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NICOLAI V. KRYLOV

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A MARTINGALE PROOF OF THE KHINCHIN ITERATED LOGARITHM LAW FOR WIENER PROCESSES

N.V. KRYLOV

Let w_t be a d -dimensional Wiener process. Our goal here is to give a martingale proof of the following celebrated Khinchin log log law:

Theorem 1. *With probability one*

$$\limsup_{t \rightarrow \infty} \frac{|w_t|}{\sqrt{2t \log \log t}} = 1.$$

The “standard” proof of this result can be found in many places (cf. for instance [1]). Our martingale proof is based on formula (4), which might be interesting by itself. In particular, this formula allows us to give a rather short proof of the Kolmogorov–Petrovskii criterion allowing one to recognize when a given function $\alpha(t)$ is an “upper” or “lower” function.

Let $\alpha(t)$ be a strictly positive continuously differentiable function on $[0, \infty)$. For $t > 0$, $x \in \mathbb{R}^d$ define

$$h(t, x) = \frac{1}{t^{d/2}} e^{|x|^2/(2t)}, \quad \beta(t) = \frac{1}{t^{d/2}} e^{\alpha^2(t)/(2t)}, \quad \gamma(t) = \left(\frac{1}{\beta(t)}\right)',$$

$$\tau(t) = \inf\{s \geq t : |w_s| \geq \alpha(s)\}.$$

Lemma 2. *For any $\varepsilon > 0$ the process*

$$m_\varepsilon(t) := h(t + \varepsilon, w_t) \beta^{-1}(t + \varepsilon) - \int_0^t h(r + \varepsilon, w_r) \gamma(r + \varepsilon) dr$$

is a martingale.

To prove the lemma it suffices to observe that for $0 \leq s \leq t$ we have

$$\beta^{-1}(t + \varepsilon) - \beta^{-1}(s + \varepsilon) = \int_s^t \gamma(r + \varepsilon) dr,$$

then take a normal $(0, (t + \varepsilon)I)$ variable ξ independent of w and notice that

$$\begin{aligned} E\{h(t + \varepsilon, w_t) | \mathcal{F}_s^w\} &= E h(t + \varepsilon, x + \xi \left(\frac{t-s}{t+\varepsilon}\right)^{1/2})|_{x=w_s} = \\ &= \frac{1}{(2\pi)^{d/2}(t+\varepsilon)^d} \int_{\mathbb{R}^d} \exp \frac{1}{2(t+\varepsilon)} [|w_s + y \left(\frac{t-s}{t+\varepsilon}\right)^{1/2}|^2 - |y|^2] dy, \\ &= \frac{1}{2(t+\varepsilon)} [|w_s + y \left(\frac{t-s}{t+\varepsilon}\right)^{1/2}|^2 - |y|^2] = \\ &= \frac{1}{2(s+\varepsilon)} |w_s|^2 - \frac{1}{2(t+\varepsilon)} |y \left(\frac{s+\varepsilon}{t+\varepsilon}\right)^{1/2} + w_t \left(\frac{t-s}{s+\varepsilon}\right)^{1/2}|^2, \\ &= E\{h(t + \varepsilon, w_t) | \mathcal{F}_s^w\} = \end{aligned}$$

$$\frac{1}{(2\pi)^{d/2}(t+\varepsilon)^d} \int_{\mathbb{R}^d} \exp\left[\frac{1}{2(s+\varepsilon)}|w|_s^2 - \frac{s+\varepsilon}{2(t+\varepsilon)^2}|y|^2\right] dy = h(s+\varepsilon, w_s) \quad (\text{a.s.}).$$

Corollary 3. *If $0 < s < T < \infty$, then*

$$P\{s < \tau(s) \leq T\} + E \frac{1}{\beta(T)} h(T, w_T) I_{\tau(s) > T} =$$

$$\frac{\alpha^d(s)}{\beta(s)s^d} \kappa_d + E \int_s^T h(r, w_r) \gamma(r) I_{\tau(s) > r} dr, \quad (1)$$

where κ_d is $(2\pi)^{-d/2}$ times the volume of the unit sphere.

Indeed, $Em_\varepsilon(s) I_{\tau(s) > s} = Em_\varepsilon(\tau(s) \wedge T) I_{\tau(s) > s}$, which means that

$$E \frac{h(s+\varepsilon, w_s)}{\beta(s+\varepsilon)} I_{\tau(s) > s} + E \int_s^{\tau(s) \wedge T} h(r+\varepsilon, w_r) \gamma(r+\varepsilon) dr =$$

$$E \frac{h(\tau(s) \wedge T + \varepsilon, w_{\tau(s) \wedge T})}{\beta(\tau(s) \wedge T + \varepsilon)} I_{\tau(s) > s}.$$

It remains only to let here $\varepsilon \downarrow 0$ and apply the dominated convergence theorem along with the observations that $|w_r| \leq \sup_{r \leq T} \alpha(r)$ for $s < r \leq \tau(s) \wedge T$,

$$E \frac{h(s, w_s)}{\beta(s)} I_{\tau(s) > s} = E \frac{h(s, w_s)}{\beta(s)} I_{|w_s| < \alpha(s)} =$$

$$\frac{1}{\beta(s)s^d(2\pi)^{d/2}} \int_{|x| < \alpha(s)} dx = \frac{\alpha^d(s)}{\beta(s)s^d} \kappa_d = \kappa_d \left[\frac{\alpha^2(s)}{s} \right]^{d/2} \exp\left[-\frac{\alpha^2(s)}{2s}\right], \quad (2)$$

$$\frac{h(\tau(s) \wedge T, w_{\tau(s) \wedge T})}{\beta(\tau(s) \wedge T)} I_{\tau(s) > s} = \frac{1}{\beta(T)} h(T, w_T) I_{\tau(s) > T} +$$

$$\frac{1}{\beta(\tau(s))} h(\tau(s), w_{\tau(s)}) I_{T \geq \tau(s) > s} = \frac{1}{\beta(T)} h(T, w_T) I_{\tau(s) > T} + I_{T \geq \tau(s) > s}.$$

By letting $T \rightarrow \infty$ in (1) and by applying the monotone convergence theorem, relations (2) and the fact that $\{\tau(s) > T\} \subset \{|w_T| < \alpha(T)\}$ we immediately get the first assertion of the following lemma

Lemma 4. *Define $f(t) = \kappa_d \left[\frac{\alpha^2(t)}{t} \right]^{d/2} \exp\left[-\frac{\alpha^2(t)}{2t}\right]$, let $\gamma(t) \geq 0$ for large t and let*

$$\lim_{t \rightarrow \infty} \frac{\alpha(t)}{\sqrt{t}} = \infty \quad (\lim_{t \rightarrow \infty} f(t) = 0). \quad (3)$$

Then for any $s > 0$

$$P\{s < \tau(s) < \infty\} = f(s) + \int_s^\infty E h(r, w_r) \gamma(r) I_{\tau(s) > r} dr; \quad (4)$$

$$\int_s^\infty r^{-d/2} \gamma(r) dr = \infty \implies \limsup_{t \rightarrow \infty} [|w_t| - \alpha(t)] \geq 0 \quad (\text{a.s.}); \quad (5)$$

$$\int_s^\infty \alpha^d(r) r^{-d} \gamma(r) dr < \infty \implies \limsup_{t \rightarrow \infty} [|w_t| - \alpha(t)] \leq 0 \quad (\text{a.s.}). \quad (6)$$

Proof. To prove (5) notice that $h(r, w_r) \geq r^{-d/2}$, so that

$$\int_s^\infty E h(r, w_r) \gamma(r) I_{\tau(s) > r} dr \geq \int_s^\infty r^{-d/2} \gamma(r) dr P\{\tau(s) = \infty\},$$

and under the condition in (5) we have $P\{\tau(s) = \infty\} = 0$. It remains only to observe that

$$\{\omega : \limsup_{t \rightarrow \infty} [|w_t| - \alpha(t)] < 0\} \subset \bigcup_{n=1}^\infty \{\omega : \tau(n) = \infty\}.$$

To prove (6) we use (2) and that $\{\tau(s) > r\} \subset \{|w_r| < \alpha(r)\}$ if $r > s$. Then

$$\int_s^\infty E h(r, w_r) \gamma(r) I_{\tau(s) > r} dr \leq \kappa_d \int_s^\infty \alpha^d(r) r^{-d} \gamma(r) dr,$$

and from (4) we see that under the condition in (6), $P\{s < \tau(s) < \infty\} \rightarrow 0$ as $s \rightarrow \infty$. Finally, for any $s > 0$

$$\{\omega : \limsup_{t \rightarrow \infty} [|w_t| - \alpha(t)] > 0\} \subset \{\omega : |w_s| \geq \alpha(s)\} \cup \{\omega : s < \tau(s) < \infty\},$$

$$P\{\limsup_{t \rightarrow \infty} [|w_t| - \alpha(t)] > 0\} \leq \lim_{s \rightarrow \infty} P\{|w_s| \geq \alpha(s)\} = \lim_{s \rightarrow \infty} P\{|w_1| \geq \frac{\alpha(s)}{\sqrt{s}}\} = 0.$$

The lemma is proved.

Proof of Theorem 1. Take $\varepsilon \in [0, 1]$ and define $\alpha(t) = ((1 + \varepsilon)2t \log \log t)^{1/2}$ if $t \geq 10$ and for $t < 10$ define $\alpha(t)$ in any way just to get a positive differentiable function on $[0, \infty)$. Then $\beta(t) = t^{-d/2}(\log t)^{1+\varepsilon}$, $\alpha(t)/\sqrt{t} \rightarrow \infty$ for any ε , and as easy to check

$$\int^\infty r^{-d/2} d\frac{1}{\beta(r)} = \infty \text{ if } \varepsilon = 0, \quad \int^\infty \alpha^d(r) r^{-d} d\frac{1}{\beta(r)} < \infty \text{ if } \varepsilon > 0;$$

from Lemma 4 it follows that

$$\limsup_{t \rightarrow \infty} \frac{|w_t|}{(2t \log \log t)^{1/2}} \geq 1, \quad \limsup_{t \rightarrow \infty} \frac{|w_t|}{(2t \log \log t)^{1/2}} \leq 1 + \varepsilon \text{ (a.s.) if } \varepsilon > 0.$$

The theorem is proved.

Next observe that

$$\begin{aligned} \int^\infty \frac{\alpha^d(r)}{r^d} \gamma(r) dr &= \int^\infty \frac{\alpha^d(r)}{r^d} d(r^{d/2} \exp[-\frac{\alpha^2(r)}{2r}]) = \\ &= \frac{d}{2} \int^\infty \frac{\alpha^d(r)}{r^{d/2}} \exp[-\frac{\alpha^2(r)}{2r}] \frac{dr}{r} - \frac{1}{2} \int^\infty \frac{\alpha^d(r)}{r^{d/2}} \exp[-\frac{\alpha^2(r)}{2r}] d\frac{\alpha^2(r)}{r}, \end{aligned}$$

and under condition (3) the integral

$$\int^\infty \frac{\alpha^d(r)}{r^{d/2}} \exp[-\frac{\alpha^2(r)}{2r}] d\frac{\alpha^2(r)}{r} = \int^\infty x^{d/2} e^{-x/2} dx$$

is finite, so that the integrals

$$\int^\infty \frac{\alpha^d(r)}{r^d} \gamma(r) dr, \quad \int^\infty \frac{\alpha^d(r)}{r^{d/2}} \exp[-\frac{\alpha^2(r)}{2r}] \frac{dr}{r}$$

converge or diverge simultaneously.

Now we see that the statement (6) in Lemma 4 implies the second statement in the following Kolmogorov–Petrovskii criterion.

Theorem 5. Assume that $t^{d/2} \exp[-\alpha^2(t)/(2t)]$ and $t\alpha^{-1}(t)$ increase for large t , and assume (3). Then

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\alpha^d(r)}{r^{d/2}} \exp\left[-\frac{\alpha^2(r)}{2r}\right] \frac{dr}{r} = \infty &\implies \limsup_{t \rightarrow \infty} \frac{|w_t|}{\alpha(t)} \geq 1 \text{ (a.s.)}; \\ \int_{-\infty}^{\infty} \frac{\alpha^d(r)}{r^{d/2}} \exp\left[-\frac{\alpha^2(r)}{2r}\right] \frac{dr}{r} < \infty &\implies \limsup_{t \rightarrow \infty} \frac{|w_t|}{\alpha(t)} \leq 1 \text{ (a.s.)}. \end{aligned} \quad (7)$$

Proof. We only need to prove (7), and to do this we come back to statement (4) of Lemma 4 but analyze it slightly more carefully. Notice that for any given $r > 0$ the process $w_t - w_r t/r$, $t \in [0, r]$, and the random vector w_r are independent. Therefore for $r > s$

$$\begin{aligned} Eh(r, w_r) I_{\tau(s) > r} &= \frac{1}{(2\pi r)^{d/2}} \int_{|x| < \alpha(r)} e^{-|x|^2/(2r)} h(r, x) P\{\tau(s) > r | w_r = x\} dx = \\ &= \frac{1}{(2\pi)^{d/2} r^d} \int_{|x| < \alpha(r)} P\{\tau(s) > r | w_r = x\} dx = \\ &= \frac{1}{(2\pi)^{d/2} r^d} \int_{|x| < \alpha(r)} P\left\{\sup_{t \in [s, r]} \left|w_t - \frac{t}{r} w_r + \frac{t}{r} x\right| \alpha^{-1}(t) < 1\right\} dx = \\ &= \frac{\alpha^d(r)}{(2\pi)^{d/2} r^d} \int_{|x| < 1} P\left\{\sup_{t \in [s, r]} \left|w_t - \frac{t}{r} w_r + \frac{t}{r} \alpha(r) x\right| \alpha^{-1}(t) < 1\right\} dx. \end{aligned}$$

From (4) it now follows that if $\int_{-\infty}^{\infty} \alpha^d(r) r^{-d} \gamma(r) dr = \infty$, then for any $s > 0$

$$\liminf_{r \rightarrow \infty} \int_{|x| < 1} P\left\{\sup_{t \in [s, r]} \left|w_t - \frac{t}{r} w_r + \frac{t}{r} \alpha(r) x\right| \alpha^{-1}(t) < 1\right\} dx = 0. \quad (8)$$

Now observe that for $\varepsilon \in (0, 1)$, $|x| \leq \varepsilon$ and s so large that $t\alpha^{-1}(t)$ increases for $t \geq s$, we have $(t/r)\alpha(r)|x|\alpha^{-1}(t) \leq \varepsilon$,

$$\begin{aligned} &P\left\{\sup_{t \in [s, r]} \left|w_t - \frac{t}{r} w_r + \frac{t}{r} \alpha(r) x\right| \alpha^{-1}(t) < 1\right\} \geq \\ &P\left\{\sup_{t \in [s, r]} \left|w_t - \frac{t}{r} \alpha(r) [w_r \alpha^{-1}(r)]\right| \alpha^{-1}(t) < 1 - \varepsilon\right\} \geq \\ &P\left\{\sup_{t \in [s, r]} \left|w_t - \frac{t}{r} \alpha(r) [w_r \alpha^{-1}(r)]\right| \alpha^{-1}(t) < 1 - \varepsilon, |w_r| \alpha^{-1}(r) \leq \varepsilon\right\} \geq \\ &P\left\{\sup_{t \in [s, r]} \frac{|w_t|}{\alpha(t)} < 1 - 2\varepsilon\right\} - P\{|w_r| > \varepsilon \alpha(r)\}. \end{aligned}$$

Since $\alpha(r)/\sqrt{r} \rightarrow \infty$, the last probability tends to zero as $r \rightarrow \infty$, and from (8) it follows that for any $\varepsilon \in (0, 1)$ we have

$$P\left\{\sup_{t \geq s} \frac{|w_t|}{\alpha(t)} < 1 - 2\varepsilon\right\} \leq \lim_{r \rightarrow \infty} P\left\{\sup_{t \in [s, r]} \frac{|w_t|}{\alpha(t)} < 1 - 2\varepsilon\right\} = 0.$$

This obviously yields the first assertion of our theorem, which is thus proved.

Remark 6. From the zero-one law it follows easily that for one-dimensional Wiener process B_t

$$\limsup_{t \rightarrow \infty} \frac{B_t}{\alpha(t)} \leq (\geq) 1 \text{ (a.s.)} \implies \limsup_{t \rightarrow \infty} \frac{|w_t|}{\alpha(t)} \leq (\geq) 1 \text{ (a.s.)}.$$

Therefore, consideration of arbitrary d does not yield any advantage, though it actually might happen that for $d = 1$ the integral in (7) converges and, say for $d = 2$ diverges. In this case \limsup in (7) simply equals 1 (a. s.).

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127 VINCENT HALL, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN, 55455

E-mail address: krylov@math.umn.edu