SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

JEAN-FRANÇOIS LE GALL

Exponential moments for the renormalized self-intersection local time of planar brownian motion

Séminaire de probabilités (Strasbourg), tome 28 (1994), p. 172-180
<http://www.numdam.org/item?id=SPS_1994__28__172_0>

Exponential moments for the renormalized self-intersection local time of planar Brownian motion

Jean-François Le Gall

Laboratoire de Probabilités - Université Paris VI - 4, place Jussieu - 3ème Etage - 75252 PARIS CEDEX 05

Let $B = (B_t, t \geq 0)$ be a planar Brownian motion with $B_o = 0$. The renormalized self-intersection local time of B, over the time interval [0,1], is the random variable γ formally defined by

$$\gamma = \iint_{0 \leq s < t \leq 1} (\delta_o(B_s - B_t) - E(\delta_o(B_s - B_t)))\, ds\, dt, \tag{1}$$

where δ_o denotes the Dirac measure at 0. A rigorous definition of γ was first provided par Varadhan [7] in the more difficult case of the Brownian bridge (see [4] and [8] for simple constructions of γ for Brownian motion). It is also known that :

$$E(\exp - \lambda\gamma) < \infty \quad , \forall\, \lambda > 0 \ . \tag{2}$$

This fact is important in order to define the so-called polymer measures

$$P^{\lambda}(d\omega) = C_{\lambda} \exp(-\lambda\gamma(\omega))\, W(d\omega), \tag{3}$$

where $W(d\omega)$ is the (two-dimensional) Wiener measure and C_{λ} is a normalizing constant. Polymer measures correspond to a model of (weakly) self-avoiding Brownian motion.

Recently, there has been some interest in self-attracting models for Brownian motion and random walks (see in particular Bolthausen [1]). In this connection, it appears natural to replace the weight $\exp(-\lambda\gamma(\omega))$ in *(3)* by $\exp(\lambda\gamma(\omega))$. This motivates the following result, which was suggested by a question of Gordon Slade (personal communication).

Theorem 1 : *There exists a constant* $\lambda_o \in (1,\infty)$ *such that*

$$E(\exp \lambda\gamma) \begin{cases} < \infty & \text{if } \lambda < \lambda_o , \\ = \infty & \text{if } \lambda > \lambda_o . \end{cases}$$

Remark : Our proof will show that

$$4 \prod_{j=1}^{\infty} (1-2^{-j}) \le \lambda_o \le 16\,\pi\, e^5/(\log 2)^2.$$

Both these bounds can be improved rather easily.

After the first version of this work had been completed we learnt of an unpublished work of M. Yor [9], who uses a different method based on his approach in [8] to check that $E[\exp \lambda\gamma] < \infty$ for $\lambda > 0$ small enough.

Before proving Theorem 1, let us briefly recall the construction of γ given in [4]. First consider another planar Brownian motion B' with $B'_o = 0$, independent of B. The random variable

$$\alpha_o := \int_0^1 \int_0^1 \delta_o(B_s - B'_t)\, ds\, dt$$

can be defined as the value at 0 of the continuous density of the random measure on $\mathbb{R}^2$

$$\mu(g) = \int_0^1 \int_0^1 g(B_s - B'_t)\, ds\, dt \tag{4}$$

(see e.g. [3]). Moreover $\alpha_o \in L^p$ for every $p < \infty$.

Then, for every integer $n \ge 1$ and for every $k \in \{1,\dots,2^{n-1}\}$, set

$$A_k^n = [(2k-2)2^{-n}, (2k-1)2^{-n}) \times ((2k-1)2^{-n}, 2k2^{-n}].$$

From the case of two independent Brownian motions, it is straightforward to define

$$\alpha(A_k^n) = \iint_{A_k^n} \delta_o(B_s - B_t)\, ds\, dt.$$

The following facts are immediate from the standard properties of Brownian motion.

(i) For every $n \ge 1$, the variables $\alpha(A_1^n),\dots,\alpha(A_{2^{n-1}}^n)$ are independent.

(ii) $\alpha(A_k^n) \overset{(d)}{=} 2^{-n}\, \alpha_o$.

One can then define γ as

$$\gamma := \sum_{n=1}^{\infty} \left(\sum_{k=1}^{2^{n-1}} (\alpha(A_k^n) - E(\alpha(A_k^n))) \right) \tag{5}$$

and, from (i) and (ii), it is easy to verify that the series converges a.s. and in L^2.

<u>**Lemma 2**</u> : *Set* $a_1 = 1/2$, $a_2 = e^{-5}(\log 2)^2/(8\pi)$. *There exist two positive constants* C_1 , C_2 *such that for every* $p \geq 1$,

$$C_2\, a_2^p\, p! \leq E((\alpha_o)^p) \leq C_1\, a_1^p\, p!.$$

<u>Proof</u> : The upper bound is essentially contained in Rosen [6], formula **(2.15)**. We give the argument for the sake of completeness and also to get an explicit constant. We start from the following identity, which is a special case of formula **(2.5)** of [3] :

$$E[(\alpha_o)^p] = (2\pi)^{-2p} \int_{(\mathbb{R}^2)^p} d\xi_1 \dots d\xi_p \int_{[0,1]^{2p}} ds_1 \dots ds_p dt_1 \dots dt_p$$

$$\times \exp - \frac{1}{2} \operatorname{var}\left(\sum_{j=1}^{p} \xi_j \cdot (B_{s_j} - B'_{t_j}) \right)$$

(to verify that $E[(\alpha_o)^p]$ is bounded above by the right side, which is all that we need for the upper bound, write

$$\alpha_o = \lim_{\varepsilon \downarrow 0} \int_0^1 \int_0^1 ds\, dt\, p_\varepsilon(B_s - B'_t) , \qquad \text{a.s.}$$

where $p_\varepsilon(\cdot)$ is the usual Gaussian kernel, express $p_\varepsilon(\cdot)$ in terms of its Fourier transform and use Fatou's lemma). Let $\mathcal{S}_p$ be the set of all permutations of $\{1,\dots,p\}$ and for $\sigma \in \mathcal{S}_p$ set

$$A_\sigma = \{(s_1,\dots,s_p,t_1,\dots,t_p);\ 0<s_1< \dots <s_p \leq 1,\ 0<t_{\sigma(1)}< \dots <t_{\sigma(p)} \leq 1 \}.$$

Then,

$$E[(\alpha_o)^p] = p!\,(2\pi)^{-2p} \sum_{\sigma \in \mathcal{S}_p} \int d\xi_1 \dots d\xi_p \int_{A_\sigma} ds_1 \dots ds_p\, dt_1 \dots dt_p$$

$$\times \exp - \frac{1}{2} \operatorname{var}\left(\sum_{j=1}^{p} \xi_j \cdot (B_{s_j} - B'_{t_j}) \right).$$

For every fixed $\sigma \in \mathcal{S}_p$, set

$$u_j = \sum_{k=j}^{p} \xi_k , \quad v_j = \sum_{k=j}^{p} \xi_{\sigma(k)} , \qquad j \in \{1,\dots,p\},$$

so that, if $(s_1,\dots,t_p) \in A_\sigma$,

$$\mathrm{var}\Big(\sum_{j=1}^{p} \xi_j\cdot(B_{s_j}-B'_{t_j})\Big) = \mathrm{var}\Big(\sum_{j=1}^{p} u_j\cdot(B_{s_j}-B_{s_{j-1}}) - \sum_{j=1}^{p} v_j\cdot(B'_{t_{\sigma(j)}}-B'_{t_{\sigma(j-1)}})\Big)$$

$$= \sum_{j=1}^{p} |u_j|^2(s_j-s_{j-1}) + \sum_{j=1}^{p} |v_j|^2(t_{\sigma(j)}-t_{\sigma(j-1)})$$

where by convention $s_o = t_{\sigma(0)} = 0$. However, by the Cauchy-Schwarz inequality, if $(s_1,\dots t_p) \in A_\sigma$,

$$\int_{(\mathbb{R}^2)^p} d\xi_1\dots d\xi_p \exp -\frac{1}{2}\Big(\sum_{j=1}^{p} |u_j|^2(s_j-s_{j-1}) + \sum_{j=1}^{p} |v_j|^2(t_{\sigma(j)}-t_{\sigma(j-1)})\Big)$$

$$\le\Big(\int d\xi_1\dots d\xi_p \exp -\sum_{j=1}^{p}|u_j|^2(s_j-s_{j-1})\Big)^{\frac{1}{2}}$$

$$\times\Big(\int d\xi_1\dots d\xi_p \exp -\sum_{j=1}^{p}|v_j|^2(t_{\sigma(j)}-t_{\sigma(j-1)})\Big)^{\frac{1}{2}}$$

$$= \pi^p \prod_{j=1}^{p} \Big((s_j-s_{j-1})^{-1/2}\,(t_{\sigma(j)}-t_{\sigma(j-1)})^{-1/2}\Big).$$

Hence, by coming back to the previous formula for $E[(\alpha_o)^p]$,

$$E[(\alpha_o)^p] \le 2^{-2p}\,\pi^{-p}\,(p!)^2\left(\int_{0<s_1<\dots<s_p\le 1}\frac{ds_1\dots ds_p}{\sqrt{s_1(s_2-s_1)\dots(s_p-s_{p-1})}}\right)^2.$$

Elementary calculations give

$$J_p = \int_{0<s_1<\dots<s_p\le 1}\frac{ds_1\dots ds_p}{\sqrt{s_1(s_2-s_1)\dots(s_p-s_{p-1})}}$$

$$= \begin{cases} \dfrac{2^p}{p\times(p-2)\times\dots\times 2}\,(\frac{\pi}{2})^{p/2} & \text{if p is even} \\[2ex] \dfrac{2^p}{p\times(p-2)\times\dots\times 3\times 1}\,(\frac{\pi}{2})^{(p-1)/2} & \text{if p is odd,} \end{cases}$$

which implies

$$J_p \underset{p\to\infty}{\sim} \left(\frac{2}{\pi}\right)^{1/4} p^{-1/4} (2\pi)^{p/2} (p!)^{-1/2}.$$

This gives the upper bound of Lemma 2.

For the lower bound, we use another equivalent formula for $E[(\alpha_o)^p]$ (see Proposition 2.1 of [5]). If $\Delta_p = \{(s_1,\dots,s_p)\in(0,\infty)^p; s_1+\dots+s_p \le 1\}$ we have

$$E[(\alpha_o^p)]=(2\pi)^{-2p}\int_{(\mathbb{R}^2)^p} dy_1..dy_p\left(\sum_{\sigma\in\mathcal{S}_p}\int_{\Delta_p}\frac{ds_1\dots ds_p}{s_1\dots s_p}\exp-\sum_{j=1}^{p}\frac{|y_{\sigma(j)}-y_{\sigma(j-1)}|^2}{2s_j}\right)^2$$

$$\ge (2\pi)^{-2p}\int_{(\mathbb{R}^2)^p} dy_1\dots dy_p\left(\sum_{\sigma\in\mathcal{S}_p}\prod_{j=1}^{p}\int_0^{1/p}\frac{ds}{s}\exp-\frac{|y_{\sigma(j)}-y_{\sigma(j-1)}|^2}{2s}\right)^2$$

$$= (2\pi)^{-2p}\, p^{-p}$$

$$\times\left(\sum_{\sigma,\tau\in\mathcal{S}_p}\int_{(\mathbb{R}^2)^p} dz_1\dots dz_p\prod_{j=1}^{p}\left(\psi\left(\frac{|z_{\sigma(j)}-z_{\sigma(j-1)}|^2}{2}\right)\psi\left(\frac{|z_{\tau(j)}-z_{\tau(j-1)}|^2}{2}\right)\right)\right)$$

where

$$\psi(r) = \int_0^1 \frac{ds}{s}\, e^{-r/s} = \int_1^\infty \frac{du}{u}\, e^{-ru}.$$

We then use the crude bound $\psi(r) \ge \psi(1) > e^{-2}\log 2$ for $r \in (0,1]$ and by integrating over $\{|z_j| \le 1/\sqrt{2}\}$ in the previous inequality, we get the lower bound of Lemma 2. □

<u>Proof of Theorem 1</u> : For simplicity, write $\alpha_{n,k} = \alpha(A_k^n)$ and $\bar{\alpha}_{n,k} = \alpha_{n,k} - E(\alpha_{n,k})$, $\bar{\alpha}_o = \alpha_o - E(\alpha_o)$. For $\lambda > 0$, set

$$\varphi(\lambda) = E[\exp \lambda\, \bar{\alpha}_o].$$

By Lemma 2, $\varphi(\lambda) < \infty$ for $\lambda < 2$. Since $\varphi'(0) = 0$ we may for every $\lambda_1\in(0,2)$ find a positive constant c such that

$$\varphi(\lambda) \le 1 + c\,\lambda^2 \quad , \quad \forall\lambda \in [0,\lambda_1].$$

Fix $\lambda_1 \in (0,2)$ and $a \in (0,1)$. For every $N \ge 1$ set

$$b_N = 2\lambda_1 \prod_{j=2}^{N} (1-2^{-a(j-1)})$$

$(b_1 = 2\lambda_1)$. Then, by the Hölder inequality, and properties (i), (ii) above, we have for $N \geq 2$,

$$E\left[\exp b_N \sum_{n=1}^{N} \sum_{k=1}^{2^{n-1}} \bar{\alpha}_{n,k}\right]$$

$$\leq E\left[\exp \frac{b_N}{1-2^{-a(N-1)}} \sum_{n=1}^{N-1} \sum_{k=1}^{2^{n-1}} \bar{\alpha}_{n,k}\right]^{1-2^{-a(N-1)}}$$

$$\times E\left[\exp 2^{a(N-1)} b_N \sum_{k=1}^{2^{N-1}} \bar{\alpha}_{n,k}\right]^{2^{-a(N-1)}}$$

$$\leq E\left[\exp b_{N-1} \sum_{n=1}^{N-1} \sum_{k=1}^{2^{n-1}} \bar{\alpha}_{n,k}\right] \varphi\left(b_N\, 2^{a(N-1)-N}\right)^{2^{(1-a)(N-1)}}$$

Notice that $b_N\, 2^{a(N-1)-N} \leq \lambda_1$. It follows that

$$\varphi\left(b_N\, 2^{a(N-1)-N}\right)^{2^{(1-a)(N-1)}} \leq \left(1 + c b_N^2\, 2^{2((a-1)N-a)}\right)^{2^{(1-a)(N-1)}}$$

$$\leq \exp(c'\, 2^{(a-1)N}),$$

for a constant c' independent of N. By induction we get

$$E\left[\exp b_N \sum_{n=1}^{N} \sum_{k=1}^{2^{n-1}} \bar{\alpha}_{n,k}\right] \leq \exp\left(c' \sum_{n=2}^{N} 2^{(a-1)n}\right) E\left(\exp b_1\, \bar{\alpha}_{1,1}\right)$$

$$\leq \exp\left(c'(1-2^{a-1})^{-1}\right) \varphi(\lambda_1).$$

Letting N tend to ∞ and using Fatou's lemma, we obtain $E[\exp b_\infty \gamma] < \infty$ for $b_\infty = 2\lambda_1 \prod_{j=1}^{\infty} (1-2^{-aj})$. Since $a \in (0,1)$ and $\lambda_1 \in (0,2)$ were arbitrary, we conclude that $E[\exp \lambda\gamma] < \infty$, for $\lambda < 4 \prod_{j=1}^{\infty} (1-2^{-j})$.

Let us now check that $E[\exp \lambda\gamma] = \infty$ for λ large enough. From the definition of γ we have

$$\gamma = \bar{\alpha}_{1,1} + \bar{\alpha}_{1,2} + \tilde{\gamma}$$

where $\alpha_{1,1}$, $\alpha_{1,2}$ are independent and distributed as $\alpha_o/2$, and $\tilde{\gamma}$ is distributed as $\gamma/2$. Using *(2)*, it follows that if $E[\exp a\,\gamma] < \infty$ for some $a > 0$ then $E[\exp b\,\alpha_o] < \infty$ for $b < a/2$. By Lemma 2 we have

$$E[\exp b\,\alpha_o] = \infty, \quad \text{if} \quad b > \frac{1}{a_2}\,.$$

It follows that $E[\exp \lambda\gamma] = \infty$ for $\lambda > \frac{2}{a_2}$. □

Remarks : (a) The first part of the proof of Theorem 1 is easily adapted to give a short proof of *(2)*. We have trivially $E[\exp - \lambda\,\bar{\alpha}_o] < \infty$ for every $\lambda > 0$ so that for every $K > 0$ there exists a constant c such that

$$E[\exp - \lambda\,\bar{\alpha}_o] \le 1 + c\,\lambda^2 \qquad , \quad \forall\lambda \in [0,K].$$

We then fix $\lambda > 0$ and take :

$$b_N = -2\lambda \prod_{j=2}^{N} (1-2^{-a(j-1)}) \;,\quad b_\infty = -2\lambda \prod_{j=1}^{\infty} (1-2^{-aj})$$

and the same calculations as in the previous proof yield $E[\exp b_\infty\gamma] < \infty$. This gives *(2)* since λ was arbitrary.

(b) In the one-dimensional case, the analogue of the variable γ is the integral

$$\int_{\mathbb{R}} dx\,(L_1^x)^2$$

where L_1^x denotes the local time at level x, at time 1 of the linear Brownian motion B started at 0 (there is no need for renormalization in dimension 1). It is easy to check that for every $\lambda > 0$

$$E\left(\exp \lambda \int_{\mathbb{R}} dx\,(L_1^x)^2\right) < \infty.$$

One may argue as follows. By Jensen's inequality,

$$\exp\left(\lambda \int dx\,(L_1^x)^2\right) \le \int dx\,L_1^x \exp \lambda\,L_1^x.$$

However, if $T_x = \inf\{t,\ B_t = x\}$,

$$E[L_1^x \exp \lambda L_1^x] = E\left[1_{\{T_x \le 1\}}\,L_1^x \exp \lambda L_1^x\right] \le P(T_x \le 1)\,E[L_1^o \exp \lambda L_1^o].$$

Hence,

$$E[\exp(\lambda \int dx\, (L_1^x)^2)] \le \left(\int dx\, P[T_x < 1]\right) E[L_1^0 \exp \lambda L_1^0] = C\, E[L_1^0 \exp \lambda L_1^0].$$

By a classical result of Lévy, L_1^0 has the same distribution as $|B_1|$. Therefore, $E[L_1^0 \exp \lambda L_1^0] < \infty$, which gives the desired result.

Another approach to *(6)*, suggested by M. Yor, would be to bound

$$\int dx\, (L_1^x)^2 \le L_1^* := \sup_{x\in\mathbb{R}} L_1^x ,$$

and then to use the fact that L_x^* has exponential moments (see Borodin [2], Theorem 1.7, it is even true that $E(\exp \lambda(L_x^*)^2)<\infty$ for $\lambda>0$ small).

Acknowledgments. I thank Gordon Slade for suggesting the problem that is treated in this note, and Marc Yor for his comments on the first version.

References

[1] **E. Bolthausen** : Localization of a two-dimensional random walk with an attractive path interaction. *Ann. Probab. to appear.*

[2] **A.N. Borodin** : On the character of convergence to Brownian local time, II. *Proba. Th. Rel. Fields* 72, *p. 251-277 (1986).*

[3] **D. Geman, J. Horowitz and J. Rosen** : A local time analysis of intersections of Brownian paths in the plane. *Ann. Probab.* 12, *p. 86-107 (1984).*

[4] **J.-F. Le Gall** : Sur le temps local d'intersection du mouvement brownien plan et la méthode de renormalisation de Varadhan. *Séminaire de Probabilités XIX. Lecture Notes in Maths* **1123**, *p.314-331. Springer (1985).*

[5] **J.-F. Le Gall** : Propriétés d'intersection des marches aléatoires, I. *Comm. Math. Physics* **104**, *p. 467-503 (1986).*

[6] **J. Rosen** : Self-intersections of random fields. *Ann. Probab.* 12, *p.108-119 (1984).*

[7] **S.R.S. Varadhan** : Appendix to : Euclidean quantum field theory, by K.Symanzik. In : Local Quantum Theory, R. Jost ed. *Academic Press (1969).*

[8] **M. Yor** : Compléments aux formules de Tanaka-Rosen. *Séminaire de Probabilités XIX. Lecture Notes in Maths.* **1123**, *p. 332-349. Springer (1985).*

[9] **M. Yor** : Remarques sur l'intégrabilité du temps local d'intersection renormalisé. *Unpublished manuscript.*