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NATHALIE EISENBAUM
HAYA KASPI

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A Counterexample for the Markov Property of Local Time for Diffusions on Graphs

by

Nathalie Eisenbaum* and Haya Kaspi**

*Laboratoire de Probabilités
Universite Pierre et Marie Curie
Tour 56, 4 Place Jussieu
75252 Paris, Cedex 05
FRANCE

**Faculty of Industrial Engineering and Management
Technion—Israel Institute of Technology
Haifa 32000
ISRAEL

1 Introduction

Let X be a transient Markov process taking values in an interval $E \subset \mathbb{R}$, admitting a local time at each point, and such that points communicate (i.e. every point may be reached from any other point). Let L^x be the total accumulated local time at x. Then $(L^x)_{x \in E}$ is a Markov process (indexed by the states of E) if, and only if X has continuous sample paths and fixed birth and death points. The sufficiency is the famous Ray-Knight theorem (see [R], [K], [W], [S] and [E]); the necessity was proved in [E.K].

In the symmetric case, the proof that Sheppard [S] and Eisenbaum [E] gave to the Ray-Knight theorem uses a centered Gaussian field $\{\phi_x : x \in E\}$ with covariance the Green function of X that was introduced in Dynkin's Isomorphism theorem [D]. They have shown that the Markov property of the local time process can be derived from the Markov property of the Gaussian field. Dynkin [D] and Atkinson [A] have shown that when $E \subset \mathbb{R}^d$ the field $(\phi_x)_{x \in E}$ is a Markov field if, and only if, X has continuous paths.

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In view of the above, it is natural to ask whether the conditions for the Markov property of the local time, for processes on $E \subset \mathbb{R}$, are also sufficient for processes taking values in $E \subset \mathbb{R}^d$, and admitting a local time at each point of E.

The following definition of the Markov property for processes indexed by $E \subset \mathbb{R}^d$ was used by Dynkin [D] and Atkinson [A] in the above-mentioned result.

Definition *A process $\{L^x : x \in E\}$ where $E \subset \mathbb{R}^d$ has the Markov property provided for every open, relatively compact set A, contained in E, $\{L^x : x \in \bar{A}\}$ and $\{L^x : x \in A^c\}$ are conditionally independent given $\{L^x : x \in \partial A\}$.*

With this definition of the Markov property, the answer to our question is that continuity and fixed birth and death points are not sufficient for $\{L^x : x \in E\}$ to be Markov. To see this, we consider the Brownian motion X on the unit circle S^1 born at $(1,0)$ and killed when the local time at $(-1,0)$ exceeds an exponential variable that is independent of X. This process has continuous sample paths and fixed birth and death points. Denoting by 1, 2, 3 and 4, respectively, the points $(1,0)$, $(0,1)$ $(-1,0)$ and $(0,-1)$ we shall show:

Proposition *Given (L^1, L^3), L^2 and L^4 are not conditionally independent.*

Remark Leuridan [L] has considered the local time process of a Brownian motion on a circle $(L^x_{\tau_r})_{x \in X^1}$, killed when the local time at $(-1,0)$ exceeds $r > 0$. He has obtained a Ray-Knight type theorem that describes the law of the process in terms of Bessel processes laws. This may be used to prove our result. It seems to us, however, that the direct computations presented here are easier.

2 Proof of the Proposition

We consider the process X restricted to the four points $\{1,2,3,4\}$. That is, we let $L_t = L_t^1 + L_t^2 + L_t^3 + L_t^4$, where, for $i = 1, \ldots, 4$, L_t^i is the local time at i up to time t, and (τ_t) is the right continuous inverse of (L_t). The process (X_{τ_t}) is a pure jump process on $S = \{1,2,3,4\}$, and for $i \in S$, L^i is also the total time (X_{τ_t}) spends at i. For $i \in S$, let $T_i = \inf\{t \geq 0 : X_{\tau_t} = i\}$, and $E_i = E(\ |X_0 = i)$. For $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$, set

$$\varphi_i(\theta) = E_i\left(e^{-\theta_1 L^1 - \theta_2 L^2 - \theta_3 L^3 - \theta_4 L^4}\right) .$$

Then

$$\varphi_1(\theta) = E_1\left(e^{-\theta_1 L^1_{T_2 \wedge T_4}} \,;\, T_2 < T_4\right) \varphi_2(\theta) + E_1\left(e^{-\theta_1 L^1_{T_2 \wedge T_4}} \,;\, T_4 < T_2\right) \varphi_4(\theta)$$

which by the symmetry is equal to

$$\frac{1}{2}\, E_1\left(e^{-\theta_1 L^1_{T_2 \wedge T_4}}\right) (\varphi_2(\theta) + \varphi_4(\theta)) .$$

Under P_1, $L^1_{T_2\wedge T_4}$ has an exponential distribution with a parameter which we denote by λ, and by symmetry again

$$\frac{1}{\lambda} = E_1(L^1_{T_2\wedge T_4}) = E_2(L^2_{T_1\wedge T_3}) = E_4(L^4_{T_1\wedge T_3}) .$$

Further, let σ be the independent exponentially distributed random variable, which, when exceeded by the local time of 3 the process is killed. Then

$$P_3(T_2 \wedge T_4 < \zeta) = P_3(L^3_{T_2\wedge T_4} < \sigma) .$$

We take the parameter of σ to be λ as well. We thus obtain the following set of equations:

$$\begin{aligned}
\varphi_1(\theta) &= \frac{\lambda}{\lambda+\theta_1}\left\{\frac{1}{2}\,\varphi_2(\theta) + \frac{1}{2}\,\varphi_4(\theta)\right\} \\
\varphi_2(\theta) &= \frac{\lambda}{\lambda+\theta_2}\left\{\frac{1}{2}\,\varphi_1(\theta) + \frac{1}{2}\,\varphi_3(\theta)\right\} \\
\varphi_4(\theta) &= \frac{\lambda}{\lambda+\theta_4}\left\{\frac{1}{2}\,\varphi_1(\theta) + \frac{1}{2}\,\varphi_3(\theta)\right\} \\
\varphi_3(\theta) &= \frac{\lambda}{2\lambda+\theta_3} + \frac{\lambda}{2\lambda+\theta_3}\left\{\frac{1}{2}\,\varphi_2(\theta) + \frac{1}{2}\,\varphi_4(\theta)\right\} .
\end{aligned}$$

We may, without loss of generality, assume that $\lambda = 1$, and solve the system. The result

$$\varphi_1(\theta) = \frac{2+\theta_2+\theta_4}{4(1+\theta_1)(2+\theta_3)(1+\theta_2)(1+\theta_4) - (2+\theta_2+\theta_4)(3+\theta_1+\theta_3)} .$$

First, note that

$$\varphi_1(\theta_1, 0, 0, 0) = \frac{1}{1+3\theta_1} ,$$

from which it follows that L^1 has an exponential distribution with parameter 1/3, and

(1) $$\varphi_1(\theta) = \int_0^\infty e^{-\theta_1\ell} E_1\left(e^{-\theta_2 L^2 - \theta_3 L^3 - \theta_4 L^4} | L^1 = \ell\right)\ P_1(L^1 \in d\ell)$$

and also

$$\varphi_1(\theta) = \frac{2+\theta_2+\theta_4}{4(2+\theta_3)(1+\theta_2)(1+\theta_4) - (2+\theta_2+\theta_4)(3+\theta_3)} \times \frac{1}{1+\theta_1\left(\frac{4(2+\theta_3)(1+\theta_2)(1+\theta_4)-(2+\theta_2+\theta_4)}{4(2+\theta_3)(1+\theta_2)(1+\theta_4)-(2+\theta_2+\theta_4)(3+\theta_3)}\right)} .$$

Set,

$$\lambda(\theta_2,\theta_3,\theta_4) = \frac{4(2+\theta_3)(1+\theta_2)(1+\theta_4) - (2+\theta_2+\theta_4)(3+\theta_3)}{4(2+\theta_3)(1+\theta_2)(1+\theta_4) - (2+\theta_2+\theta_4)} .$$

Then it can be easily checked that $\lambda(\theta_2,\theta_3,\theta_4) > 0$ for all $(\theta_2,\theta_3,\theta_4)$ and that

$$\varphi_1(\theta) = \frac{2+\theta_2+\theta_4}{4(2+\theta_3)(1+\theta_2)(1+\theta_4) - (2+\theta_2+\theta_4)(3+\theta_3)} \int_0^\infty e^{-\theta_1\ell}\lambda(\theta_2,\theta_3,\theta_4)e^{-\lambda(\theta_2,\theta_3,\theta_4)\ell}d\ell ,$$

which, when compared with (1) gives

$$E_1\left(e^{-\theta_2 L^2-\theta_3 L^3-\theta_4 L^4}|L^1=\ell\right)$$
$$=\frac{3(2+\theta_2+\theta_4)}{4(2+\theta_3)(1+\theta_2)(1+\theta_4)-(2+\theta_2+\theta_4)}\ e^{-\{\lambda(\theta_2,\theta_3,\theta_4)-\frac{1}{3}\}\ell}\ .$$

In particular, for $\theta_2=\theta_4=0$ we obtain

$$E_1\left(e^{-\theta_3 L^3}|L^1=\ell\right)=\frac{3/2}{3/2+\theta_3}\ e^{-\frac{1}{6}\ell\left(\frac{\theta_3}{3/2+\theta_3}\right)}\ .$$

But

$$e^{-\frac{1}{6}\ell\left(\frac{\theta_3}{3/2+\theta_3}\right)}=e^{-\frac{1}{6}\ell\left(1-\frac{3/2}{3/2+\theta_3}\right)}$$

which is the Laplace transform of a compound Poisson process with rate 1/6 and jump distribution exponential (3/2) at time ℓ. Therefore, given $L^1=\ell$ the law of L^3 is the convolution of an exponential law with the above compound Poisson at time ℓ.

Set

$$\beta(\theta_2,\theta_4)=\frac{6+7(\theta_2+\theta_4)+8\theta_2\theta_4}{4(1+\theta_2)(1+\theta_4)}$$
$$\lambda_\ell(\theta_2,\theta_4)=\frac{\ell}{6}\ \frac{2+5(\theta_2+\theta_4)+8\theta_2\theta_4}{2(1+\theta_2)(1+\theta_4)}\ ,$$

and for λ and β positive, let $f_{(\beta,\lambda)}(m)$ be the density of a convolution of an exponential law with parameter β and of a compound Poisson law with jump distribution exponential with parameter β, and Poisson parameter λ. Then

(2)
$$E_1\left(e^{-\theta_2 L^2-\theta_3 L^3-\theta_4 L^4}|L^1=\ell\right)$$
$$=\int_0^\infty E_1\ \left(e^{-\theta_2 L^2-\theta_4 L^4}|\,L^1=\ell,\ L^3=m\right)\ e^{-\theta_3 m}P_1(L^3\in dm|\,L^1=\ell)$$
$$=\int_0^\infty E_1\ \left(e^{-\theta_2 L^2-\theta_4 L^4}|\,L^1=\ell,\ L^3=m\right)\ f_{\beta(0,0),\lambda_\ell(0,0)}(m)\ e^{-\theta_3 m}dm$$

On the other hand, by our above computation

(3)
$$E_1\left(e^{-\theta_2 L^2-\theta_3 L^3-\theta_4 L^4}|L^1=\ell\right)$$
$$=\frac{3(2+\theta_2+\theta_4)}{6+7(\theta_2+\theta_4)+8\theta_2\theta_4}\ \exp\left\{-\frac{\ell}{3}\ \frac{8(\theta_2+\theta_4)+16\theta_2\theta_4}{6+7(\theta_2+\theta_4)+8\theta_2\theta_4}\right\}$$
$$\times\ \frac{1}{1+\frac{\theta_3}{\beta(\theta_2,\theta_4)}}\ \exp\left\{-\lambda_\ell(\theta_2,\theta_4)\left(1-\frac{1}{1+\frac{\theta_3}{\beta(\theta_2,\theta_4)}}\right)\right\}$$
$$=\frac{3(2+\theta_2+\theta_4)}{6+7(\theta_2+\theta_4)+8\theta_2\theta_4}\ \exp\left\{-\frac{\ell}{3}\ \frac{8(\theta_2+\theta_4)+16\theta_2\theta_4}{6+7(\theta_2+\theta_4)+8\theta_2\theta_4}\right\}$$
$$\times\int_0^\infty e^{-\theta_3 m}f_{\beta(\theta_2,\theta_4),\lambda_\ell(\theta_2,\theta_4)}(m)dm\ .$$

Comparing (2) and (3) we obtain

$$E\left(e^{-\theta_2 L^2-\theta_4 L^4}\,|\,L^1=\ell,\ L^3=m\right)=$$

$$\frac{3(2+\theta_2+\theta_4)}{6+7(\theta_2+\theta_4)+8\theta_2\theta_4}\exp\left\{-\frac{\ell}{3}\,\frac{8(\theta_2+\theta_4)+16\theta_2\theta_4}{6+7(\theta_2+\theta_4)+8\theta_2\theta_4}\right\}\frac{f_{\beta(\theta_2,\theta_4),\lambda_\ell(\theta_2,\theta_4)}(m)}{f_{\beta(0,0),\lambda_\ell(0,0)}(m)}\ .$$

Suppose now that conditioned on (L^1,L^3), L^2 and L^4 are independent, that is for almost every $(m,\ell)\in\mathbb{R}^2$

$$E_1\left(e^{-\theta_2 L^2-\theta_4 L^4}\,|\,L^1=\ell,\ L^3=m\right)=E_1\left(e^{-\theta_2 L^2}|L^1=\ell,\ L^3=m\right)E_1\left(e^{-\theta_4 L^4}\,|\,L^1=\ell,\ L^3=m\right)$$

Using the above computation, this is equivalent to

$$\frac{3(2+\theta_2+\theta_4)}{6+7(\theta_2+\theta_4)+8\theta_2\theta_4}\exp\left\{-\frac{\ell}{3}\,\frac{8(\theta_2+\theta_4)+16\theta_2\theta_4}{6+7(\theta_2+\theta_4)+8\theta_2\theta_4}\right\}f_{\beta(\theta_2,\theta_4),\lambda_\ell(\theta_2,\theta_4)}(m)\cdot f_{\beta(0,0),\lambda_\ell(0,0)}(m)$$

$$=\frac{3(2+\theta_2)}{6+7\theta_2}\cdot\frac{3(2+\theta_4)}{6+7\theta_4}\exp\left\{-\frac{\ell}{3}\left[\frac{8\theta_2}{6+7\theta_2}+\frac{8\theta_4}{6+7\theta_4}\right]\right\}f_{\beta(\theta_2,0),\lambda_\ell(\theta_2,0)}f_{\beta(0,\theta_4),\lambda_\ell(0,\theta_4)}\ . \tag{4}$$

Recall that

$$f_{\beta,\lambda}(m)=\beta e^{-(\lambda+\beta m)}\sum_{n=0}^{\infty}\frac{(\lambda\beta m)^n}{(n!)^2}\ .$$

Thus, since (4) has to hold for every (m,ℓ), the constant terms have to be equal, which amounts to

$$\frac{3(2+\theta_2+\theta_4)}{6+7(\theta_2+\theta_4)+8\theta_2\theta_4}\cdot\frac{6+7(\theta_2+\theta_4)+8\theta_2\theta_4}{4(1+\theta_2)(1+\theta_4)}\cdot\frac{3}{2}$$

$$=\frac{9(2+\theta_2)(2+\theta_4)}{(6+7\theta_2)(6+7\theta_4)}\cdot\frac{6+7\theta_4}{4(1+\theta_4)}\cdot\frac{6+7\theta_2}{4(1+\theta_2)}\ .$$

This is equivalent to

$$2(2+\theta_2+\theta_4)=(2+\theta_2)(2+\theta_4)$$

which is absurd. □

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