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HOW LONG DOES IT TAKE A TRANSIENT BESSEL PROCESS TO REACH ITS FUTURE INFIMUM?

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Summary. We establish an iterated logarithm law for the location of the future infimum of a transient Bessel process.

1. Introduction.

Let $\{R(t); t \geq 0\}$ be a d -dimensional Bessel process, and let

$$(1.1) \quad \nu = \frac{d}{2} - 1,$$

be the “index” of R (see Revuz & Yor [R-Y] Chap. XI). When d is an integer, R can be realized as the radial part of an $\mathbb{R}^d$ -valued Brownian motion. We refer to [R-Y] (Chap. XI) for a detailed account of general properties of Bessel processes. It is known ([R-Y] p.423) that R is transient (i.e. $\lim_{t \rightarrow \infty} R(t) = \infty$ almost surely) if and only if $d > 2$. Unless stated otherwise, this condition will be taken for granted throughout the note.

Define for $t > 0$,

$$\xi(t) = \inf\{u \geq t : R(u) = \inf_{s \geq t} R(s)\}.$$

In words, for any given $t > 0$, $\xi(t)$ denotes the (almost surely unique) location of

the infimum of R over $[t, \infty)$. Such random times have been first studied by Williams ([W1] and [W2]), who proved a path decomposition theorem at $\xi(t)$ respectively in case of Brownian motion and linear diffusions. Generalizations of Williams' result have since been established for Lévy and more general Markov processes. See for example Millar [M], Pitman [P], Bertoin [B] and Chaumont [C], and the references therein.

This note is concerned with $\xi(t)$ as a process of t , and more particularly, we are interested in the path property of $t \mapsto \xi(t)$. Of course, it is meaningless to study its liminf behaviour, since there are infinitely many large t 's such that $\xi(t) = t$. Instead, we ask: what can be said about the limsup behaviour of $\xi(t)$?

Theorem 1. *For any non-decreasing function $f > 0$, we have*

$$\limsup_{t \rightarrow \infty} \frac{\xi(t)}{t f(t)} = 0 \quad \text{or} \quad \infty, \quad \text{a.s.,}$$

according as

$$\int^{\infty} \frac{dt}{t f^{\nu}(t)}$$

converges or diverges, where ν is defined in (1.1).

Remark. In case $R(0) = 0$, there is also a “local” version of Theorem 1 for small times t .

Theorem 1 is proved in Section 2. Some related problems are raised in Section 3.

2. Proof of Theorem 1.

Without loss of generality, we assume $R(0) = 0$. Throughout the note, $\{X(t); t \geq 0\}$ stands for a generic d -dimensional Bessel process starting from 1, independent of R , and we denote by V the (almost surely) unique time when X reaches the infimum over $(0, \infty)$. Observe that R and X almost have the same law, except that $R(0) = 0$ whereas $X(0) = 1$. The process X being a linear diffusion with scale function $-x^{-2\nu}$ (Revuz & Yor [R-Y] p.426), we obviously have

$$(2.1) \quad \mathbb{P}\left(\inf_{u \geq 0} X(u) < x\right) = x^{2\nu}, \quad 0 < x < 1.$$

In order to prove Theorem 1, some preliminary results are needed. In the sequel, $K > 1$, $K_1 > 1$ and $K_2 > 1$ denote unimportant finite constants. Their values, which may change from line to line, depend only on d .

Lemma 1. *For any $t \geq 1$, we have*

$$(2.2) \quad K^{-1}t^{-\nu} \leq \mathbb{P}(V > t) \leq Kt^{-\nu},$$

where ν is defined in (1.1).

Proof of Lemma 1. We have

$$\begin{aligned} \mathbb{P}(V > t) &= \mathbb{P}\left(\inf_{s \geq t} X(s) < \inf_{0 \leq u \leq t} X(u)\right) \\ &= \mathbb{E}\left[\mathbb{P}\left(\inf_{s \geq t} X(s) < \inf_{0 \leq u \leq t} X(u) \mid X(u); 0 \leq u \leq t\right)\right]. \end{aligned}$$

Given the value of $X(t)$, the post- t process $\{X(s+t); s \geq 0\}$ is a d -dimensional Bessel process starting from $X(t)$, independent of $\{X(u); 0 \leq u \leq t\}$. Thus by scaling and (2.1), we obtain

$$\mathbb{P}\left(\inf_{s \geq t} X(s) < \inf_{0 \leq u \leq t} X(u) \mid X(u); 0 \leq u \leq t\right) = \left(\frac{1}{X(t)} \inf_{0 \leq u \leq t} X(u)\right)^{2\nu}.$$

Consequently,

$$(2.3) \quad \mathbb{P}(V > t) = \mathbb{E}\left[\left(\frac{1}{X(t)} \inf_{0 \leq u \leq t} X(u)\right)^{2\nu}\right].$$

Since $\inf_{0 \leq u \leq t} X(u) \leq 1$, we have

$$\mathbb{E}\left[\left(\frac{1}{X(t)} \inf_{0 \leq u \leq t} X(u)\right)^{2\nu}\right] \leq \mathbb{E}(X^{-2\nu}(t)).$$

Applying a diffusion comparison theorem ([R-Y] Theorem IX.3.7) to square Bessel processes, it is seen that $X(t)$ is stochastically bigger than $R(t)$ (which is intuitively obvious, of course). Thus by scaling, this implies

$$\mathbb{P}(V > t) \leq \mathbb{E}(R^{-2\nu}(t)) = t^{-\nu} \mathbb{E}(R^{-2\nu}(1)),$$

which yields the upper bound in Lemma 1, since $\mathbb{E}(R^{-2\nu}(1)) < \infty$. To show the lower bound, observe that by (2.3), for any $\lambda > 0$,

$$\begin{aligned} \mathbb{P}(V > t) &\geq \mathbb{E}\left[\left(\frac{1}{X(t)} \inf_{0 \leq u \leq t} X(u)\right)^{2\nu} \mathbf{1}_{\{\inf_{u \geq 0} X(u) > 1/2; X(t) < \lambda\sqrt{t}\}}\right] \\ &\geq (2\lambda)^{-2\nu} t^{-\nu} \mathbb{P}\left(\inf_{u \geq 0} X(u) > 1/2; X(t) < \lambda\sqrt{t}\right) \\ &\geq (2\lambda)^{-2\nu} t^{-\nu} \left(\mathbb{P}\left(\inf_{u \geq 0} X(u) > 1/2\right) - \mathbb{P}(X(t) > \lambda\sqrt{t})\right). \end{aligned}$$

Since $\mathbb{P}(X(t) > \lambda\sqrt{t}) \leq \mathbb{P}(R(t) > \lambda\sqrt{t} - 1) = \mathbb{P}(X(1) > \lambda - 1/\sqrt{t})$, we can choose λ so large that this probability is smaller than $\frac{1}{2}\mathbb{P}(\inf_{u \geq 0} X(u) > 1/2)$. The lower bound in Lemma 1 is proved. $\square$

Lemma 1 will be used to obtain accurate estimates of the law of some functionals of ξ . Define for $r > 0$

$$\sigma(r) = \inf\{t > 0 : R(t) = r\},$$

the first hitting time of R at level r , which is (almost surely) finite. Since $R(0) = 0$, the scaling property immediately yields that for any given $r > 0$, $\sigma(r)$ has the same law as $r^2\sigma(1)$. For notational convenience, we write in the sequel

$$\begin{aligned}\sigma &\equiv \sigma(1); \\ \xi_\sigma &\equiv \xi(\sigma(1)).\end{aligned}$$

The random variables σ and ξ_σ play an important rôle in our proof of Theorem 1. Here we give a résumé of their basic properties. The equivalence for the lower tail of σ is known. Recall that $([G-S]) \lim_{s \rightarrow 0} s^\nu e^{1/(2s)} \mathbb{P}(\sigma < s) = 2^{1-\nu}/\Gamma(1+\nu)$. Therefore,

$$(2.4) \quad K^{-1}s^{-\nu} \exp\left(-\frac{1}{2s}\right) \leq \mathbb{P}\left(\sigma < s\right) \leq Ks^{-\nu} \exp\left(-\frac{1}{2s}\right), \quad 0 < s \leq 1.$$

The exact upper tail of σ , which involves Bessel functions and their positive zeros, was evaluated respectively by Ciesielski & Taylor [C-T] for integer dimensions d , and by Kent [Ke] and Ismail & Kelker [I-K] for any $d > 0$. Their result implies the following useful estimate for $x \geq 1$:

$$(2.5) \quad \mathbb{P}(\sigma > x) \leq \exp\left(-\frac{x}{K}\right).$$

For the variable ξ_σ , it follows from the strong Markov property of R that $\{R(\sigma + t); t \geq 0\}$ is a d -dimensional Bessel process starting from 1 (thus behaving like the process X), independent of σ . Since V is the location of the infimum of X over $(0, \infty)$, this yields:

$$(2.6) \quad \xi_\sigma - \sigma \text{ is independent of } \sigma;$$

$$(2.7) \quad \xi_\sigma - \sigma \stackrel{(d)}{=} V;$$

(“ $\stackrel{(d)}{=}$ ” denoting identity in distribution). Our next preliminary result is on the joint tail of ξ_σ and σ .

Lemma 2. *Let $x \geq 2$ and $y \geq 1$. Then*

$$(2.8) \quad \mathbb{P}\left(\frac{\xi_\sigma}{\sigma} > x\right) \leq Kx^{-\nu},$$

$$(2.9) \quad \mathbb{P}\left(\frac{\xi_\sigma}{\sigma} > x; y > \sigma > 1\right) \geq K^{-1}x^{-\nu} - e^{-y/K}.$$

Proof of Lemma 2. According to our notation, V is independent of R (thus of σ). We have, by (2.6) and (2.7),

$$\begin{aligned} \mathbb{P}\left(\frac{\xi_\sigma}{\sigma} > x\right) &= \mathbb{P}\left(V > (x-1)\sigma\right) \\ &= \mathbb{P}\left(V > (x-1)\sigma; \sigma \geq \frac{1}{x-1}\right) + \mathbb{P}\left(V > (x-1)\sigma; \sigma < \frac{1}{x-1}\right). \end{aligned}$$

Using (2.2) and (2.4), the above expression is

$$\begin{aligned} &\leq K_1(x-1)^{-\nu} \mathbb{E}\left(\sigma^{-\nu} \mathbf{1}_{\{\sigma \geq 1/(x-1)\}}\right) + \mathbb{P}\left(\sigma < \frac{1}{x-1}\right) \\ &\leq K_1(x-1)^{-\nu} \mathbb{E}(\sigma^{-\nu}) + K_1(x-1)^{-\nu} \exp\left(-\frac{x-1}{2}\right) \\ &\leq Kx^{-\nu}, \end{aligned}$$

the last inequality due to the fact that $\mathbb{E}(\sigma^{-\nu}) < \infty$ (this is easily seen from (2.4)). Therefore (2.8) is proved. To show (2.9), observe that by (2.6), (2.7), (2.2) and (2.5), we have

$$\begin{aligned} \mathbb{P}\left(\frac{\xi_\sigma}{\sigma} > x; y > \sigma > 1\right) &\geq \mathbb{P}\left(\frac{\xi_\sigma}{\sigma} > x; \sigma > 1\right) - \mathbb{P}\left(\sigma \geq y\right) \\ &= \mathbb{P}\left(V > (x-1)\sigma; \sigma > 1\right) - \mathbb{P}\left(\sigma \geq y\right) \\ &\geq K_1(x-1)^{-\nu} \mathbb{E}\left(\sigma^{-\nu} \mathbf{1}_{\{\sigma > 1\}}\right) - e^{-y/K} \\ &\geq K^{-1}x^{-\nu} - e^{-y/K}. \end{aligned}$$

Lemma 2 is proved. □

Lemma 3. *For any $x \geq 2$, we have*

$$\mathbb{P}\left(\xi(1) > x\right) \leq Kx^{-\nu}.$$

Proof of Lemma 3. Conditioning on $R(1) = x$, $\xi(1) - 1$ has the same distribution as x^2V (this is easily seen from the Markov and scaling properties of R). Thus by Lemma 1,

$$\begin{aligned} \mathbb{P}\left(\xi(1) > x\right) &= \mathbb{P}\left(R^2(1)V > x - 1\right) \\ &\leq \mathbb{P}\left(R(1) > \sqrt{x-1}\right) + \mathbb{P}\left(R^2(1)V > x - 1; R(1) \leq \sqrt{x-1}\right) \\ &\leq \mathbb{P}\left(\sup_{0 \leq t \leq 1} R(t) > \sqrt{x-1}\right) \\ &\quad + K_1(x-1)^{-\nu} \mathbb{E}\left(R^{-2\nu}(1) \mathbf{1}_{\{R(1) \leq \sqrt{x-1}\}}\right). \end{aligned}$$

Since $\mathbb{E}(R^{-2\nu}(1)) < \infty$, the proof of Lemma 3 is reduced to showing the following estimate:

$$(2.10) \quad \mathbb{P}\left(\sup_{0 \leq t \leq 1} R(t) > \sqrt{x-1}\right) \leq K_2 x^{-\nu}.$$

This is easily verified. Indeed, by scaling, we have, for any $\lambda > 0$,

$$\mathbb{P}\left(\sup_{0 \leq t \leq 1} R(t) > \lambda\right) = \mathbb{P}\left(\sup_{0 \leq t \leq 1/\lambda^2} R(t) > 1\right) = \mathbb{P}\left(\sigma < \frac{1}{\lambda^2}\right).$$

Taking $\lambda = \sqrt{x-1}$ and using (2.4), we obtain

$$\begin{aligned} \mathbb{P}\left(\sup_{0 \leq t \leq 1} R(t) > \sqrt{x-1}\right) &= \mathbb{P}\left(\sigma < \frac{1}{x-1}\right) \\ &\leq K(x-1)^\nu \exp\left(-\frac{x-1}{2}\right) \\ &\leq K_2 x^{-\nu}, \end{aligned}$$

which yields (2.10). □

Proof of Theorem 1. We begin with the convergent part. Let $f > 0$ be non-decreasing such that $\int^\infty dt/t f^\nu(t) < \infty$. Thus f increases to infinity. Choose a large initial value n_0 and define $t_n = e^n$ for $n \geq n_0$. By scaling and Lemma 3, we have

$$\begin{aligned} \mathbb{P}\left(\xi(t_{n+1}) > t_n f(t_n)\right) &= \mathbb{P}\left(\xi(1) > \frac{t_n}{t_{n+1}} f(t_n)\right) \\ &= \mathbb{P}\left(\xi(1) > \frac{1}{e} f(t_n)\right) \\ &\leq K f^{-\nu}(t_n). \end{aligned}$$

Since

$$\begin{aligned} \sum_{n=n_0+1}^{\infty} f^{-\nu}(t_n) &= \sum_{n=n_0+1}^{\infty} \int_{t_{n-1}}^{t_n} \frac{dt}{t_n - t_{n-1}} f^{-\nu}(t_n) \leq \frac{e}{e-1} \sum_{n=n_0+1}^{\infty} \int_{t_{n-1}}^{t_n} \frac{dt}{t} f^{-\nu}(t) \\ &= \frac{e}{e-1} \int_{t_{n_0}}^{\infty} \frac{dt}{t} f^{-\nu}(t) < \infty, \end{aligned}$$

the Borel–Cantelli lemma tells us that

$$\limsup_{n \rightarrow \infty} \frac{\xi(t_{n+1})}{t_n f(t_n)} \leq 1 \quad \text{a.s.}$$

Since replacing f by a multiple of f does not change the test, an argument by monotonicity readily yields $\limsup_{t \rightarrow \infty} \xi(t)/t f(t) = 0$ almost surely. To verify the divergent part of Theorem 1, pick an f such that $\int^{\infty} dt/t f^{\nu}(t)$ diverges. Obviously we only have to treat the case that $f(\infty) = \infty$. Choose a large k_0 , and define $t_k = e^k$ as before (it will be seen that t_k is rather a space variable than a time variable, but the notation should not cause any trouble). We shall consider a sequence of *random* times in order to avoid dependence difficulty. Let

$$E_k = \left\{ \xi(\sigma(t_k)) > \sigma(t_k) f(t_k^3); t_k^3 > \sigma(t_k) > t_k^2 \right\},$$

for $k \geq k_0$. By scaling and (2.9), we have

$$\mathbb{P}(E_k) = \mathbb{P}\left(\frac{\xi_{\sigma}}{\sigma} > f(t_k^3); t_k > \sigma > 1\right) \geq K^{-1} f^{-\nu}(t_k^3) - \exp(-t_k/K).$$

Accordingly,

$$(2.11) \quad f^{-\nu}(t_k^3) \leq K \mathbb{P}(E_k) + K e^{-t_k/K}.$$

Since $\sum_{k=k_0}^{\infty} f^{-\nu}(t_k^3) = \infty$ and $\sum_k e^{-t_k/K} < \infty$, the above estimate clearly implies

$$(2.12) \quad \sum_k \mathbb{P}(E_k) = \infty.$$

To apply the Borel–Cantelli lemma, we need to check that the measurable events E_k are almost independent. Let $k_0 \leq k < \ell$. Denote by $\xi(s, t)$ the time when R reaches its minimum over (s, t) (thus $\xi(t) = \xi(t, \infty)$ according to our notation). Write

$$\begin{aligned} \mathbb{P}(E_k E_{\ell}) &= \mathbb{P}\left(E_k; E_{\ell}; \xi(\sigma(t_k)) < \sigma(t_{\ell})\right) + \mathbb{P}\left(E_k; E_{\ell}; \xi(\sigma(t_k)) \geq \sigma(t_{\ell})\right) \\ &\equiv \Delta_1 + \Delta_2, \end{aligned}$$

with obvious notation. Then

$$\begin{aligned}
\Delta_1 &\leq \mathbb{P}\left(\xi(\sigma(t_k), \sigma(t_\ell)) > \sigma(t_k)f(t_k^3); \sigma(t_k) > t_k^2; \right. \\
&\quad \left. \xi(\sigma(t_\ell)) - \sigma(t_\ell) > \sigma(t_\ell)(f(t_\ell^3) - 1); \sigma(t_\ell) > t_\ell^2\right) \\
&\leq \mathbb{P}\left(\xi(\sigma(t_k), \sigma(t_\ell)) - \sigma(t_k) > t_k^2(f(t_k^3) - 1); \right. \\
&\quad \left. \xi(\sigma(t_\ell)) - \sigma(t_\ell) > t_\ell^2(f(t_\ell^3) - 1)\right).
\end{aligned}$$

Using the strong Markov property of R , it is seen that $\xi(\sigma(t_k), \sigma(t_\ell))$ is independent of $\xi(\sigma(t_\ell)) - \sigma(t_\ell)$. Thus by scaling and (2.7),

$$\begin{aligned}
\Delta_1 &\leq \mathbb{P}\left(\xi(\sigma(t_k), \sigma(t_\ell)) - \sigma(t_k) > t_k^2(f(t_k^3) - 1)\right) \mathbb{P}\left(\xi_\sigma - \sigma > f(t_\ell^3) - 1\right) \\
&\leq \mathbb{P}\left(\xi(\sigma(t_k)) - \sigma(t_k) > t_k^2(f(t_k^3) - 1)\right) \mathbb{P}\left(V > f(t_\ell^3) - 1\right) \\
&= \mathbb{P}\left(V > f(t_k^3) - 1\right) \mathbb{P}\left(V > f(t_\ell^3) - 1\right) \\
(2.13) \quad &\leq K_1 f^{-\nu}(t_k^3) f^{-\nu}(t_\ell^3),
\end{aligned}$$

the last inequality following from Lemma 1. Now let us evaluate Δ_2 . Clearly we have

$$\begin{aligned}
\Delta_2 &\leq \mathbb{P}\left(\xi(\sigma(t_k)) > \sigma(t_\ell)f(t_\ell^3); \sigma(t_\ell) > t_\ell^2\right) \\
&\leq \mathbb{P}\left(\xi(\sigma(t_k)) - \sigma(t_k) > \sigma(t_\ell)(f(t_\ell^3) - 1); \sigma(t_\ell) > t_\ell^2\right) \\
&\leq \mathbb{P}\left(\xi(\sigma(t_k)) - \sigma(t_k) > t_\ell^2(f(t_\ell^3) - 1)\right),
\end{aligned}$$

which, by scaling and (2.7) and (2.2), implies

$$\begin{aligned}
\Delta_2 &\leq \mathbb{P}\left(\xi_\sigma - \sigma > \left(\frac{t_\ell}{t_k}\right)^2 (f(t_\ell^3) - 1)\right) \leq K_1 \left(\frac{t_k}{t_\ell}\right)^{2\nu} f^{-\nu}(t_\ell^3) \\
(2.14) \quad &\leq K_1 f^{-\nu}(t_k^3) e^{-2\nu(\ell-k)}.
\end{aligned}$$

Since $\mathbb{P}(E_k E_\ell) = \Delta_1 + \Delta_2$, combining (2.13), (2.14) and (2.11) gives

$$\begin{aligned}
\mathbb{P}(E_k E_\ell) &\leq K_2 (\mathbb{P}(E_k) + e^{-t_k/K}) (\mathbb{P}(E_\ell) + e^{-t_\ell/K}) \\
&\quad + K_2 (\mathbb{P}(E_k) + e^{-t_k/K}) e^{-2\nu(\ell-k)}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\sum_{k_0 \leq k < \ell \leq n} \mathbb{P}(E_k E_\ell) &\leq K_2 \left(\sum_{k=k_0}^n (\mathbb{P}(E_k) + e^{-t_k/K}) \right)^2 \\
&\quad + \frac{K_2}{1 - e^{-2\nu}} \sum_{k=k_0}^n (\mathbb{P}(E_k) + e^{-t_k/K}).
\end{aligned}$$

Since $\sum_k \mathbb{P}(E_k) = \infty$ and $\sum_k e^{-t_k/K} < \infty$, this yields

$$\limsup_{n \rightarrow \infty} \sum_{k=k_0}^n \sum_{\ell=k_0}^n \mathbb{P}(E_k E_\ell) \bigg/ \left(\sum_{k=k_0}^n \mathbb{P}(E_k) \right)^2 \leq K_1 < \infty.$$

According to a well-known version of the Borel–Cantelli lemma (cf. [K-S]), this implies $\mathbb{P}(E_k, \text{ i.o. }) \geq 1/K_1$. In particular, we have

$$\mathbb{P} \left(\limsup_{t \rightarrow \infty} \frac{\xi(t)}{t f(t)} \geq 1 \right) \geq \frac{1}{K_1}.$$

Using Bessel time inversion (which tells that $\{tR(1/t); t > 0\}$ is again a Bessel process of dimension d) and Blumenthal’s 0–1 law, this probability equals 1. Since replacing f by a multiple of f does not change the test, we have established the divergent part of Theorem 1. $\square$

3. Some related problems.

3.1. Theorem 1 is concerned with the location of the future infimum of R . A natural question is to study also the location of the past supremum of a Bessel process R of dimension $d > 0$. Let

$$\eta(t) = \sup \left\{ u \leq t : R(u) = \sup_{0 \leq s \leq t} R(s) \right\}.$$

Thus $\eta(t)$ is the location of the maximum of R over $[0, t]$. Of course there are infinitely many large t ’s such that $\eta(t) = t$. What about the liminf behaviour of $\eta(t)$? In case $d = 1$ (thus R is a reflecting Brownian motion), the answer to this question can be found in Csáki, Földes & Révész [Cs-F-R]:

$$(3.1) \quad \liminf_{t \rightarrow \infty} \frac{(\log \log t)^2}{t} \eta(t) = \frac{\pi^2}{4} \quad \text{a.s.}$$

The corresponding problem for arbitrary dimension d remains open. Nonetheless, some heuristic arguments suggest that the following Chung-type law of the iterated logarithm might hold:

Conjecture. *For any $d > 0$, we have*

$$\liminf_{t \rightarrow \infty} \frac{(\log \log t)^2}{t} \eta(t) = j_\nu^2 \quad \text{a.s.,}$$

where j_ν denotes the smallest positive zero of the Bessel function J_ν of index $\nu \equiv d/2 - 1$.

If the above Conjecture is true, by taking $d = 1$ we would recover (3.1), since $j_{-1/2} = \pi/2$.

3.2. There has been several recent papers devoted to the so-called Bessel gap, i.e. the difference between the past supremum and future infimum of R ($d > 2$). See for example Khoshnevisan [Kh]. It also seems interesting to investigate the difference between the locations of the past supremum and future infimum of R , i.e. we propose to study the process $t \mapsto \xi(t) - \eta(t)$. Since $\eta(t) \leq t$, it is seen that $\xi(t) - \eta(t)$ have the same upper functions as $\xi(t)$, i.e. Theorem 1 holds for $\xi(t) - \eta(t)$ in the place of $\xi(t)$.

What about the liminf behaviour of $\xi(t) - \eta(t)$? Obviously for any $t > 0$, $\xi(t) - \eta(t)$ is (strictly) positive. A little more thinking convinces that with probability one,

$$\liminf_{t \rightarrow \infty} (\xi(t) - \eta(t)) = 0.$$

It would therefore be natural to look for a liminf iterated logarithm law for $t \mapsto \xi(t) - \eta(t)$. This problem is raised by Omer Adelman (personal communication).

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