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Some remarks on perturbed reflecting Brownian motion

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0. Introduction

Let B denote a one-dimensional Brownian motion started from 0 and L its local time process at level 0. For fixed $\mu > 0$, the perturbed reflecting Brownian motion X is defined for all $t \geq 0$ by

$$X_t = |B_t| - \mu L_t.$$

It has aroused some interest in the last few years (see Le Gall-Yor [7], Yor [13], chapters 8 and 9, Carmona-Petit-Yor [2], Perman [8]). We are going to make a few remarks concerning this process and give short elementary proofs of some known results, such as the generalized Ray-Knight Theorems for X . Let us just stress that none of the results derived here is new, and that our modest aim is to shed a new light on them, which we hope can improve our understanding of these identities.

We now recall a few relevant facts: For all $a \in \mathbb{R}$, $T_a = \inf\{t \geq 0; X_t = a\}$ will denote the hitting time of a by X . Except when $\mu = 1$, X is not Markovian; however, for $a > 0$, T_{-a} is the hitting time of a/μ by L and hence a stopping time for B . The strong Markov property then yields that the processes $(X_t, t \geq 0)$ and $(a + X_{t+T_{-a}}, t \geq 0)$ have the same law. We will refer to this property as the ‘strong Markov property’ for X .

Note also that for $\mu = 1$, Lévy’s identity (that is: if $S_t = \sup_{s \leq t} B_s$, then the processes $(S, S - B)$ and $(L, |B|)$ have the same law) shows that X is in fact a Brownian motion.

1. A hitting time property

In [11], we used the following result: For all $a > 0$, $b > 0$,

$$P(T_{-a} < T_b) = \left(\frac{b}{a+b} \right)^{1/\mu}. \quad (1)$$

This is a generalization of the classical hitting time property for Brownian motion (which is in fact (1) for $\mu = 1$):

$$P(\sigma_{-a} < \sigma_b) = \frac{b}{a+b}, \quad (2)$$

where $\sigma_x = \inf\{t > 0, B_t = x\}$.

In [11], we derived (1) from the explicit law of L_{T_1} derived by Carmona-Petit-Yor [2] (corollary 3.4.1 there) (one has $P(T_{-a} < T_b) = P(L_{T_b} > a/\mu)$). As briefly pointed out in [2], the law of L_{T_1} (and therefore (1)) is in fact also a direct consequence of the explicit solution to Skorokhod's problem by Azéma and Yor [1] (see also exercise (5.9) chapter VI in Revuz-Yor [10]) in a very special case: One just has to compute the right-hand side of (5.9) in [10] for an affine function γ and then use Lévy's identity.

We now give an alternative elementary short proof of (1): First, for all $x \geq 1$ we put

$$g(x) = P(T_{1-x} < T_1).$$

For $x > 1$ and $y > 1$, one has immediately $T_{1-x} < T_{1-xy}$. The 'strong Markov property' at time T_{1-x} and the scaling property imply that

$$g(xy) = P(T_{1-x} < T_1)P(T_{x-xy} < T_x) = g(x)g(y).$$

Moreover, g is continuous decreasing on $[1, \infty)$ and $g(1) = 1$. Hence, for some fixed $c = c(\mu)$,

$$g(x) = x^{-c}. \quad (3)$$

It now remains to show that $c = 1/\mu$: We look at the asymptotic behaviour of

$$f(x) = P(T_{-1} > T_x) = 1 - g(1 + 1/x)$$

as $x \rightarrow \infty$. (3) implies that $f(x) = 1 - (1 + 1/x)^{-c} \sim c/x$ as $x \rightarrow \infty$. On the other hand, Lévy's identity implies that

$$P(\sigma_{-(x+1/\mu)} < \sigma_{1/\mu}) \leq f(x) \leq P(\sigma_{-x} < \sigma_{1/\mu}),$$

and consequently (using (2)), $f(x) \sim 1/(\mu x)$ as $x \rightarrow \infty$, and (1) follows.

2. The generalized second Ray-Knight Theorem as a consequence of (1)

In [2] (see also Yor [13], chapter 9), Carmona-Petit-Yor have derived a generalized second Ray-Knight Theorem for the local times of X (Theorem 3.3 in [2], Theorem 9.1 in [13]; we refer to Yor [13], chapter 3 or Revuz-Yor [10], Chapter XI for the Ray-Knight Theorems for Brownian motion). They then derive the law of L_{T_1} (which

implies (1)) as a consequence of this Theorem. We now briefly point out how this generalized second Ray-Knight Theorem for X can in fact be derived ‘backwards’, as a consequence of (1), using a general result of Lamperti [5] on semi-stable Markov processes we first recall.

Suppose $(Y_t, t \geq 0)$ is a non-deterministic continuous Markov process in $[0, \infty)$, started from $x \in [0, \infty)$ under the probability measure P_x . Suppose furthermore that Y is semi-stable of index 1 (in the sense of [5]), that is $(c^{-1}Y_{ct}, t \geq 0)$ under P_x and $(Y_t, t \geq 0)$ under $P_{x/c}$ have the same law for all $c > 0$. Then, Theorem 5.1 in Lamperti [5] implies that Y is a multiple of a squared Bessel process (of index $\delta \in \mathbb{R}$), which is either absorbed or reflected at 0. This result is the key to our approach.

We now put down some notation and state the generalized second Ray-Knight Theorem. Let ℓ_t^a denote the local time of X at level a and time t , and let τ denote the right-continuous inverse process of ℓ^0 . Then:

Theorem (Carmona-Petit-Yor). *The processes $(\ell_{\tau_1}^a, a \geq 0)$ and $(\ell_{\tau_1}^{-a}, a \geq 0)$ are independent and:*

- (i) $(\ell_{\tau_1}^a, a \geq 0)$ *is a squared Bessel process of dimension 0 started from 1 and absorbed at 0.*
- (ii) $(\ell_{\tau_1}^{-a}, a \geq 0)$ *is a squared Bessel process of dimension $2 - 2/\mu$ started from 0 and absorbed at 0.*

Let $A^+(t) = \int_0^t 1_{\{X_s > 0\}} ds$ and $A^-(t) = \int_0^t 1_{\{X_s < 0\}} ds$. Let also σ^+ (respectively σ^-) denote the right-continuous inverse of A^+ (resp. A^-). We put for all $u \geq 0$, $X_u^+ = X_{\sigma_u^+}$ and $X_u^- = X_{\sigma_u^-}$. In other words and loosely speaking: X^+ (resp. X^-) is obtained by glueing the positive (resp. negative) excursions of X together. Then:

Lemma *The two-processes X^+ and X^- are independent. Moreover X^+ is a reflected Brownian motion.*

There are various possible proofs of this lemma. Yor ([13], Chapter 8) indicates a proof based upon Knight’s Theorem on orthogonal martingales. Mihael Perman suggested an excursion-theoretical approach. The last part of this note provides yet another possible justification.

This lemma shows immediately that $(\ell_{\tau_1}^a, a \geq 0)$ and $(\ell_{\tau_1}^{-a}, a \geq 0)$ are independent; (i) then follows from the second Ray-Knight Theorem for Brownian motion (it actually also follows from (ii) with $\mu = 1$). It remains to show (ii).

The ‘Markov property’ for X and the lemma show that $(\ell_{\tau_1}^a, a \geq 0)$ is a Markov process (one just has to apply the Lemma to $(a + X_{T_{-a}+t}, t \geq 0)$). As X is a continuous semi-martingale, Theorem (1.7) in Chapter VI of Revuz-Yor [10] yields that $(\ell_{\tau_1}^a, a \geq 0)$ is continuous. The scaling property for B (which is also the scaling property for X) implies that $(\ell_{\tau_1}^a, a \geq 0)$ is a semi-stable Markov process of index 1

in the sense of Lamperti [5]. Hence, Lamperti's result mentioned at the beginning of this section shows that $(\ell_{\tau_1}^{-a}, a \geq 0)$ is a multiple of a squared Bessel process Y . Let δ denote its dimension and $y = Y_0$. Y is absorbed at 0 since otherwise, $\ell_{\tau_1}^{-a}$ is not identically 0 for all sufficiently large a . It remains to identify δ and y , which can be done using section 1: As ℓ^0 increases,

$$P(T_{-a} < T_b) = P(\ell_{T_{-a}}^0 < \ell_{T_b}^0).$$

But $\ell_{T_b}^0$ depends only on X^+ whereas $\ell_{T_{-a}}^0$ depends only on X^- ; hence, these two random variables are independent. It is well-known that $\ell_{T_b}^0$ is an exponential random variable of parameter $1/2b$ (see e.g. Proposition (4.6), Chapter 6 in Revuz-Yor [10]). Consequently, if ξ denotes an exponential random variable of parameter $\lambda = 1/(2b)$, if ρ denotes the hitting time of 0 by Y and Z_γ a Gamma-random variable of index $\gamma > 0$ (that is with density $z^{\gamma-1}e^{-z}/\Gamma(\gamma)$ on R_+),

$$\begin{aligned} E(e^{-\lambda/\rho}) &= P(1/\rho < \xi) = P(\inf(X_s, s \leq \tau_1) < -\frac{1}{\xi}) \\ &= P(\ell_{T_{-1}}^0 < \ell_{T_b}^0) = \left(\frac{b}{1+b}\right)^{1/\mu} = (1+2\lambda)^{-1/\mu} \end{aligned} \quad (4)$$

which is the Laplace transform of $(2Z_{1/\mu})$. Hence, ρ has the same law as $1/(2Z_{1/\mu})$. On the other hand, if $\rho(\alpha, x)$ is the hitting time of 0 by a squared Bessel process of dimension $2 - 2\alpha$ started from $x > 0$, then it is well-known that $\rho(\alpha, x)$ has the same law as $x^2/(2Z_\alpha)$ (one can for instance compare the Laplace transforms, using the results of Kent [4], and equation (15) section 6.22 in Watson [12]; alternatively, one can note by time-reversal that $\rho(\alpha, x)$ is the last passage time at x by a Bessel process of index $2 + 2\alpha$ started from 0, and use the results of Gettoor [3], see also Yor [14]). Hence, $y = 1$ and $\delta = 2 - 2/\mu$, which completes the proof of the theorem.

3. The generalized first Ray-Knight Theorem

We now briefly point out how the same approach also yields the generalization of the first Ray-Knight Theorem for perturbed reflecting Brownian motion derived by Le Gall-Yor [6] (see also Yor [13], Section 3.3). However, in this case, the original proofs are shortish anyway.

Theorem (Le Gall-Yor). $(\ell_{T_{-1}}^{-1+a}, 0 \leq a \leq 1)$ is a squared Bessel process of dimension $2/\mu$ started from 0 and reflected at 0.

By time-reversal (since $(B_t, t \leq T_{-1})$ and $(B_{T_{-1}-t}, t \leq T_{-1})$ have the same law), one can consider the process $\tilde{X}_u = |B_u| + \mu L_u$ and its local times taken at infinite time: $\tilde{\ell}^a = \ell_\infty^a(\tilde{X})$, for $a \geq 0$, and remark that $(\ell_{T_{-1}}^{-1+a}, 0 \leq a \leq 1)$ and $(\tilde{\ell}^a, 0 \leq a \leq 1)$ have the same law. As previously, $\tilde{\ell}$ is a continuous Markov process, which is self-similar of index 1 because of the scaling property of $\tilde{X}$. $\tilde{\ell}$ is henceforth (using again

Theorem 5.1 in Lamperti [5]) a multiple of a squared Bessel process β of dimension δ : $\tilde{\ell} = \alpha\beta$, with $\alpha > 0$. This time, β has to be reflected at 0, since almost surely, for all rational $a > 0$, $\tilde{\ell}^a \neq 0$. We now identify δ and α using (4). On the one hand, one has for all $\lambda > 0$ (see e.g. Revuz-Yor [10], line before Corollary (1.4) in Chapter XI):

$$E(e^{-\lambda\alpha\beta_1}) = (1 + 2\lambda\alpha)^{-\delta/2}.$$

On the other hand, (4) implies that (using the same notations ξ , λ as in (4)),

$$E(e^{-\lambda\alpha\beta_1}) = P(\xi > \alpha\beta_1) = P(\ell_{T_1/(2\lambda)}^0 > \ell_{T_1}^0) = P(T_1/(2\lambda) > T_1) = (1 + 2\lambda)^{-1/\mu}$$

and the Theorem follows.

4. The discrete approach

We now mention an approximation of X by a random walk, which converges towards perturbed reflecting Brownian motion as the simple random walk does towards Brownian motion. We define $(S_n, n \geq 0)$ as follows: We fix $\mu > 0$ and we put $q = (1 + \mu)^{-1} \in (0, 1)$.

Let $I_n = \min\{S_0, S_1, \dots, S_n\}$ and $S_0 = 0$. By induction, for all $n \geq 0$, if $S_0, \dots, S_n$ are defined, then the law of S_{n+1} is the following:

$$P(S_{n+1} = S_n + 1) = P(S_{n+1} = S_n - 1) = 1/2 \text{ if } S_n \neq I_n$$

and

$$P(S_{n+1} = S_n + 1) = q, \quad P(S_{n+1} = S_n - 1) = 1 - q \text{ if } S_n = I_n.$$

Using for instance Lévy's identity and Proposition 2 page 137-138 in Révész [4], one can show that the processes

$$(n^{-1/2}S_{[nt]}, t \in [0, 1])$$

converge weakly towards $(X_t, t \in [0, 1])$ as $n \rightarrow \infty$, where $[x]$ denotes the integer part of x . With little extra work, this approach provides another possible proof of the Lemma, since the independence of the positive and negative parts of the random walk $(S_n, n \geq 0)$ is trivial. Equation (1) can also be deduced, since (if $N_p = \inf\{n \geq 0, S_n = p\}$), $P(N_{-1} < N_p)$ and consequently $P(N_{-p'} < N_p)$ and $P(N_{-[ap]} < N_{[bp]})$ can be very easily explicitly computed, when $p > 0$, $p' > 0$, $a > 0$, $b > 0$: Indeed, it is a good undergraduate exercise to see that

$$P(N_{-1} < N_p) = (1 - q) \sum_{k \geq 0} \left(\frac{q(p-1)}{p} \right)^k = (1 + 1/(\mu p))^{-1},$$

and consequently as $P(N_{-p'} < N_p) = P(N_{-1} < N_p)P(N_{-1} < N_{p+1}) \dots P(N_{-1} < N_{p+p'-1})$,

$$\log P(N_{-[ap]} < N_{[bp]}) = - \sum_{k=[bp]}^{k=[ap]+[bp]-1} \log(1 + 1/(\mu k)) \sim \frac{1}{\mu} \log \left(\frac{b}{a+b} \right)$$

as $p \rightarrow \infty$, which yields (1).

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Note added in proof. I would like to mention the two recent preprints by Burgess Davis [15] and Darryl Nester [16], which are very closely related with Section 4 of this note.

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