 t^2} J_\nu(at),$$

with the help of the F_1 hypergeometric functions ; such formulae are also found in Lebedev ([2], p. 278, Exercise 12).

For clarity and future reference, we write again the equalities (2) and (6) in the particular case where $M_t = \frac{1}{R_t^{d-2}}$, under $P_r^{(\nu)}$, the law of R , starting from $r > 0$.

Proposition 1 : *The 4 following quantities are equal :*

$$\begin{aligned}
& \lim_{\alpha \rightarrow \infty} \left\{ \alpha P_r^{(\nu)} \left(\frac{1}{(\inf_{s \leq t} R_s^{d-2})} \geq \alpha \right) \right\} \\
& \lim_{y \rightarrow \infty} \left\{ \sqrt{\frac{\pi}{2}} y P_r^{(\nu)} \left((d-2) \left(\int_0^t \frac{ds}{R_s^{2(d-1)}} \right) \geq y \right) \right\} \\
(11) \quad E_r^{(\nu)} \left(\frac{1}{r^{d-2}} - \frac{1}{R_t^{d-2}} \right) &= \frac{1}{2^\nu \Gamma(\nu)} \int_0^t \frac{du}{u^{1+\nu}} \exp \left(-\frac{r^2}{2u} \right).
\end{aligned}$$

(2.3) Associated with the 2-dimensional Bessel process $(R_t, t \geq 0)$, starting from $r > 0$, there is also the strictly local martingale $M_t = \log \frac{1}{R_t}$, which satisfies the hypothesis of our Theorem 1'.

In order to obtain the corresponding values of the quantities in (1') in the present case, it suffices to divide both sides of (11) by (2ν) , and to let $\nu \rightarrow 0$.

Thus, we obtain the following

Proposition 2 : Let $P_r^{(0)}$ be the law of $(R_t, t \geq 0)$, the 2-dimensional Bessel process starting from $r > 0$. Then, the following 4 quantities are equal :

$$\begin{aligned}
& \lim_{\alpha \rightarrow \infty} \left\{ \alpha P_r^{(0)} \left(\log \frac{1}{(\inf_{s \leq t} R_s)} \geq \alpha \right) \right\} ; \lim_{y \rightarrow \infty} \left\{ \sqrt{\frac{\pi}{2}} y P_r^{(0)} \left(\left(\int_0^t \frac{ds}{R_s^2} \right)^{1/2} \geq y \right) \right\} \\
(12) \quad E_r^{(0)} \left[\log R_t - \log r \right] &= \frac{1}{2} \int_0^t \frac{du}{u} e^{-\frac{r^2}{2u}}.
\end{aligned}$$

We again remark that the identity (12), may be expressed as an integral identity involving the Bessel function I_0 , i.e. see (2.4) below.

On the other hand, if we particularize our argument in the proof of Theorem 1' concerning the quadratic variation of M , we obtain, in the present case :

$$(13) \quad \frac{1}{\nu} E_r^{(0)} \left\{ 1 - \exp \left(-\frac{\nu^2}{2} \int_0^t \frac{ds}{R_s^2} \right) \right\} \xrightarrow{\nu \rightarrow 0} E_r^{(0)} [\log R_t - \log r]$$

But, the following formula is known (see, e.g., Yor [8]) :

$$(14) \quad E_r^{(0)} \left\{ \exp \left(-\frac{\nu^2}{2} \int_0^t \frac{ds}{R_s^2} \right) \middle| R_t \right\} = \frac{I_\nu}{I_0} \left(\frac{r R_t}{t} \right).$$

Hence, it is deduced from (13) that :

$$(15) \quad E_r^{(0)} \left\{ - \frac{\partial}{\partial \nu} \Big|_{\nu=0} \left(\frac{I_\nu}{I_0} \right) \left(\frac{r R_t}{t} \right) \right\} = E_r^{(0)} \left[\log R_t - \log r \right].$$

It now follows from a classical integral representation of $I_\nu(\xi)$ that :

$$- \frac{\partial}{\partial \lambda} \Big|_{\lambda=0} I_\lambda(\xi) = \int_0^\infty du e^{-\xi(\cosh u)} \equiv K_0(\xi) \quad (11).$$

Hence, we deduce from (15) and the explicit formula for the semi-group

$Q_t^{(0)}(r, d\rho)$ that :

$$(16) \quad \frac{1}{t} \int_0^\infty d\rho \rho e^{-\frac{r^2 + \rho^2}{2t}} \int_0^\infty du e^{-\frac{r\rho}{t}(\cosh u)} = \frac{1}{2} \int_0^t \frac{du}{u} e^{-\frac{r^2}{2u}}.$$

(2.4) Some remarks following Proposition 2, and formulae (13) through (16).

i) We can write formula (12) in the form :

$$(12') \quad \int_0^\infty \frac{d\rho}{t} \rho \exp\left(-\frac{r^2 + \rho^2}{2t}\right) I_0\left(\frac{r\rho}{t}\right) (\log \rho - \log r) = \frac{1}{2} \int_0^t \frac{du}{u} e^{-(r^2/2u)}$$

and, as above with (11) and (12) may deduce this identity (12') from the corresponding identity involving I_ν , and deduced from (11).

ii) From (14), we can obtain a result similar to, but deeper than, (13), namely :

$$\lim_{\nu \rightarrow 0} \frac{1}{\nu} E_r^{(0)} \left[1 - \exp\left(-\frac{\nu^2}{2} \int_0^t \frac{ds}{R_s^2}\right) \Big|_{R_t = \rho} \right] = - \frac{\partial}{\partial \nu} \Big|_{\nu=0+} \frac{I_\nu(\xi)}{I_0(\xi)} = \frac{K_0(\xi)}{I_0(\xi)}$$

(11) In fact, the equality : $K_0(\xi) = - \frac{\partial}{\partial \lambda} \Big|_{\lambda=0} (I_\lambda(\xi))$ also follows immediately from the formula : $K_\lambda(\xi) = \frac{\pi}{2} \frac{I_{-\lambda}(\xi) - I_\lambda(\xi)}{\sin(\pi\lambda)}$ (see, e.g., formula (5.7.2),

p. 108 in Lebedev [2]). Also, the formula : $K_0(\xi) = \int_0^\infty du e^{-\xi(\cosh u)}$ is a

particular case of : $K_\nu(\xi) = \int_0^\infty du e^{-\xi(\cosh u)} \cosh(\nu u).$

where $\xi = \frac{r\rho}{t}$, on one hand, and, on the other hand, thanks to the Tauberian theorem, we find that the same quantity is equal to :

$$\sqrt{\frac{\pi}{2}} \lim_{y \rightarrow \infty} \left\{ y P_r^0 \left(\left(\int_0^t \frac{ds}{R_s^2} \right)^{1/2} \geq y \mid R_t = \rho \right) \right\}$$

(2.5) An important part of the results presented in Propositions 1 and 2 is well-known ; in fact, some of these results, namely those concerning

$$\lim_{\alpha \rightarrow \infty} \left\{ \alpha P \left(\sup_{s \leq T} M_s \geq \alpha \right) \right\}$$

in their applications to Bessel processes, form the core of the arguments of the proof of the main asymptotics of the Wiener sausage, i.e : Le Gall [1], Theorem 1.1, and, in part, Spitzer [5].

The results about the asymptotics of $\sqrt{\frac{\pi}{2}} y P(<M>_T^{1/2} \geq y)$ are perhaps less known, although they also appear in Werner [7].

An interesting consequence of Proposition 2 is the following

Corollary : Let $(\theta_u, u \geq 0)$ be a continuous determination of the argument around 0 of the 2-dimensional Brownian motion $(Z_u, u \geq 0)$, starting from $z_0 \neq 0$. Then, we have :

$$\lim_{\alpha \rightarrow \infty} \{ \alpha P(|\theta_t| \geq \alpha) \} = \lim_{\alpha \rightarrow \infty} \left\{ \alpha P \left(\sup_{s \leq t} \theta_s \geq \alpha \right) \right\} = \frac{1}{\pi} \int_0^t \frac{du}{u} \exp(-\frac{r^2}{2u})$$

$$\lim_{\alpha \rightarrow \infty} \left\{ \alpha P \left(\sup_{s \leq t} |\theta_s| \geq \alpha \right) \right\} = \frac{1}{2} \int_0^t \frac{du}{u} \exp(-\frac{r^2}{2u}).$$

Proof : Thanks to the skew-product representation of Z , there exists a 1-dimensional Brownian motion $(\gamma_t, t \geq 0)$, independent of $(R_u, u \geq 0)$ such that :

$$\theta_t = \gamma \left(\int_0^t \frac{ds}{R_s^2} \right).$$

Hence, we have : $|\theta_t| \stackrel{(\text{law})}{=} \sup_{s \leq t} \theta_s \stackrel{(\text{law})}{=} \left(\int_0^t \frac{ds}{R_s^2} \right)^{1/2} \left(\sup_{u \leq 1} \gamma_u \right) \stackrel{(\text{law})}{=} \left(\int_0^t \frac{ds}{R_s^2} \right)^{1/2} |\gamma_1|$

thanks to the reflection principle. Thus, we have :

$$\alpha P(|\theta_t| \geq \alpha) = \alpha P\left(\sup_{s \leq t} \theta_s \geq \alpha\right) = \alpha P\left(\left(\int_0^t \frac{ds}{R_s^2}\right)^{1/2} \geq \frac{\alpha}{|\gamma_1|}\right),$$

so that, by dominated convergence, we find :

$$\lim_{\alpha \rightarrow \infty} \alpha P\left(\sup_{s \leq t} \theta_s \geq \alpha\right) = E(|\gamma_1|) \lim_{\beta \rightarrow \infty} \beta P\left(\left(\int_0^t \frac{ds}{R_s^2}\right)^{1/2} \geq \beta\right)$$

and, since : $E(|\gamma_1|) = \sqrt{\frac{2}{\pi}}$, we obtain, from Proposition 2, that :

$$\lim_{\alpha \rightarrow \infty} \alpha P\left(\sup_{s \leq t} \theta_s \geq \alpha\right) = \frac{2}{\pi} \left(\frac{1}{2} \int_0^t \frac{du}{u} e^{-r^2/2u}\right).$$

Likewise, we obtain :

$$\begin{aligned} \lim_{\alpha \rightarrow \infty} \left\{ \alpha P\left(\sup_{s \leq t} |\theta_s| \geq \alpha\right) \right\} &= E[\gamma_1^*] \lim_{\beta \rightarrow \infty} \left\{ \beta P\left(\left(\int_0^t \frac{ds}{R_s^2}\right)^{1/2} \geq \beta\right) \right\} \\ &= \left(E[\gamma_1^*] \sqrt{\frac{2}{\pi}} \right) \left(\frac{1}{2} \int_0^t \frac{du}{u} e^{-r^2/2u} \right) \end{aligned}$$

and the desired result follows from the next

Lemma 2 : Define $\gamma_1^* = \sup_{s \leq 1} |\gamma_s|$. Then, one has : $E[\gamma_1^*] = \sqrt{\frac{\pi}{2}}$.

Proof : Define $\hat{T}_1 = \inf\{t : |\gamma_t| = 1\}$. Then, from the scaling property of

Brownian motion, we deduce : $\gamma_1^* \stackrel{(law)}{=} 1/(\hat{T}_1)^{1/2}$, so that :

$$\begin{aligned} E[\gamma_1^*] &= E\left[\frac{1}{(\hat{T}_1)^{1/2}}\right] = \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty du u^{-1/2} E\left[e^{-u\hat{T}_1}\right] \\ &= \sqrt{\frac{2}{\pi}} \int_0^\infty dv E\left[e^{-\frac{v^2}{2}\hat{T}_1}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{dv}{(\text{ch } v)} \end{aligned}$$

$$\text{Now, we have : } \int_0^\infty \frac{dv}{(\text{ch } v)} = 2 \int_0^\infty \frac{e^{-v}}{(1+e^{-2v})} dv = 2 \int_0^1 \frac{dx}{1+x^2} = 2 \text{Arctg}(1) = \frac{\pi}{2},$$

so that, finally : $E[\gamma_1^*] = \sqrt{\frac{\pi}{2}}$. \square

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