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ON THE SPITZER AND CHUNG LAWS OF THE ITERATED LOGARITHM FOR BROWNIAN MOTION

by

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1. Introduction.

Let $\{W(t); t \geq 0\}$ be a two-dimensional Brownian motion. It is well-known that the Brownian path is almost surely dense in the plane, but never hits a given point at positive time. A natural question is thus to study the rate with which the small values of $\|W - x\|$ (the symbol " $\|\cdot\|$ " denoting the usual Euclidean modulus) approach 0 for any $x \in \mathbb{R}^2$. Without loss of generality, we assume $W(0) = (1, 0)$ and $x = (0, 0)$. Let

$$X(t) = \inf_{0 \leq s \leq t} \|W(s)\|, \quad t > 0.$$

The following celebrated Spitzer (1958) integral test characterizes the lower functions of X :

Theorem A (Spitzer 1958). *For any non-decreasing function $f > 1$,*

$$\mathbb{P}\left[X(t) < \frac{t^{1/2}}{f(t)}, \text{ i.o.}\right] = \begin{cases} 0 \\ 1 \end{cases} \iff \int^{\infty} \frac{dt}{t \log f(t)} \begin{cases} < \infty \\ = \infty \end{cases}.$$

Here and in the sequel, "i.o." stands for "infinitely often" as t tends to infinity.

Spitzer's Theorem A answers the how-small-are-the-small-values-of- $\|W\|$ question. We propose to study the corresponding "how big" problem for the small values. Our Theorem 1, stated as follows, provides a characterization of the upper functions of X .

Theorem 1. *If $g > 1$ is non-decreasing, then*

$$\mathbb{P}\left[X(t) > \exp\left(-\frac{\log t}{g(t)}\right), \text{ i.o.}\right] = \begin{cases} 0 \\ 1 \end{cases} \iff \int^{\infty} \frac{dt}{t g(t) \log t} dt \begin{cases} < \infty \\ = \infty \end{cases}.$$

Theorems A and 1 together give an accurate description of the almost sure asymptotic behaviours of X . For example, it is immediately seen from the above theorems that

$$\mathbb{P}\left[X(t) > \exp\left(-\frac{\log t}{(\log \log t)^a}\right), \text{ i.o.}\right] = \begin{cases} 0 & \text{if } a > 1; \\ 1 & \text{otherwise.} \end{cases}$$

$$\mathbb{P}\left[X(t) < \exp\left(-(\log t)(\log \log t)^a\right), \text{ i.o.}\right] = \begin{cases} 0 & \text{if } a > 1; \\ 1 & \text{otherwise.} \end{cases}$$

What about the lower functions of the big values of Brownian motion? Let us recall the classical Chung (1948) integral test for linear Brownian motion.

Theorem B (Chung 1948). *Let B be a real-valued Brownian motion. For every non-decreasing function $h > 0$ such that $t^{-1/2}h(t)$ is non-increasing, we have*

$$\mathbb{P}\left[\sup_{0 \leq s \leq t} |B(s)| < \frac{t^{1/2}}{h(t)}, \text{ i.o.}\right] = \begin{cases} 0 \\ 1 \end{cases} \iff \int^{\infty} \frac{dt}{t} h^2(t) \exp\left(-\frac{\pi^2}{8} h^2(t)\right) \begin{cases} < \infty \\ = \infty \end{cases}.$$

Chung's Theorem B was obtained for linear Brownian motion. The following natural question was raised by Révész (1990, p.195): for the planar Brownian motion W , what can be said on the liminf behaviour of $\sup_{0 \leq s \leq t} \|W(s)\|$?

The same question can be asked for a Brownian motion of any dimension. Let $\{V(t); t \geq 0\}$ denote a d -dimensional Brownian motion, and let

$$Y(t) = \sup_{0 \leq s \leq t} \|V(s)\|, \quad t > 0.$$

Our answer to the problem is the following

Theorem 2. *Let $d \geq 1$ and let $h > 0$ be a non-decreasing function. Then*

$$\mathbb{P}\left[Y(t) < \frac{t^{1/2}}{h(t)}, \text{ i.o.}\right] = \begin{cases} 0 \\ 1 \end{cases} \iff \int^\infty dt \frac{h^2(t)}{t} \exp\left(-\frac{j_\nu^2}{2} h^2(t)\right) \begin{cases} < \infty \\ = \infty \end{cases},$$

where j_ν denotes the smallest positive zero of the Bessel function $J_\nu(x)$ of index $\nu \equiv (d-2)/2$.

Remarks. (i) Since $j_{-1/2} = \pi/2$, Theorem B is a special case of the above result.

(ii) An interesting feature in Theorem 2 is that we do not suppose $t^2/h(t)$ to be non-decreasing. Thus the latter condition can be removed from Chung's Theorem B.

(iii) As usual, Theorem 2 has a "local" version for t tending to 0, of which the statement and proof are omitted.

Corollary 1. *We have, for $d \geq 1$,*

$$(1.1) \quad \liminf_{t \rightarrow \infty} \left(\frac{\log \log t}{t} \right)^{1/2} Y(t) = \frac{j_\nu}{2^{1/2}} \quad \text{a.s.},$$

with rate of convergence

$$\liminf_{t \rightarrow \infty} \frac{(\log \log t)^{3/2}}{t^{1/2} \log \log t} \left(Y(t) - \frac{j_\nu}{2^{1/2}} \frac{t^{1/2}}{(\log \log t)^{1/2}} \right) = -\frac{j_\nu}{2^{1/2}}, \quad \text{a.s.}$$

Remark. The LIL (1.1) was previously obtained by Lévy (1953) for $d = 2$ and by Ciesielski & Taylor (1962) for any dimension d .

Theorem 1 is proved in Section 2, and Theorem 2 in Section 3.

2. The proof of Theorem 1.

Let W be as before a Brownian motion in the plane, starting from $(1, 0)$. Define

$$H(x) = \inf\{t > 0 : \|W(t)\| = x\}, \quad 0 < x \leq 1,$$

the first hitting time of $\|W\|$ at level x . Obviously the process $y \mapsto H(1/y)$ (for $y \geq 1$) is increasing, and it has independent increments by using the strong Markov property of $\|W\|$. Since

$$H(2^{-n}) = \sum_{k=1}^n \left(H(2^{-k}) - H(2^{-(k-1)}) \right),$$

using Brownian scaling gives

$$H(2^{-n}) = \sum_{k=1}^n 2^{-2(k-1)} \xi_k,$$

where $(\xi_k)_{k \geq 1}$ is an i.i.d. sequence of random variables having the same law as $H(1/2)$. Consequently,

$$(2.1) \quad 2^{-2(n-1)} \sum_{k=1}^n \xi_k \leq H(2^{-n}) \leq \sum_{k=1}^n \xi_k.$$

Let us first establish a preliminary result for the partial sum of (ξ_k) :

Lemma 1. *Let $\{\Lambda(t); t \geq 0\}$ be a subordinator, and assume that $\Lambda(1)$ has the same law as $H(1/2)$. Then for any function $f > 1$ such that $f(t)/t$ is non-decreasing, we have*

$$\limsup_{t \rightarrow \infty} \frac{\Lambda(t)}{f(t)} = \begin{cases} 0 \\ \infty \end{cases}, \quad \text{a.s.} \iff \int^{\infty} \frac{dt}{\log f(t)} \begin{cases} < \infty \\ = \infty \end{cases}.$$

Proof of Lemma 1. The Laplace transform of $H(1/2)$ is well-known (see Kent (1978 Theorem 3.1)):

$$\mathbb{E} \exp(-\lambda H(1/2)) = \frac{K_0(\sqrt{2\lambda})}{K_0(\sqrt{\lambda/2})}, \quad \forall \lambda > 0,$$

where K_0 is the modified Bessel function. Recall that $\Lambda(1)$ has the same law as $H(1/2)$. Write

$$\mathbb{E} \exp(-\lambda \Lambda(1)) = \exp(-\Psi(\lambda)).$$

Thus

$$\Psi(\lambda) = \log K_0(\sqrt{\lambda/2}) - \log K_0(\sqrt{2\lambda}).$$

Using elementary asymptotics of K_0 , we immediately arrive at the following estimate

$$\Psi(\lambda) - \lambda\Psi'(\lambda) \sim \frac{2\log 2}{\log(1/\lambda)}, \quad \lambda \rightarrow 0,$$

(the usual symbol " $a(x) \sim b(x)$ " ($x \rightarrow x_0$) means $\lim_{x \rightarrow x_0} a(x)/b(x) = 1$). Now the statement of Lemma 1 follows by applying a general result for subordinators (see for example Fristedt (1974 Theorem 6.1)) which tells that $\limsup_{t \rightarrow \infty} \Lambda(t)/f(t) = 0$ or ∞ (almost surely) according as

$$\int_0^\infty \left(\Psi\left(\frac{1}{f(t)}\right) - \frac{1}{f(t)} \Psi'\left(\frac{1}{f(t)}\right) \right) dt$$

converges or diverges. □

Lemma 2. *If $h > 1$ is a non-decreasing function with $\int_0^\infty dt/h(t) = \infty$, then*

$$\int_0^\infty \frac{dt}{t + h(t)} = \infty.$$

Proof of Lemma 2. The proof is briefly sketched, since it involves only elementary computations. Set $\mathcal{A} = \{t : h(t) \leq t\}$ and $\mathcal{B} = \{t : h(t) > t\}$. Obviously, we have

$$\frac{1}{t + h(t)} \geq \frac{1}{2} \left(\frac{1}{t} \mathbf{1}_{\mathcal{A}}(t) + \frac{1}{h(t)} \mathbf{1}_{\mathcal{B}}(t) \right).$$

Assume $\int_0^\infty \mathbf{1}_{\mathcal{A}}(t)(dt/t) < \infty$. We only have to show $\int_0^\infty \mathbf{1}_{\mathcal{B}}(t)(dt/h(t)) = \infty$. Write $F_{\mathcal{A}}(t) \equiv \int_1^t \mathbf{1}_{\mathcal{A}}(s)ds$. Using integration by parts for $\int \mathbf{1}_{\mathcal{A}}(s)(ds/s)$, it is seen that $t \mapsto F_{\mathcal{A}}(t)/t$ is a Cauchy family for $t > 1$. Thus $\lim_{t \rightarrow \infty} F_{\mathcal{A}}(t)/t$ exists. If $\lim_{t \rightarrow \infty} F_{\mathcal{A}}(t)/t > 0$, then $\int_1^t F_{\mathcal{A}}(s)(ds/s^2)$ would diverge, which contradicts the convergence of $\int_1^\infty \mathbf{1}_{\mathcal{A}}(s)(ds/s)$ (the latter is obviously greater than $\int_1^t F_{\mathcal{A}}(s)(ds/s^2)$ by integration by parts). Consequently, $\lim_{t \rightarrow \infty} F_{\mathcal{A}}(t)/t = 0$. Thus $\int_1^t \mathbf{1}_{\mathcal{B}}(s)ds \geq t/2$ for sufficiently large t . Again using integration by parts, we obtain

$$\int_1^t \mathbf{1}_{\mathcal{B}}(s) \frac{ds}{h(s)} \geq \frac{1}{2} \int_1^t \frac{ds}{h(s)} + \text{a finite term},$$

which diverges as t tends to infinity. Lemma 2 is proved. □

Proof of Theorem 1. Pick $0 < x < 1$, and let us write $2^{-n} \leq x \leq 2^{-(n-1)}$ (which means $(n-1)\log 2 \leq \log(1/x) \leq n\log 2$). Then $H(2^{-(n-1)}) \leq H(x) \leq H(2^{-n})$. Using (2.1), we have

$$(2.2) \quad x^2 \Lambda \left(\log(1/x)/\log 2 - 1 \right) \leq H(x) \leq \Lambda \left(\log(1/x)/\log 2 + 1 \right).$$

First, we show the following integral test for H :

$$(2.3) \quad \limsup_{x \rightarrow 0^+} \frac{H(x)}{f(x)} = \begin{cases} 0 \\ \infty \end{cases}, \quad \text{a.s.} \iff \int_{0^+} \frac{dx}{x \log f(x)} \begin{cases} < \infty \\ = \infty \end{cases},$$

for any function $f > 1$ such that $\log f(x)/\log(1/x)$ is non-increasing. Indeed, assume that $\int_{0^+} dx/(x \log f(x))$ converges. Define $\hat{f}(t) = f(2^{-t/2})$. Then $\hat{f}(t)/t$ is non-decreasing, with $\int_0^\infty dt/\log \hat{f}(t) < \infty$. By Lemma 1, we have $\limsup_{t \rightarrow \infty} \Lambda(t)/\hat{f}(t) = 0$, with probability 1. Thus $\limsup_{t \rightarrow \infty} \Lambda(t)/f(2^{-(t-1)}) = 0$. Using the second part of (2.2), this implies $\limsup_{x \rightarrow 0^+} H(x)/f(x) = 0$. It remains to verify the divergent half of (2.3). Suppose $\int_{0^+} dx/(x \log f(x)) = \infty$. Then $\int_0^\infty dt/\log f(2^{-2(t+1)})$ diverges as well. According to Lemma 2, this implies

$$\int_0^\infty \frac{dt}{\log \tilde{f}(t)} = \infty,$$

for $\tilde{f}(t) \equiv 2^{2(t+1)} f(2^{-2(t+1)})$. Applying Lemma 1 gives $\limsup_{t \rightarrow \infty} \Lambda(t)/\tilde{f}(t) = \infty$, which, by means of the first part of (2.2), yields $\limsup_{x \rightarrow 0^+} H(x)/f(x) = \infty$. Hence (2.3) is proved. By noting $[H(x) > t] = [X(t) > x]$ (for any $0 < x < 1$ and $t > 0$), several lines of standard calculation readily confirm that the integral test (2.3) is equivalent to that in Theorem 1. \square

3. The proof of Theorem 2.

In this section, V denotes a d -dimensional Brownian motion, which, without loss of generality, is assumed to start from 0. Let $H(x) = \inf\{t > 0 : \|V(t)\| = x\}$ (for $x > 0$). The proof of Theorem 2 is essentially based on the following exact density function of $H(1)$ due to Ciesielski & Taylor (1962):

$$\mathbb{P}[H(1) \in dt]/dt = \frac{1}{2^\nu \Gamma(\nu+1)} \sum_{n=1}^{\infty} \frac{j_{\nu,n}^{\nu+1}}{J_{\nu+1}(j_{\nu,n})} \exp\left(-\frac{j_{\nu,n}^2}{2}t\right), \quad t > 0,$$

where $\nu = (d-2)/2$, and $0 < j_{\nu,1} < j_{\nu,2} < \dots$ are the positive zeros of the Bessel function J_ν (and of course $J_{\nu+1}$ denotes the Bessel function of index $\nu+1$). Let Y be as before the supremum process of $\|V\|$. By Brownian scaling, we have, for any $x > 0$,

$$\begin{aligned}\mathbb{P}[Y(1) < x] &= \mathbb{P}[H(x) > 1] = \mathbb{P}[H(1) > 1/x^2] \\ &= \frac{2^{1-\nu}}{\Gamma(\nu+1)} \sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^{1-\nu} J_{\nu+1}(j_{\nu,n})} \exp\left(-\frac{j_{\nu,n}^2}{2x^2}\right),\end{aligned}$$

which implies

$$(3.1) \quad \mathbb{P}[Y(1) < x] \sim \frac{2^{1-\nu}}{\Gamma(\nu+1)j_\nu^{1-\nu}J_{\nu+1}(j_\nu)} \exp\left(-\frac{j_\nu^2}{2x^2}\right), \quad \text{as } x \rightarrow 0,$$

(recall that $j_\nu \equiv j_{\nu,1}$ is the smallest positive zero of J_ν). Write in the sequel $\rho \equiv j_\nu^2/2$.

Let $h > 0$ be a non-decreasing function. In the rest of the note, generic constants will be denoted by K_i ($1 \leq i \leq 9$).

We begin with the convergent part of Theorem 2, which is an immediate consequence of the tail estimation (3.1). Indeed, pick a sufficiently large initial value t_0 and define the sequence $(t_k)_{k \geq 1}$ by $t_{k+1} = (1 + h^{-2}(t_k))t_k$ for $k = 0, 1, 2, \dots$, and write $h_k \equiv h(t_k)$ for notational convenience. Obviously t_k increases to infinity (as $k \rightarrow \infty$). Assume that $\int^\infty (dt/t)h^2(t)\exp(-\rho h^2(t))$ converges. This implies, by several lines of elementary calculation, the convergence of $\sum_k \exp(-\rho h_k^2)$. From (3.1) and scaling it follows that

$$\begin{aligned}\mathbb{P}\left[Y(t_k) < \frac{t_k^{1/2}}{h_k - 1/h_k}\right] &= \mathbb{P}\left[Y(1) < \frac{1}{h_k - 1/h_k}\right] \\ &\leq K_1 \exp(-\rho(h_k - 1/h_k)^2) \\ &\leq K_2 \exp(-\rho h_k^2),\end{aligned}$$

which sums. According to Borel-Cantelli lemma, (almost surely) for large k , $Y(t_k) \geq t_k^{1/2}/(h_k - 1/h_k)$. Let $t \in [t_k, t_{k+1}]$. Then by our construction of (t_k) ,

$$Y(t) \geq Y(t_k) \geq \frac{t_k^{1/2}}{h_k - 1/h_k} = \frac{t_{k+1}^{1/2}}{(1 + 1/h_k^2)(h_k - 1/h_k)} \geq \frac{t_{k+1}^{1/2}}{h_k} \geq \frac{t^{1/2}}{h(t)},$$

which yields the convergent part of Theorem 2.

It remains to show the divergent part. Let h be such that

$$(3.2) \quad \int^\infty \frac{dt}{t} h^2(t) \exp(-\rho h^2(t)) = \infty.$$

In view of (1.1), we assume without loss of generality that

$$(3.3) \quad \frac{(\log \log t)^{1/2}}{(2\rho)^{1/2}} \leq h(t) \leq \frac{(2 \log \log t)^{1/2}}{\rho^{1/2}}.$$

Define $t_k = \exp(k/\log k)$ (for $k \geq k_0$) and write as before $h_k \equiv h(t_k)$. In what follows, we only deal with the index k tending ultimately to infinity. Therefore our results, sometimes without further mention, are to be understood for sufficiently large k 's. Obviously (3.2) is equivalent to the following

$$(3.4) \quad \sum_k \exp(-\rho h_k^2) = \infty.$$

Using (3.3) gives

$$(3.5) \quad \frac{(\log k)^{1/2}}{(3\rho)^{1/2}} \leq h_k \leq \log k.$$

Fix an $\varepsilon > 0$, then

$$(3.6) \quad k^{-1/2} h_k^2 \leq \frac{\varepsilon}{\rho},$$

(for $k \geq k_0$). Consider the measurable event $A_k = \{Y(t_k) < t_k^{1/2}/h_k\}$. From (3.1) it follows that for $k \geq k_0$,

$$(3.7) \quad \mathbb{P}(A_k) = \mathbb{P}\left[Y(1) < \frac{1}{h_k}\right] \geq (1 - \varepsilon) \frac{2^{1-\nu}}{\Gamma(\nu+1) j_\nu^{1-\nu} J_{\nu+1}(j_\nu)} \exp(-\rho h_k^2),$$

which, by means of (3.4), yields

$$(3.8) \quad \sum_k \mathbb{P}(A_k) = \infty.$$

Let $k < \ell$. Since V has independent and stationary increments, we have

$$\begin{aligned} \mathbb{P}(A_k A_\ell) &= \mathbb{P}\left[\sup_{0 \leq t \leq t_k} \|V(t)\| < \frac{t_k^{1/2}}{h_k}, \sup_{0 \leq t \leq t_\ell} \|V(t)\| < \frac{t_\ell^{1/2}}{h_\ell}\right] \\ &\leq \mathbb{P}(A_k) \sup_{\|x\| \leq t_k^{1/2}/h_k} \mathbb{P}\left[\sup_{0 \leq t \leq t_\ell - t_k} \|V(t) + x\| < \frac{t_\ell^{1/2}}{h_\ell}\right]. \end{aligned}$$

Using a general property of Gaussian measures (see for example Ledoux & Talagrand (1991 p.73)), it follows that

$$\begin{aligned} \mathbb{P}(A_k A_\ell) &\leq \mathbb{P}(A_k) \mathbb{P}\left[\sup_{0 \leq t \leq t_\ell - t_k} \|V(t)\| < \frac{t_\ell^{1/2}}{h_\ell}\right] \\ (3.9) \quad &= \mathbb{P}(A_k) \mathbb{P}\left[Y(1) < \frac{t_\ell^{1/2}}{(t_\ell - t_k)^{1/2} h_\ell}\right]. \end{aligned}$$

For every $n > k_0$, define

$$\mathcal{E}(n) = \{k_0 \leq k < \ell \leq n : \ell - k \leq (\log k)^3\},$$

$$\mathcal{F}(n) = \{k_0 \leq k < \ell \leq n : \ell - k > (\log k)^3\}.$$

It is seen that when $k < \ell < k + (\log k)^3$,

$$\begin{aligned} \frac{\ell}{\log \ell} - \frac{k}{\log k} &= \frac{\ell \log k - k \log \ell}{\log k \log \ell} \\ &= \frac{(\ell - k) \log k - k \log(1 + (\ell - k)/k)}{\log k \log \ell} \\ &\sim \frac{\ell - k}{\log k}, \quad (k \rightarrow \infty), \end{aligned}$$

which implies

$$\frac{t_k}{t_\ell} \leq \exp\left(-\frac{\ell - k}{2 \log k}\right).$$

Let $(k, \ell) \in \mathcal{E}(n)$. From the above estimate it follows that

$$\frac{t_\ell^{1/2}}{(t_\ell - t_k)^{1/2}} \leq \left(1 - \exp\left(-\frac{\ell - k}{2 \log k}\right)\right)^{-1/2} \leq \left[K_3 \min\left((\ell - k)/\log k, 1\right)\right]^{-1/2},$$

which, by means of (3.5), yields

$$\frac{t_\ell^{1/2}}{(t_\ell - t_k)^{1/2} h_\ell} \leq \frac{t_\ell^{1/2}}{(t_\ell - t_k)^{1/2} h_k} \leq \left[K_4 \min(\ell - k, \log k)\right]^{-1/2}.$$

From (3.9) and (3.1) it follows that

$$\mathbb{P}(A_k A_\ell) \leq K_5 \mathbb{P}(A_k) \exp(-K_6(\ell - k)) + K_5 \mathbb{P}(A_k) k^{-K_6}.$$

Obviously,

$$\begin{aligned} \sum_{\ell > k} \exp(-K_6(\ell - k)) &\leq K_7, \\ \sum_{k < \ell < k + (\log k)^3} k^{-K_6} &\leq k^{-K_6} (\log k)^3 \leq K_8. \end{aligned}$$

Therefore,

$$(3.10) \quad \sum_{(k, \ell) \in \mathcal{E}(n)} \mathbb{P}(A_k A_\ell) \leq K_9 \sum_{k=k_0}^n \mathbb{P}(A_k).$$

Now let $(k, \ell) \in \mathcal{F}(n)$. In this case, $\ell - (\log \ell)^2 \geq k + (\log k)^3 - (\log(k + (\log k)^3))^2 > k$, thus $\ell - k > (\log \ell)^2$. Since

$$\frac{\ell}{\log \ell} - \frac{k}{\log k} = \frac{(\ell - k) \log k - k \log(1 + (\ell - k)/k)}{\log k \log \ell} \sim \frac{\ell - k}{\log \ell},$$

we have

$$\frac{t_\ell^{1/2}}{(t_\ell - t_k)^{1/2} h_\ell} \leq \frac{1}{[1 - \exp(-(\ell - k)/2 \log \ell)]^{1/2} h_\ell} \leq \frac{1}{(1 - \ell^{-1/2})^{1/2} h_\ell}.$$

By means of (3.9), (3.1), (3.6) and (3.7), this implies

$$\begin{aligned} \mathbb{P}(A_k A_\ell) &\leq \mathbb{P}(A_k)(1 + \varepsilon) \exp(\rho \ell^{-1/2} h_\ell^2) \frac{2^{1-\nu}}{\Gamma(\nu + 1) j_\nu^{1-\nu} J_{\nu+1}(j_\nu)} \exp(-\rho h_k^2) \\ &\leq (1 + 3\varepsilon) e^\varepsilon \mathbb{P}(A_k) \mathbb{P}(A_\ell). \end{aligned}$$

Combining the above estimate together with (3.10) and (3.8) yields

$$\liminf_{n \rightarrow \infty} \sum_{k=k_0}^n \sum_{\ell=k_0}^n \mathbb{P}(A_k A_\ell) \Big/ \left(\sum_{k=k_0}^n \mathbb{P}(A_k) \right)^2 \leq 1.$$

According to a well-known version of Borel-Cantelli lemma (see for example Révész (1990 p.28)), we have $\mathbb{P}(A_k, \text{i.o.}) = 1$. The proof of the divergent part of Theorem 2 is completed. \square

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