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Onsager–Machlup Functionals for Solutions of Stochastic Boundary Value Problems

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Abstract. The purpose of this paper is to compute the asymptotic probability that the solution of a stochastic differential equation with boundary conditions belongs to a small tube of radius $\epsilon > 0$ centered around the solution of the deterministic equation without drift.

1 Introduction

Let $\{X_t, 0 \le t \le 1\}$ be a continuous stochastic process defined on the Wiener space $(\Omega, \mathcal{F}, P)$. Given a smooth function ϕ belonging to the support of the law of X, we are interested in the asymptotic behaviour as ϵ tends to zero of the probability that X belongs to a tube of radius ϵ around ϕ.

If X is the standard d-dimensional Wiener process, and ϕ has a square integrable derivative, we know that the probability $P(\|X-\phi\|_\infty < \epsilon)$ is equivalent to

$$c_1 e^{-\lambda_1/\epsilon^2} \exp\left(-\frac{1}{2}\int_0^1 \dot{\phi}_s^2 ds\right),$$

as ϵ tends to zero, where c_1 is a constant, and λ_1 is the first eigenvalue of the Laplacian in the unit ball. More generally, if X is the Wiener process with drift b, then we have

$$P(\|X-\phi\|_\infty < \epsilon) \sim c_1 e^{-\lambda_1/\epsilon^2} \exp\left(\int_0^1 L(\dot{\phi}_s, \phi_s) ds\right), \quad \epsilon \downarrow 0, \tag{1.1}$$

where

$$L(x,y) = -\frac{1}{2}|b(y) - x|^2 - \frac{1}{2}(\operatorname{div} b)(y).$$

This is true if the drift b belongs to $C_b^2(\mathbf{R}^d)$, and ϕ is an arbitrary function in the Cameron–Martin space (cf. Ikeda and Watanabe [3]) and [10]).

The functional $L(\dot{\phi}, \phi)$ appearing in (1.1) is called the Onsager–Machlup functional of the process X. In [1] we have computed this functional for the Brownian motion with drift, and assuming that the initial value is a random functional of the Brownian path. In this paper our aim is to compute the functional $L(\dot{\phi}, \phi)$ when the process

X satisfies a stochastic differential equation with boundary conditions, of the type studied in the references [7] and [8]. Like in [1], the main ingredient in computing the Onsager–Machlup functional will be a noncausal version of Girsanov theorem, due to Kusuoka (cf. [4]). The next two sections will be devoted to the computation of the Onsager–Machlup functional for first order and second order stochastic differential equations, respectively.

2 Onsager–Machlup functional for stochastic differential equations with boundary conditions

Let $\omega = \{\omega_t, 0 \le t \le 1\}$ be a d–dimensional Brownian motion defined on the canonical probability space $(\Omega, \mathcal{F}, P)$, where $\Omega = C_0([0,1], \mathbf{R}^d)$. Suppose that $f, g : \mathbf{R}^d \to \mathbf{R}^d$ are two continuous functions. Consider the following stochastic differential equation:

$$\begin{cases} X_t = X_0 - \int_0^t f(X_s)ds + \omega_t \quad , \quad 0 \le t \le 1 \\ X_0 = g(X_1 - X_0) \end{cases} \tag{2.1}$$

That means, instead of giving the initial condition X_0, we impose some nonlinear relation between X_0 and X_1. This functional relation could have been written in the more general form $h(X_0, X_1) = 0$, but the above condition is more suitable for our purpose. The existence and uniqueness of a solution to equation (2.1), and the Markov property of this process have been studied in [7].

Using Theorem 2.3 below, it is easy to prove that the support of the law of X is the set of continuous functions u in $C([0,1], \mathbf{R}^d)$ satisfying the boundary condition $u_0 = g(u_1 - u_0)$. Then, smooth functions in the support are of the form $\phi - g(\phi_1)$ where ϕ belongs to the Cameron–Martin space H^1. We recall that H^1 is the subspace of functions $\phi_t = \int_0^t \dot{\phi}_s ds$, and $\dot{\phi} \in L^2([0,1], \mathbf{R}^d)$.

Given an arbitrary function ϕ in H^1, we are interested in the evaluation of the asymptotic behaviour as ϵ tends to zero of the probability that X belongs to a tube of radius ϵ around a smooth functions of the support, namely

$$J^\epsilon(\phi) = P\left(\|X - \phi - g(\phi_1)\|_\infty < \epsilon\right), \tag{2.2}$$

where $\|\cdot\|_\infty$ denotes the supremum norm in Ω. This will provide the computation of the Onsager–Machlup functional.

In order to compute the asymptotic behaviour of $J^\epsilon(\phi)$, we will apply a generalized version of Girsanov theorem that has been used in [7] to study of the Markov property. For this purpose we introduce the process $Y = \{Y_t, 0 \le t \le 1\}$ which is the solution to equation (2.1) when f is zero. That means,

$$\begin{cases} Y_t = Y_0 + \omega_t, \quad , 0 \le t \le 1 \\ Y_0 = g(Y_1 - Y_0) \end{cases} \tag{2.3}$$

Clearly, $Y_t = \omega_t + g(\omega_1)$.

We define the transformation $T : \Omega \to \Omega$ by

$$\begin{aligned} T(\omega)_t &= \omega_t + \int_0^t f(\omega_s + g(\omega_1))ds \\ &= \omega_t + \int_0^t f(Y_s(\omega))ds. \end{aligned} \tag{2.4}$$

We will assume the following conditions.

(h1) $f, g : \mathbf{R}^d \to \mathbf{R}^d$ are continuously differentiable functions.

(h2) The transformation T given by (2.4) is bijective.

(h3) We have

$$\det(I - \Phi_1 g'(\omega_1) + g'(\omega_1)) \neq 0,$$

a.s., where $\{\Phi_t, 0 \leq t \leq 1\}$ is the fundamental solution of the linear system

$$\begin{cases} d\Phi_t = -f'(Y_t)\Phi_t dt \\ \Phi_0 = I \end{cases}$$

In [7] some sufficient conditions on the functions f and g are given for the hypotheses (h1) to (h3) to hold. An example of such conditions would be:

a) f is a monotone function of class C^1 such that

$$\lim_{a \uparrow +\infty} \frac{1}{a} \sup_{|x| \leq a} |f(x)| = 0.$$

b) g is a monotone function of class C^1 and there exists a constant $C > 0$ such that $|g(x)| \leq C(1 + |x|)$.

The bijective property of T is equivalent to the existence and uniqueness of the solution to equation (2.1).

As it is proved in [7, Theorem 3.6], under hypotheses **(h1)** to **(h3)** one can apply a generalized version of Girsanov theorem due to Kusuoka [4]:

Proposition 2.1 *Suppose that hypotheses* **(h1)** *to* **(h3)** *hold. There exists a probability Q on $(\Omega, \mathcal{F})$ such that $Q \circ T^{-1} = P$ (namely, $T(\omega)$ is a Brownian motion under Q), and Q is given by*

$$\frac{dQ}{dP} = |\det(I - \Phi_1 g'(\omega_1) + g'(\omega_1))| \times \exp\left\{\frac{1}{2}\int_0^1 \mathrm{Tr} f'(Y_t)dt - \int_0^1 f(Y_t) \circ d\omega_t - \frac{1}{2}\int_0^1 |f(Y_t)|^2 dt\right\},$$

where $\int_0^1 f(Y_t) \circ d\omega_t$ denotes the extended Stratonovich integral (cf. [6]).

We can state now the main result of this section.

Theorem 2.1 *Suppose that the above hypotheses* **(h1)** *to* **(h3)** *hold, and, in addition, the following conditions are satisfied:*

(h4) $f \in C_b^2(\mathbf{R}^d; \mathbf{R}^d)$.

(h5) $\det(I - g'(x)) \neq 0$, *a.e.*

(h6) *Either g is linear ($g(x) = Gx$) or ϕ is of class C_b^2.*

Then, for any $\phi \in H^1$, we have

$$P\left(\|X-\phi-g(\phi_1)\|_\infty<\epsilon\right) \sim L_1(\phi_1)e^{-\frac{\lambda_1}{\epsilon^2}}$$
$$\times|\det(I-\Phi_1(\phi)g'(\phi_1)+g'(\phi_1))|$$
$$\times\exp\left\{\frac{1}{2}\int_0^1 \mathrm{Tr} f'(\phi_t+g(\phi_1))dt-\frac{1}{2}\int_0^1|f(\phi_t+g(\phi_1))+\dot{\phi}_t|^2dt\right\},$$

as $\epsilon \downarrow 0$, where λ_1 is the first eigenvalue of the Laplacian in the unit ball, f_1 is the associated eigenfunction, and

$$L_1(\phi_1)=\int_{\{|g'(\phi_1)u|<1,|(I+g'(\phi_1))u|<1\}} f_1(g'(\phi_1)u)f_1((I+g'(\phi_1))u)du.$$

Proof: Fix a function $\phi \in H^1$. Consider the transformation $T^\phi:\Omega\to\Omega$ defined by

$$\begin{aligned} T^\phi(\omega)_t &= \omega_t+\phi_t+\int_0^t f(\omega_s+\phi_s+g(\omega_1+\phi_1))ds \qquad (2.5)\\ &= \omega_t+\phi_t+\int_0^t f(Y_s(\omega+\phi))ds. \end{aligned}$$

In other words, $T^\phi(\omega)=T(\omega+\phi)$. Applying Proposition 2.2 and the ordinary Girsanov theorem one can show that there exists a probability Q^ϕ on $(\Omega,\mathcal{F})$ such that $Q^\phi\circ(T^\phi)^{-1}=P$ (namely, $T^\phi(\omega)$ is a Brownian motion under Q^ϕ) and

$$\frac{dQ^\phi}{dP}=|\det(I-\Phi_1(\omega+\phi)g'(\omega_1+\phi_1)+g'(\omega_1+\phi_1))|$$
$$\times\exp\left\{\frac{1}{2}\int_0^1 \mathrm{Tr} f'(Y_t(\omega+\phi))dt-\int_0^1 f(Y_t(\omega+\phi))\circ d\omega_t\right.$$
$$\left.-\int_0^1\dot{\phi}_t d\omega_t-\frac{1}{2}\int_0^1|f(Y_t(\omega+\phi))+\dot{\phi}_t|^2dt\right\}.$$

The process $Y(\omega+\phi)$ satisfies the equation

$$\left\{\begin{array}{ll} Y_t(\omega+\phi)=Y_0(\omega+\phi)+T^\phi(\omega)_t-\int_0^t f(Y_s(\omega+\phi))ds, & 0\le t\le 1\\ Y_0=g(Y_1-Y_0) & \end{array}\right. \qquad (2.6)$$

Therefore, the law of the process X under the original probability P coincides with the law of the process $Y(\omega+\phi)$ under the probability Q^ϕ. This allows us to write the functional $J^\epsilon(\phi)$ in the following form

$$\begin{aligned} J^\epsilon(\phi) &= P\left(\|X-\phi-g(\phi_1)\|_\infty<\epsilon\right)\\ &= Q^\phi(\|Y(\omega+\phi)-\phi-g(\phi_1)\|_\infty<\epsilon)\\ &= Q^\phi(\|\omega+g(\omega_1+\phi_1)-g(\phi_1)\|_\infty<\epsilon). \end{aligned}$$

Define the set

$$G_\epsilon=\{\|\omega+g_\phi(\omega_1)\|_\infty<\epsilon\},$$

where $g_\phi(x)=g(x+\phi_1)-g(\phi_1)$. Then $J^\epsilon(\phi)$ can be written as

$$\begin{aligned} J^\epsilon(\phi) &= E\left(\frac{dQ^\phi}{dP}|G_\epsilon\right)P(G_\epsilon)\\ &= a_1(\epsilon)a_2(\epsilon). \end{aligned}$$

In order to prove the theorem it suffices to prove the following facts:

(C1)

$$\lim_{\epsilon\downarrow 0} a_1(\epsilon) = |\det(I-(\Phi_1(\phi)-I)g'(\phi_1)| \times \exp\left\{\frac{1}{2}\int_0^1 \mathrm{Tr} f'(Y_t(\phi)dt - \frac{1}{2}\int_0^1 |f(Y_t(\phi)) + \dot{\phi}_t|^2 dt\right\}.$$

(C2) $\lim_{\epsilon\downarrow 0} e^{\frac{\lambda_1}{\epsilon^2}} a_2(\epsilon) = L_1(\phi_1)$.

Proof of (C1): Consider the following Wiener functionals:

$$A_1(\omega) = |\det(I-(\Phi_1(\omega)-I)g'(\omega_1)| \times \exp\left\{\frac{1}{2}\int_0^1 \mathrm{Tr} f'(Y_t(\omega)dt - \frac{1}{2}\int_0^1 |f(Y_t(\omega)) + \dot{\phi}_t|^2 dt\right\}.$$

$$A_2(\omega) = \exp\left\{-\int_0^1 f(Y_t(\omega+\phi)) \circ d\omega_t\right\}$$

$$A_3(\omega) = \exp\left\{-\int_0^1 \dot{\phi}_t d\omega_t\right\}.$$

We have

$$\begin{aligned} &|E[(a_1(\epsilon) - A_1(\phi))|G_\epsilon]| \\ &\quad = |E[(A_1(\omega+\phi)A_2(\omega)A_3(\omega) - A_1(\phi)) \mid G_\epsilon]| \\ &\le \left(E\left[(A_1(\omega+\phi) - A_1(\phi))^2 \mid G_\epsilon\right]\right)^{1/2} \left(E[A_2(\omega)^2 A_3(\omega)^2 \mid G_\epsilon]\right)^{1/2} \\ &+ A_1(\phi)\, |E[A_2(\omega)A_3(\omega) \mid G_\epsilon] - 1|. \end{aligned}$$

Therefore, the proof of the convergence (C1) reduces to establish the following three facts:

$$\lim_{\epsilon\downarrow 0} E\left[(A_1(\omega+\phi) - A_1(\phi))^2 \mid G_\epsilon\right] = 0, \tag{2.7}$$

$$\sup_{\epsilon>0} E[(A_2A_3)^2 \mid G_\epsilon] < \infty, \tag{2.8}$$

$$\lim_{\epsilon\downarrow 0} E[A_2A_3 \mid G_\epsilon] = 1. \tag{2.9}$$

The convergence (2.7) is immediate because A_1 is a continuous functional of ω, and on the set G_ϵ we have $|g(\omega_1+\phi_1) - g(\phi_1)| < \epsilon$ and, therefore, $\|\omega\|_\infty < 2\epsilon$. We are going to prove the relation (2.9) and the proof of (2.8) would follow the same lines. Using the arguments of Ikeda and Watanabe ([3], page 449), which are still true if we condition by the set G_ϵ instead of $\{\|\omega\|_\infty < \epsilon\}$, in order to prove (2.9) it suffices to show that for any c in $\mathbf{R}$, we have

$$\limsup_{\epsilon\downarrow 0} E\left[\exp\left(c\int_0^1 f(Y_t(\omega+\phi)) \circ d\omega_t\right) \mid G_\epsilon\right] \le 1, \tag{2.10}$$

$$\limsup_{\epsilon\downarrow 0} E\left[\exp\left(c\int_0^1 \dot{\phi}_t d\omega_t\right) \mid G_\epsilon\right] \le 1. \tag{2.11}$$

The inequality (2.11) follows from the results of Shepp and Zeitouni (cf. [10]) if we assume that the function g is linear. In fact, in this case the set G_ϵ is convex and symmetric, and we can use the arguments based on the correlation inequalities. On the other hand, this inequality is also true by an integration by parts argument if the function ϕ is of class C_b^2.

In order to show the inequality (2.10) we will use the same arguments as in [1]. We write

$$f(Y_t(\omega+\phi)) = f(\phi_t + g(\omega_1+\phi_1)) + f'(\phi_t + g(\omega_1+\phi_1))\omega_t + R_t,$$

and hypotheses (h4) implies that $|R_t| \le C\epsilon^2$ on the set G_ϵ. Then, using again the arguments of Ikeda and Watanabe, it suffices to show that

$$\limsup_{\epsilon\downarrow 0} E\left[\exp\left(c\int_0^1 f(\phi_t + g(\omega_1+\phi_1)) \circ d\omega_t\right) \mid G_\epsilon\right] \le 1, \tag{2.12}$$

$$\limsup_{\epsilon\downarrow 0} E\left[\exp\left(c\int_0^1 f'(\phi_t + g(\omega_1+\phi_1))\omega_t \circ d\omega_t\right) \mid G_\epsilon\right] \le 1, \tag{2.13}$$

and

$$\limsup_{\epsilon\downarrow 0} E\left[\exp\left(c\int_0^1 R_t \circ d\omega_t\right) \mid G_\epsilon\right] \le 1, \tag{2.14}$$

for all $c \in \mathbf{R}$. The proof of these inequalities would follow the same lines as the proof of Theorem 2.1 in [1]. For this reason we omit the details of this proof and we just indicate the method of proof. The inequality (2.12) is obtained using the integration by parts formula of the nonadapted Stratonovich calculus (see [6]). The inequality (2.13) follows by the same arguments as in Ikeda and Watanabe ([3], page 451) with the help of the Lévy area and exponential inequalities. In order to handle inequality (2.14) one uses uniform exponential estimates and the susbstitution theorem for the Stratonovich integral (see [6]).

Proof of (C2)*:* We have to estimate the probability

$$a_2(\epsilon) = P\left(\sup_{0\le t\le 1} |\omega_t + g_\phi(\omega_1)| < \epsilon\right)$$

when ϵ goes to zero. Note that g_ϕ is continuously differentiable and $g_\phi(0) = 0$, $g'_\phi(0) = g'(\phi_1)$. Using an idea of Le Gall we can express $a_2(\epsilon)$ in terms of the exit time of Brownian motion. More precisely, let us denote by $B_a(r)$ the open ball in $\mathbf{R}^d$ with center a and radius $r > 0$. Then, if $T_{a,r}$ is the exit time of this ball for the Brownian motion starting at zero, the above probability can be written as

$$a_2(\epsilon) = P\left(T_{-g_\phi(\omega_1),\epsilon} > 1\right). \tag{2.15}$$

For any $b \in \mathbf{R}^d$, $|b| < \epsilon$ and any Borel set $A \subset B_b(\epsilon)$ we can write

$$P(T_{b,\epsilon} > 1, \omega_1 \in A) = \int_A \psi_{b,\epsilon}(x)dx,$$

where the function $\psi_{b,\epsilon}$ is obtained as follows. Let Q_t be the semigroup whose generator is $-\frac{1}{2}\Delta$ on the ball $B_b(\epsilon)$ with Dirichlet boundary conditions. Then, for any bounded measurable function f on $B_b(\epsilon)$, and for any $x \in B_b(\epsilon)$, we have

$$E_x\left(f(\omega_t)\mathbf{1}_{\{T_{b,\epsilon}>t\}}\right) = Q_t f(x).$$

Therefore,

$$P\left(T_{b,\epsilon} > 1, \omega_1 \in A\right) = Q_1 \mathbf{1}_A(0).$$

If we denote by $q_t(x,y)$ the kernel of Q_t, we have $\psi_{b,\epsilon}(x) = q_1(0,x)$.

Suppose that $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ are the eigenvalues, and $\{f_n, n \geq 1\}$ an orthonormal sequence of associated eigenfunctions of the problem

$$\begin{cases} \frac{1}{2}\Delta f_n + \lambda_n f_n = 0 \\ f_n|_{\partial B_0(1)} = 0 \end{cases} \tag{2.16}$$

Then,

$$q_t(x,y) = \sum_{n=1}^{\infty} e^{-\frac{\lambda_n t}{\epsilon^2}} \epsilon^{-d} f_n\left(\frac{x-b}{\epsilon}\right) f_n\left(\frac{y-b}{\epsilon}\right), \tag{2.17}$$

and

$$\psi_{b,\epsilon}(x) = \sum_{n=1}^{\infty} e^{-\frac{\lambda_n}{\epsilon^2}} \epsilon^{-d} f_n\left(-\frac{b}{\epsilon}\right) f_n\left(\frac{x-b}{\epsilon}\right). \tag{2.18}$$

It holds that

$$P\left(T_{-g_\phi(\omega_1),\epsilon} > 1\right) = \int_{\{|g_\phi(x)|<\epsilon, |x+g_\phi(x)|<\epsilon\}} \psi_{-g_\phi(x),\epsilon}(x)dx. \tag{2.19}$$

In fact, we have that

$$\psi_{b,\epsilon}(x) = P\left(T_{b,\epsilon} > 1 | \omega_1 = x\right) f_{\omega_1}(x), \tag{2.20}$$

where f_{ω_1} is the density of ω_1. Moreover, $\psi_{b,\epsilon}(x)$ is a continuous function of the variables (b,ϵ) on the set $\{|b| < \epsilon, |x-b| < \epsilon\}$ (this follows from (2.17)). Consequently, we can substitute b by $-g_\phi(x)$ in (2.19) and we obtain

$$\begin{aligned} \psi_{-g_\phi(x),\epsilon}(x) &= P\left(T_{-g_\phi(x),\epsilon} > 1 | \omega_1 = x\right) f_{\omega_1}(x) \\ &= P\left(T_{-g_\phi(\omega_1),\epsilon} > 1 | \omega_1 = x\right) f_{\omega_1}(x), \end{aligned}$$

which implies (2.18).

Finally, from (2.15), (2.18) and (2.17) we get

$$a_2(\epsilon) = \sum_{n=1}^{\infty} e^{-\frac{\lambda_n}{\epsilon^2}} \epsilon^{-d} \int_{\{|g_\phi(x)|<\epsilon, |x+g_\phi(x)|<\epsilon\}} f_n\left(\frac{g_\phi(x)}{\epsilon}\right) f_n\left(\frac{x+g_\phi(x)}{\epsilon}\right) dx. \tag{2.21}$$

Set

$$L_n^\epsilon(\phi_1) = \epsilon^{-d} \int_{\{|g_\phi(x)|<\epsilon, |x+g_\phi(x)|<\epsilon\}} f_n\left(\frac{g_\phi(x)}{\epsilon}\right) f_n\left(\frac{x+g_\phi(x)}{\epsilon}\right) dx.$$

Using the change of variables $x = u\epsilon$ we can write

$$L_n^\epsilon(\phi_1) = \int_{\{|\frac{1}{\epsilon}g_\phi(u\epsilon)|<1, |u+\frac{1}{\epsilon}g_\phi(u\epsilon)|<1\}} f_n\left(\frac{1}{\epsilon}g_\phi(u\epsilon)\right) f_n\left(u + \frac{1}{\epsilon}g_\phi(u\epsilon)\right) du.$$

Taking into account that the functions f_n are bounded we deduce that the limit

$$L_n(\phi_1) = \lim_{\epsilon \downarrow 0} L_n^\epsilon(\phi_1)$$

exists and it is expressed as

$$L_n(\phi_1) = \int_{\{|g'_\phi(0)u|<1, |(I+g'_\phi(0))u|<1\}} f_n(g'_\phi(0)u) f_n((I+g'_\phi(0))u)du.$$

It is well-known that f_1 is strictly positive (see [2]). Consequently, $L_1(\phi_1) > 0$ and from (2.21) we obtain the exact asymptotic behaviour

$$a_2(\epsilon) \sim L_1(\phi_1) e^{-\frac{\lambda_1}{\epsilon^2}},$$

and the proof of the theorem is complete. □

Remarks:

1. When $g \equiv 0$ we get

$$L_1(\phi_1) = f_1(0) \int_{\{|u|<1\}} f_1(u)du,$$

which is the usual constant for the Onsager-Machlup functional with given initial value.

2. The periodic boundary condition $X_0 = X_1$ is not covered by the equation $X_0 = g(X_1 - X_0)$. Nevertheless with minor modifications and using the ideas of [7] one can also obtain the Onsager-Machlup functional for this and related cases.

3 Onsager-Machlup functional for second order stochastic differential equations

In this section we will assume that $d = 1$, namely, Ω is the space of real valued functions on $[0,1]$ which vanish at zero. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function and consider the equation

$$\begin{cases} \dot{X}_t = \dot{X}_0 - \int_0^t f(X_s)ds + \omega_t & ,0 \le t \le 1 \\ X_0 = X_1 = 0 \end{cases} \tag{3.1}$$

This equation has been studied in [8] and [9]. There exists a unique solution for each $\omega \in \Omega$ provided the function f is continuously differentiable and $f' \le 0$.

Let us denote by $\{Y_t(\omega), 0 \le t \le 1\}$ the solution of the above equation when $f \equiv 0$, that is,

$$\begin{cases} \dot{Y}_t = \dot{Y}_0 + \omega_t & ,0 \le t \le 1 \\ Y_0 = Y_1 = 0 \end{cases} \tag{3.2}$$

Clearly, $Y_t = \int_0^t \omega_s ds - t\int_0^1 \omega_s ds$ and $\dot{Y}_t = \omega_t - \int_0^1 \omega_s ds$.

From Proposition 3.1 below it follows that the support of the law of X is the set of continuously differentiable functions on $[0,1]$ which vanish at 0 and 1. Given a function ϕ in the Cameron-Martin space H^1, the path

$$Y_t(\phi) = \int_0^t \phi_s ds - t\int_0^1 \phi_s ds$$

belongs to the support of the law of X and we are interested in the asymptotic behaviour as ϵ tends to zero of

$$J^{\epsilon}(\phi) = P(\|X - Y(\phi)\|_{1,\infty} < \epsilon). \tag{3.3}$$

In the above expression we have used the seminorm $\|\cdot\|_{1,\infty}$ defined by

$$\|y\|_{1,\infty} = \sup_{t\in[0,1]} |\dot{y}(t)|,$$

for $y \in C^1([0,1])$, instead of the usual supremum norm. This seminorm is well fitted to the process X because $\dot{X}_t$ behaves as a Brownian motion.

As in the previous section the computation of the asymptotic behaviour of $J^{\epsilon}(\phi)$ is based on the application of the extended Girsanov theorem to a suitable transformation T^{ϕ} on the Wiener space. This transformation will be defined as follows

$$T^{\phi}(\omega)_t = \omega_t + \phi_t + \int_0^t f(Y_s(\omega+\phi))ds. \tag{3.4}$$

From the results of [8] and [9] we have the following result:

Proposition 3.1 *Suppose that f is continuously differentiable and $f' \leq 0$. Fix $\phi \in H^1$. Then the transformation T^{ϕ} defined by (3.4) is bijective and there exists a probability Q^{ϕ} such that $Q^{\phi} \circ (T^{\phi})^{-1} = P$ and*

$$\frac{dQ^{\phi}}{dP} = |Z_1(\omega+\phi)| \exp\left\{-\int_0^1 f(Y_t(\omega+\phi)) \circ d\omega_t - \int_0^1 \dot{\phi}_t d\omega_t - \frac{1}{2}\int_0^1 |f(Y_t(\omega+\phi)) + \dot{\phi}_t|^2 dt\right\},$$

where Z_1 is the solution at time $t = 1$ of the second order differential equation

$$\begin{cases} \ddot{Z}_t + f'(Y_t)Z_t = 0, & ,0 \leq t \leq 1 \\ Z_0 = 0, \dot{Z}_1 = 0 \end{cases} \tag{3.5}$$

The process $Y(\omega+\phi)$ satisfies the equation

$$\dot{Y}_t(\omega+\phi) - \dot{Y}_0(\omega+\phi) = T^{\phi}(\omega)_t - \int_0^t f(Y_s(\omega+\phi))ds.$$

As a consequence, the law of the process X solution to (3.1) coincides with the law of $Y(\omega+\phi)$ under Q^{ϕ}. This allows to write the functional $J^{\epsilon}(\phi)$ in the following form

$$\begin{aligned} J^{\epsilon}(\phi) &= P(\|X - Y(\phi)\|_{1,\infty} < \epsilon) \\ &= Q^{\phi}(\|Y(\omega+\phi) - Y(\phi)\|_{1,\infty} < \epsilon) \\ &= Q^{\phi}(\|\omega - \int_0^1 \omega_t dt\|_{\infty} < \epsilon). \end{aligned}$$

In the sequel we will use the notation

$$H_{\epsilon} = \{\|\omega - \int_0^1 \omega_t dt\|_{\infty} < \epsilon\}.$$

Now we can state the main result of this section.

Theorem 3.1 *Suppose that f is continuously differentiable and $f' \leq 0$. For any $\phi \in H^1$ we have, as ϵ tends to zero,*

$$P(\|X - Y(\phi)\|_{1,\infty} < \epsilon) \sim P(H_\epsilon) Z_1(\phi) \exp\left\{-\frac{1}{2}\int_0^1 |f(Y_t(\phi)) + \dot{\phi}_t|^2 dt\right\}.$$

Proof: We have

$$J^\epsilon(\phi) = E\left(\frac{dQ^\epsilon}{dP}|H_\epsilon\right) P(H_\epsilon).$$

So in order to prove the theorem it suffices to show that

$$\lim_{\epsilon \downarrow 0} E\left(\frac{dQ^\epsilon}{dP}|H_\epsilon\right) = |Z_1(\phi)| \exp\left\{-\frac{1}{2}\int_0^1 |f(Y_t(\phi)) + \dot{\phi}_t|^2 dt\right\}.$$

Consider the following Wiener functionals

$$B_1(\omega) = |Z_1(\omega)| \exp\left\{-\frac{1}{2}\int_0^1 |f(Y_t(\omega)) + \dot{\phi}_t|^2 dt\right\},$$

$$B_2(\omega) = \exp\{-\int_0^1 f(Y_t(\omega + \phi)) \circ d\omega_t\}$$

$$B_3(\omega) = \exp\{-\int_0^1 \dot{\phi}_t d\omega_t\}.$$

We want to show that

$$\lim_{\epsilon \downarrow 0} E\left(\frac{dQ^\epsilon}{dP}|H_\epsilon\right) = B_1(\phi).$$

Using the same technique as in the previous paragraph we have

$$\begin{aligned}
&\left|E\left[\frac{dQ^\epsilon}{dP} - B_1(\phi)|H_\epsilon\right]\right| \\
&= |E\left[B_1(\omega + \phi)B_2(\omega)B_3(\omega) - B_1(\phi) \mid H_\epsilon\right]| \\
&\leq \left(E\left[(B_1(\omega + \phi) - B_1(\phi))^2 \mid H_\epsilon\right]\right)^{1/2} \left(E[B_2^2 B_3^2 \mid G_\epsilon]\right)^{1/2} \\
&+ B_1(\phi)\left|E[B_2(\omega)B_3(\omega) \mid H_\epsilon] - 1\right|,
\end{aligned}$$

and it suffices to show that

$$\lim_{\epsilon \downarrow 0} E\left[(B_1(\omega + \phi) - B_1(\phi))^2 \mid H_\epsilon\right] = 0, \tag{3.6}$$

$$\sup_{\epsilon > 0} E[(B_2 B_3)^2 \mid H_\epsilon] < \infty, \tag{3.7}$$

$$\lim_{\epsilon \downarrow 0} E[B_2 B_3 \mid H_\epsilon] = 1. \tag{3.8}$$

The convergence (3.6) is immediate because B_1 is a continuous functional of ω, and on the set H_ϵ we have $\|\omega\|_\infty < 2\epsilon$. Using the arguments of Ikeda and Watanabe

([3], page 449), in order to prove (3.7) and (3.8) it suffices to show that for any c in $\mathbf{R}$, we have

$$\limsup_{\epsilon\downarrow 0} E\left[\exp\left(c\int_0^1 f(Y_t(\omega+\phi))\circ d\omega_t\right) \mid H_\epsilon\right] \leq 1, \tag{3.9}$$

$$\limsup_{\epsilon\downarrow 0} E\left[\exp\left(c\int_0^1 \dot{\phi}_t d\omega_t\right) \mid H_\epsilon\right] \leq 1. \tag{3.10}$$

The inequality (2.11) follows from the results of Shepp and Zeitouni because the set H_ϵ is convex and symmetric with respect to the origin.

In order to show the inequality (3.9) we will use the same arguments as in the proof of (2.10)

The following lemma, whose proof was given to us by Wenbo Li (cf. [5]), provides the asymptotic behaviour of the probability of the set H_ϵ as ϵ tends to zero.

Lemma 3.1 *We have*

$$\lim_{\epsilon\downarrow 0} \epsilon^2 \log P\left(\sup_{0\leq t\leq 1} |\omega_t - \int_0^1 \omega_s ds| < \epsilon\right) = -\frac{\pi^2}{8}.$$

Proof: Set $p_\epsilon = P(H_\epsilon)$. We have

$$\begin{aligned} p_\epsilon &\geq P\left(\sup_{0\leq t\leq 1} |\omega_t - \int_0^1 \omega_s ds| < \epsilon, |\int_0^1 \omega_s ds| < \epsilon^3\right) \\ &\geq P\left(\sup_{0\leq t\leq 1} |\omega_t| < (1-\epsilon^2)\epsilon, |\int_0^1 \omega_s ds| < \epsilon^3\right) \\ &\geq P\left(\sup_{0\leq t\leq 1} |\omega_t| < (1-\epsilon^2)\epsilon\right) P\left(|\int_0^1 \omega_s ds| < \epsilon^3\right), \end{aligned}$$

where the last step is due to the correlation inequalities (see Shepp and Zeitouni [10]). Finally, we deduce the inequality

$$p_\epsilon \geq C\epsilon^3 e^{-\frac{\pi^2}{8\epsilon^2}}, \tag{3.11}$$

for some constant $C > 0$ and for ϵ small enough.

To obtain an upper bound we write

$$\begin{aligned} p_\epsilon &= \sum_{|k|\leq [\frac{1}{2}\epsilon^{-2}]} P\left(\sup_{0\leq t\leq 1} |\omega_t - \int_0^1 \omega_s ds| < \epsilon, |\int_0^1 \omega_s ds - 2k\epsilon^3| < \epsilon^3\right) \\ &\leq \sum_{|k|\leq [\frac{1}{2}\epsilon^{-2}]} P\left(\sup_{0\leq t\leq 1} |\omega_t - 2k\epsilon^3| < (1+\epsilon^2)\epsilon, |\int_0^1 \omega_s ds - 2k\epsilon^3| < \epsilon^3\right) \\ &\leq \sum_{|k|\leq [\frac{1}{2}\epsilon^{-2}]} P\left(\sup_{0\leq t\leq 1} |\omega_t - 2k\epsilon^3| < (1+\epsilon^2)\epsilon\right) \\ &\leq (2[\frac{1}{2}\epsilon^{-2}]+1) P\left(\sup_{0\leq t\leq 1} |\omega_t| < (1+\epsilon^2)\epsilon\right). \end{aligned}$$

This implies that

$$p_\epsilon \leq C\epsilon^{-2}e^{-\frac{\pi^2}{8\epsilon^2}}, \tag{3.12}$$

for some constant $C > 0$ and for ϵ small enough. □

We conjecture that the estimates (3.11) and (3.12) can be improved in the sense that the polynomial term in ϵ can be replaced by a constant.

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