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On the differentiability of functions of an operator

by YaoZhong HU

Introduction. Let f be a continuous function on $\mathbb{R}$. Then it is well known how to define $f(A)$ when A is a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$, using the spectral decomposition of A . But if A, H are two non-commuting self-adjoint operators, no explicit computation of $f(A+tH)$ is known. Our problem here is to study the regularity of $f(A+tH)$ under some regularity assumptions on f . We will assume that $\mathcal{H}$ is finite-dimensional. This note is a complement to our article “Some operator inequalities” in volume XXVIII¹, and answers a question of P.A. Meyer.

Notation. Given real numbers $\lambda_i \neq \lambda_j$, we put (divided differences : see the first pages of Donoghue [1] for more detail)

$$\{\lambda_2, \lambda_1\}f = \frac{f(\lambda_2) - f(\lambda_1)}{\lambda_2 - \lambda_1}$$

$$\{\lambda_{k+1}, \dots, \lambda_1\}f = \{\lambda_{k+1}, \lambda_k\}\{\cdot, \lambda_{k-1}, \dots, \lambda_1\}f$$

LEMMA. Let f have continuous derivatives up to of order $n+1$. Let $R_{n+1}(a, b)$ be the corresponding Taylor remainder

$$R_{n+1}(a, b)f = f(b) - f(a) - (b-a)f'(a) - \frac{(b-a)^{n+1}}{(n+1)!}f^{(n+1)}(a).$$

Then $\{\cdot, \lambda\}f$ has derivatives up to order n , and we have

$$\frac{d^n}{dt^n}\{t, \lambda\}f = \frac{n!}{(\lambda - x)^{n+1}} R_{n+1}(\lambda, x)f.$$

One can deduce that, if f is of class $\mathcal{C}^k$, the function $\{\lambda_{k+1}, \dots, \lambda_1\}f$ can be extended by continuity to $\mathbb{R}^{k+1}$ including the diagonals, and that

$$|\{\lambda_{k+1}, \dots, \lambda_1\}f| \leq \gamma(k) \|f\|_{k;T}$$

where the last norm is the $\mathcal{C}^k$ norm of f on any interval T containing $\lambda_1, \dots, \lambda_{k+1}$. Then many results proved for polynomials $f(t) = t^d$ can be extended by density to $\mathcal{C}^k$ functions. In particular, the divided differences are symmetric in all their arguments from the following lemma, which is the crucial point of the calculation.

LEMMA. When $f(t) = t^d$, we have

$$\{\lambda_k, \dots, \lambda_1\}f = \sum_{m_k + \dots + m_1 = d-k+1} \lambda_k^{m_k} \dots \lambda_1^{m_1}.$$

PROOF. By induction on k .

¹ La rédaction du Séminaire regrette ce retard de publication, dû à une erreur de transmission du manuscrit.

Computation of derivatives. When f is a polynomial, the operator function $f(A)$ is infinitely differentiable at every A and we can write its partial derivatives

$$\frac{\partial^k}{\partial t_1 \dots \partial t_k} f(A + t_1 H^1 + \dots + t_k H^k) \Big|_{t_1 = \dots = t_k = 0} = \Phi(A; H^1, \dots, H^k)$$

where $\Phi(A; \cdot)$ is a symmetric k -linear functional. The problem is to give a uniform estimate of these derivatives knowing the C^k norm of f , which will allow us to extend the result from polynomials to C^k functions. Since Φ arises from polarization of the function $\Phi(H, \dots, H)$, it will be sufficient to estimate this function.

When $f(t) = t^d$ we have

$$\frac{d^k}{dt^k} (A + tH)^d \Big|_{t=0} = \sum_{m_1 + \dots + m_{k+1} = d-k} A^{m_1} H \dots H A^{m_{k+1}}$$

Choose a basis in which A is diagonal with eigenvalues λ_i . Then the matrix of this operator D is

$$D_{ij} = \sum_{u_1, \dots, u_{k+1}} \delta_{iu_1} h_{u_1 u_2} \dots h_{u_k u_{k+1}} \delta_{u_{k+1} j} \sum_{m_1 + \dots + m_{k+1} = d-k} \lambda_{u_1}^{m_1} \dots \lambda_{u_{k+1}}^{m_{k+1}}$$

and this last coefficient is $\{\lambda_{u_{k+1}}, \dots, \lambda_{u_1}\} f$, in which the explicit form of f no longer appears. It follows that, if f is a polynomial, whenever the spectrum of A is contained in some interval T , we have a domination in Hilbert-Schmidt norm

$$\left\| \frac{d^k}{dt^k} f(A + tH) \Big|_{t=0} \right\|_{HS} \leq C \|f\|_{k,T} \|H^k\|_{HS}$$

This is no longer basis dependent, and shows that, approximating locally a C^k function by polynomials in C^k norm, the function $f(A + tH)$ is k -times continuously differentiable in t .

This reasoning suggests that in infinite dimensions $f(A)$ is differentiable at A bounded, but along Hilbert-Schmidt directions.

REFERENCES

- [1] DONOGHUE (W.F.). *Monotone Matrix Functions and Analytic Continuation*, Springer (Grundlehren 207), 1974.
- [2] NÖRLUND (N.). *Vorlesungen über Differenzenrechnung*, Berlin 1924.

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