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# On the predictable representation property for superprocesses.

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**Abstract**

In this note a simple proof of the equivalence of the predictable representation property of a martingale with respect to a filtration associated with an orthogonal martingale measure and the extremality of the underlying probability measure $P$ is given. The representation property enables us to characterize all measures which are locally absolutely continuous with respect to $P$. We apply this to superprocesses and remark on a related property of the excursion filtration of the Brownian motion.

Keywords: Predictable Representation, Orthogonal Martingale Measures, Superprocesses, Absolute Continuity.

## 1 Introduction

In this note we first extend the simple proof of the predictable representation property for superprocesses given in [EP1] to all orthogonal martingale measures provided the underlying probability measure $P$ is extremal in the convex set of all solutions of the martingale problem which defines the martingale measure. The predictable representation says that every martingale of the underlying filtration can be written uniquely as a stochastic integral with respect to the orthogonal martingale measure. The proof follows easily from well-known techniques of Stochastic Calculus, cf. [JS, JY]. In the case of the historical process the predictable integrand is identified in [EP2] for a large class of martingales.

As our main new result we show in the Section 2.2 that every measure which is absolutely continuous with respect to $P$ arises as a Girsanov transformation like in Dawson's lecture notes, [D].

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Applied to superprocesses this means that every process which is absolutely continuous with respect to a superprocess is a superprocess with an additional (interacting) immigration, cf. Section 3.1.

The Fleming-Viot process is an example that the predictable representation does not hold if the martingale measure is not orthogonal, cf. Section 3.2.

Finally, in Section 3.3 we show that a related representation property of the excursion filtration of the Brownian motion (cf. [RW]) can (at least partially) deduced from the predictable representation property of a special superprocesses, namely of that with the trivial one-particle-motion.

## 2 Predictable Representation for orthogonal martingale measures

For the basic definition of martingale measures and their stochastic integrals we refer to Walsh [W]. We fix a worthy martingale measure $M(ds,dx)$ over a measurable space $(E,\mathcal{E})$ defined on the stochastic base $(\Omega,\mathcal{F},(\mathcal{F}_t),P)$ where the starting $\sigma$-field $\mathcal{F}_0$ is $P$-trivial and $\mathcal{F}=\vee_{t\geq 0}\mathcal{F}_t$. The quadratic variation measure and its dominating measure are denoted by $Q$ and $K$, resp. The set of $(M-)$integrable functions $\mathcal{P}_M$ equals the closure of simple predictable functions on $\Omega\times[0,\infty)\times E$ with respect to the norm $(\cdot,\cdot)_K^{1/2}$. The stocastic integral $\int_0^\cdot n(s,x)M(ds,dx)$ is denoted by $n.M$. We restrict to orthogonal martingale measures, i.e.

$$Q([0,t],A,B) \;=\; 0, \text{ if } A\cap B=\emptyset. \tag{2.1}$$

It follows that such a martingale measure is worthy with $K=$ extension of $Q$, cf. [W].

### *2.1 Representation Theorem*

We denote the set of square-integrable martingales over $(\Omega,\mathcal{F},(\mathcal{F}_t),P)$ by $\mathbf{M}^2$.

**Proposition 2.1** *Let $N$ be in $\mathbf{M}^2$. Then there exists a unique function $n\in\mathcal{P}_M$ such that*

$$N_t \;=\; N_0+\int_0^t\int_E n(s,x)M(ds,dx)+L_t, \tag{2.2}$$

*where $L$ is an $L^2$-martingale with $<L,\int_0^\cdot\int_E b(s,x)M(ds,dx)>=0$ for every $b\in\mathcal{P}_M$.*

**Proof:** We define

$$\mathbf{M}_0^2:=\{N\in\mathbf{M}^2 | N=\int_0^\cdot\int_E b(s,x)M(ds,dx), b\in\mathcal{P}_M\}.$$

By the orthogonality of $Q$ we have

$$E[((b_k-b_l).M_\infty)^2]=(b_k-b_l,b_k-b_l)_K$$

for a sequence $(b_k)_{k\in\mathbb{N}}\subset\mathcal{P}_M$ and so this sequence is Cauchy in $\mathcal{P}_M$ iff the sequence $(b_k\cdot M_\infty)_{k\in\mathbb{N}}$ is Cauchy in $L^2(\Omega,P)$. Therefore $\mathbf{M}_0^2$ is a closed subspace of $L^2$ and the assertion follows. ◇

The next theorem gives conditions which are equivalent to the predictable representation property and well-known in the case of $d$-dimensional martingales, cf. [JS, Chapter 3]. But first, we extend the integration with respect to a martingale measure to the space of $\mathcal{P}_M$-valued measures.

**Definition.** Let $(E', \mathcal{E}')$ be a measurable space with generating field $\mathcal{A}'$. Let $M'$ be a martingale measure over $\mathcal{A}'$. A function

$$m' : \Omega \times \mathbb{R}^+ \times E \times \mathcal{E}' \to \mathbb{R}$$

is called a $\mathcal{P}_M$-valued measure over $\mathcal{E}'$ iff it is finitely additive, $\sigma$-finite and continuous in $\emptyset$ as a function from $\mathcal{E}'$ to the Banach space $\mathcal{P}_M$. We then define a new martingale measure $m' \otimes M$ over $\mathcal{A}'$ by

$$m' \otimes M_t(A') \ := \ \int_0^t \int_E m'(s,x,A') M(ds,dx). \tag{2.3}$$

**Theorem 2.2** *The following statements are equivalent:*

*(i) The measure $P$ is extremal in the convex set of all measures $P^*$ on $(\Omega, \mathcal{F})$ such that $M$ is a martingale measure with covariation $Q$ under $P^*$.*

*(ii) Every local martingale $N$ has a unique predictable representation*

$$N_t \ = \ N_0 + \int_0^t \int_E n(s,x) M(ds,dx) \tag{2.4}$$

*where $n \in \mathcal{P}_M^{loc}$. (Here "loc " means that there is a sequence of stopping times $(S_n)$ such that $n(s \wedge T_n, x) \in \mathcal{P}_M$.)*

*(iii) Every martingale measure $M'$ defined on $(\Omega, \mathcal{F})$ with respect to the filtration $(\mathcal{F}_t)$ over some measurable space $(E', \mathcal{E}')$ with generating field $\mathcal{A}'$ has a unique predictable representation*

$$M' \ = \ m' \otimes M \tag{2.5}$$

*where $m'$ is a $\mathcal{P}_M$-valued measure over $\mathcal{E}'$.*

**Proof.** (i) $\Rightarrow$ (ii): Follows by Proposition 2.1, cf. Theorem 38 in [Pr].

(ii) $\Rightarrow$ (iii): Let $m'(\omega, s, x, A')$ be the representing function of the martingale $(M_t'(A'))_{t \in \mathbb{R}^+}$. We just have to prove that the mapping $A' \to m'(\cdot,\cdot,\cdot,A')$ defines a $\mathcal{P}_M$-valued measure. The additivity of this mapping is obvious by the uniqueness of the predictable representation. Because

$$E[\int_0^\infty \int_E \int_E m'(s,x,A') m'(s,y,A') Q(ds,dx,dy)] = E[M_\infty'(A')^2],$$

the continuity of $m'$ in $\emptyset$ follows by the continuity of $M'$ in $\emptyset$.

(iii) $\Rightarrow$ (ii): Take as $(E', \mathcal{E}')$ a one point set.

(ii) $\Rightarrow$ (i): Cf. [Pr, Theorem 37]. $\diamond$

**Remark.** Assume condition (i) in Theorem 2.2. Then, according to [J],[JY], the set of all martingales coincides with $\mathcal{L}(M)$, the smallest closed subspace of martingales $N$ with norm $E[\sup_{t\geq 0} |N_t|^2]^{1/2}$ which is closed under stopping and which contains all (ordinary) stochastic integrals $n.M(A)$ where $n$ is $M(A)$ integrable and $A \in \mathcal{E}$. Hence we only have shown that $\{n.M | n \in \mathcal{P}_M\} = \mathcal{L}(M)$. This identification can however fail if $M$ is not orthogonal, see Section 3.2.

### *2.2 Necassary condition of absolute continuity*

In this section we want to show the converse of Dawson's Giransov transformation [D, ch.7]. In the present setting his result reads as follows:

For $r$ be in $\mathcal{P}_M^{loc}$ we denote the corresponding exponential local martingale

$$\exp(r \cdot M_t - \frac{1}{2}\int_{[0,t)\times E\times E} r(s,x)r(s,y)Q(ds,dx,dy))$$

by $\mathcal{E}(r)$. If $\mathcal{E}(r)$ is a martingale, e.g. if $E[\exp(\frac{1}{2}\int_0^\cdot \int_E \int_E r(s,x)r(s,y)Q(ds,dx,dy))] < \infty$ or $\exp(r.M)$ is uniformly integrable, then we can define a new measure $P^r$ on $(\Omega, \mathcal{F})$ by $\frac{dP^r}{dP}\Big|_{\mathcal{F}_t} := \mathcal{E}(r)_t$.

A modification of Dawson's argument shows that under $P^r$ the process $M^r$ defined by

$$M^r((0,t],A) \;:=\; M((0,t],A) + \int_0^t \int_E \int_E \mathbf{1}_A(x) r(s,y) Q(ds,dx,dy) \tag{2.6}$$

is a martingale measure with covariation measure $Q$.

We shall show that every probability measure which is absolutely continuous with repect to $P$ arises as a suitable $P^r$.

**Theorem 2.3** *Let $P$ be a measure on $(\Omega, \mathcal{F})$ such that $M$ is a martingale measure with covariation goverened by $Q$ which has the predictable representation property.*

*Let $P' << P$.*

- *There exits a predictable function $r \in \mathcal{P}_M^{loc.}$ such that the process $M(A)$ is a semimartingale with increasing process*

$$\int_0^t \int_{E\times E} \mathbf{1}_A(x) r(s,y) Q(ds,dx,dy). \tag{2.7}$$

- *If we assume additionally that $M$ is a continuous martingale measure, i.e. $h.M$ is continuous $\forall h \in \mathcal{P}_M$, then the density has the exponential form*

$$\frac{dP'}{dP}\Big|_{\mathcal{F}_t} = \exp(\int_0^t \int_E r(s,x) M(ds,dx) - \frac{1}{2}\int_0^t (\int_{E\times E} r(s,x) r(s,y) Q(ds,dx,dy))) \tag{2.8}$$

*Hence $P' = P^r$.*

**Proof.** Let $Z_t = \frac{dP'}{dP}\Big|_{\mathcal{F}_t}$. The Girsanov transformation for one-dimensional martingales ([JS, Chapter 3]) implies that for every $A \in \mathcal{A}$ there exists a local $P'$-martingale $M'(A)$ such that

$$M_t(A) \quad = \quad M_t'(A) + \int_0^t \frac{1}{Z_s} d < Z, M(A) > s.$$

By the representation of martingales under $P$ we have that

$$Z_t = 1 + \int_0^t \int_E z(s,x) M(ds,dx) \tag{2.9}$$

and hence

$$\int_0^t \frac{1}{Z_{s-}} d < Z, M(A) >_s = \int_0^t \int_{E \times E} \mathbf{1}_A(x) \frac{z(s,y)}{Z_{s-}} Q(ds,dx,dy).$$

The function $r$ equals therefore $Z_s^{-1} z(s,x)$.

In order to prove the second assertion we notice that by (2.9) up to $T_n = \inf\{t | Z_t < \frac{1}{n}\}$ we have

$$\begin{aligned} Z_t &= 1 + \int_0^t Z_s \int_E r(s,x) M(ds,dx) \qquad (2.10) \\ &= 1 + \int_0^t \int_E Z_s r(s,x) M(ds,dx). \end{aligned}$$

Hence by the exponential formula for martingales the assertion is proved for all $(t,\omega)$ such that $t \in [0, T_n(\omega)]$ for some $n \in I\!N$. Because the process

$$V := \int_0^\cdot \int_E \int_E r(s,x) r(s,y) Q(ds,dx,dy)$$

is continuous it "does not jump to infinity" in the terminology of [JS, Chapter 3.5a] and so the assertion is valid $P$-almost surely for all $t$. ◇

# 3 Examples

*3.1 Superprocess.*

The basic example motivating the present note is the (interacting) superprocess, cf. [D,P]. It is a process $X$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P_{\mu_0}^{A,c})$ which takes values in the space $\mathcal{M}(E)$ of positive finite measures over a Polish space $E$. The basic data are a familiy of linear operators $A = (A(\omega,s))_{\omega \in \Omega, s \in [0,\infty)}$ with common domain $D \subset C_b([0,\infty) \times E)$, a positive bounded branching variation function $c$ defined on $\Omega \times [0,\infty) \times E$ and a starting point $\mu_0 \in M(E)$. Then $(X, P_{\mu_0}^{A,c})$ satisfies by definition that

$$M_t[f] \quad := \quad X_t(f(t)) - \mu_0(f(0)) - \int_0^t X_s(A(s)f(s))ds \tag{3.1}$$

is a martingale under $P_{\mu_0}^{A,c}$ with quadratic variation

$$< M[f] >_t \quad = \quad \int_0^t X_s(c(s) f^2(s)) ds \tag{3.2}$$

for all $f \in D$ (where we use the notation $\mu(f) := \int_E f(x)\mu(dx), \mu \in \mathcal{M}(E), f \in C_b(E)$).

These linear martingales give rise to an orthogonal martingale measure $M^A$ defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P_{\mu_0}^{A,c})$ with covariation measure

$$Q^c(ds, dx, dy) \quad = \quad c(s,x)\delta_x(dy)X_s(dx)ds, \tag{3.3}$$

cf.[D]. Let us now assume that $P_{\mu_0}^{A,c}$ is extremal under all measures under which (3.1) is a martingale with quadratic variation (3.2). This is in particular the case if there is no interaction, i.e. $A(\omega, s) = A_0$ for one fixed operator $A_0$, which generates a Hunt process with state space $E$, and $c(\omega, s, x) = c(s, x)$. The unique solution of the martingale problem (3.1,3.2) is the superprocess over the one-partical-motion generated by $(A_0, D)$. The martingale problem is also well-posed if $A(\omega, s)f(x) = A_0 f(x) + b(s, \omega, x)f(x)$ for some nice $b \in \mathcal{P}_{M^{A_0}}$, cf. [D]. The question of uniqueness of a general $P_{\mu_0}^{A,c}$ is intensively studied in [P].

Our results imply that every extremal $P_{\mu_0}^{A,c}$ has the predictable representation property and additionally that for every measure $P'$ which is absolutely continuous with respect to $P_{\mu_0}^{A,c}$ there exists a $r \in \mathcal{P}_{M^A}$ such that $P' = P_{\mu_0}^{A^r,c}$, where $A^r(\omega, s)f(s, x) = A(\omega, s)f(s, x) + r(\omega, s, x)c(\omega, s, x)f(s, x)$, i.e. $P'$ is a superprocess with additional immigration parameter $rc$.

If $P_{\mu_0}^{A,c}$ is a superprocess without interaction every process which is absolutely continuous with respect to $P_{\mu_0}^{A,c}$ is therefore a superprocess with immigration term

$c(s,x)r(\omega, s, x)$, i.e. the immigration term of a particle at place $x$ depends on the history of the population $\omega$ up to time $s$.

### *3.2 Fleming-Viot-process*

The Fleming-Viot-process $X$ is a process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ taking values in the space $\mathcal{M}_1(E)$ of probability measures over a Polish space $E$. Its distribution is by definition the unique solution of the martingale problem characterize by the linear martingales (3.1) and their quadratic covariation $< M[f], M[g] >_t = \int_0^t X_s(fg) - X_s(f)X_s(g)ds$. Hence the associated martingale measure $M$ is not orthogonal. Because the $L^2$-norm of a stochastic integral $a.M$ differs from the $\mathcal{P}_M$-norm of $a$ the arguments in Section 2 do not work. Moreover, every predictable function $g(\omega, s)$, which does not depend on the space variable $x$ has $g.M = 0$. Therefore,

$$F(\omega) := \int_0^T \int_E a(\omega, s, x)M(\omega, ds, dx) = \int_0^T \int_E (a(\omega, s, x) + g(\omega, s))M(\omega, ds, dx)$$

where we can choose the two representing functions $a$ and $a + g$ different in $\mathcal{P}_M$, by assuming that

$$0 \neq E[\int_0^T g^2(s)ds] = (g, g)_K < \infty.$$

Hence already the uniqueness in Proposition 2.1 does not hold.

*3.3 Excursion filtration*

Rogers and Walsh consider in [RW] the following situation:

Let $B_t$ be a Brownian motion on a complete probability space $(\Omega, \mathcal{F}, P)$ started in 0 and $L(t,x)$ its local time. For every $x \in I\!R$ define the increasing process $\tau(\cdot, x)$ by $\tau(t,x) = \inf\{u : \int_0^x L(u,y)dy > t\}$. $\tau$ is the inverse of the occupation time $A(u,x) := \int_0^x L(u,y)dy = \int_0^u \mathbf{1}_{B_s \le x} ds$. Define the $\sigma$-field $\mathcal{E}_x$ by the completion of

$$\sigma(B_{\tau(t,x)}, t \ge 0).$$

The family $(\mathcal{E}_x)_{x\in I\!R}$ is called the filtration of *excursion fields.* A function $\tilde{\phi}$ on $\Omega \times [0,\infty) \times I\!R$ is called $(\mathcal{E}_x)$-predictable if it is measurable with repect to the $\sigma$-algebra generated by all $(\mathcal{E}_x \times \mathcal{B}((0,\infty)))_{x\in I\!R}$-adapted processes which are left-continuous in $t$.

It is proved in [RW] that every $F \in L^2(\Omega, \mathcal{F}, P)$ can be written as

$$F \;=\; E[F] + \int_0^\infty \int_{I\!R} \phi(t,x) L(dt,dx) \qquad (3.4)$$

with an *identifiable* $\phi$ satisfying $4E[\int_0^\infty \phi^2(t, B_t)dt] < \infty$. The property *identifiable* means that $\phi = \tilde{\phi} \circ \Gamma$ with a $(\mathcal{E}_x)$-predictable $\tilde{\phi}$ and $\Gamma(\omega,t,x) = (\omega, A(t,x,\omega), x)$. The integral above is defined by

$$\int\!\!\int \phi(t,x)L(dt,dx) := Z\left[L(T,b) - L(T,a) - L(S,b) + L(S,a)\right]$$

for a simple function $\phi(t,x) = Z\mathbf{1}_{(a,b]}(x)\mathbf{1}_{(S,T]}(t)$, where $Z \in b\mathcal{E}_a$ and $S = \tau(S^e, a), T = \tau(T^e, a)$ with $\mathcal{E}_a$-measurable times $S^e$ and $T^e$ and by a standard extension for all identifiable functions $\phi$.

We will now point out how this result follows, at least partially, from the predictable representation property for superprocesses.

In their proof Rogers and Walsh use the following result from the Ray-Knight theory:

Suppose that $S < T < U < V$ are $\mathcal{E}_0$-identifiable times (, i.e. constructed as $S$ and $T$ above). Let $M_x := L(T,x) - L(S,x)$ and $N_x := L(V,x) - L(U,x)$. Then $(M_x)_{x\ge0}$ is a continuous local $(\mathcal{E}_x)$-martingale with increasing process $4\int_0^x M_y dy$. Moreover, $M$ and $N$ are orthogonal and $(M_x)_{x\ge0}$ is an $L^p$-martingale iff $M_0 \in L^p$.

Here we easily notice the connection to a superprocess, namely that $(M_x)_{x\ge0}$ is the superprocess with state space $\mathcal{M}([0,\infty))$ over the one-particle *motion* $Af = 0$ , if we impose a time change of the Brownian motion $B$, i.e. a space transformation for '$L(\cdot, x)$' as a measure-valued process. Define the measure-valued process $M$ by

$$M_x((a,b]) := L(\tau(b,0),x) - L(\tau(a,0),x).$$

By the covariation of $M_x$ this can be extended to an orthogonal martingale measure with covariation $\delta_a(db)M_x(da)dx$. Hence every $F \in L^2(\Omega, \vee_{t\ge0}\mathcal{E}_t, P)$ can be written as

$$F = E[F|\mathcal{E}_0] + \int_0^\infty \int_0^\infty \psi(a,x)M(dx,da).$$

By the definition of the different integrales $\psi.M$ and $\phi.L$ it is clear that $\psi.M = \phi.L$ iff $\phi(\omega, r, x) = \psi \circ \Gamma^0(\omega, r, x)$ with $\Gamma^0(\omega, r, x) = (\omega, A(\omega, r, 0), x)$. Hence we have to show that $\psi \circ \Gamma^0$ is identifiable if $\psi \in \mathcal{P}_M$. This follows for a simple function $\psi \in \mathcal{P}_M$ because in that case $\psi \circ \Gamma^0$ satisfies the conditions of Proposition 2.4 in [RW] and is therefore identifiable. For a general function $\psi \in \mathcal{P}_M$ the identifiability follows then by a monotone class argument.

Hence at least for $F \in L^2(\Omega, \vee_{x \geq 0} \mathcal{E}_x, P)$ the assertion (3.4) which is formula (2.1) in Theorem 2.1 of [RW] follows easily from the predictable representation property for superprocesses.

This remark should makes it plausible that in the case where we consider the reflecting Brownian motion $|B|$ instead of the Brownian motion the analog result of Rogers and Walsh follows completely from the predictable representation for orthogonal martingale measures, cf. also Remark 1.3b in [EP1].

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