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Exponential moments for the renormalized self-intersection

local time of planar Brownian motion

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Let $B = (B_t, t \geq 0)$ be a planar Brownian motion with $B_0 = 0$. The renormalized self-intersection local time of B , over the time interval $[0,1]$, is the random variable γ formally defined by

$$\gamma = \iint_{0 \leq s < t \leq 1} (\delta_0(B_s - B_t) - E(\delta_0(B_s - B_t))) ds dt, \quad (1)$$

where δ_0 denotes the Dirac measure at 0. A rigorous definition of γ was first provided par Varadhan [7] in the more difficult case of the Brownian bridge (see [4] and [8] for simple constructions of γ for Brownian motion). It is also known that :

$$E(\exp - \lambda \gamma) < \infty, \quad \forall \lambda > 0. \quad (2)$$

This fact is important in order to define the so-called polymer measures

$$P^\lambda(d\omega) = C_\lambda \exp(-\lambda \gamma(\omega)) W(d\omega), \quad (3)$$

where $W(d\omega)$ is the (two-dimensional) Wiener measure and C_λ is a normalizing constant. Polymer measures correspond to a model of (weakly) self-avoiding Brownian motion.

Recently, there has been some interest in self-attracting models for Brownian motion and random walks (see in particular Bolthausen [1]). In this connection, it appears natural to replace the weight $\exp(-\lambda \gamma(\omega))$ in (3) by $\exp(\lambda \gamma(\omega))$. This motivates the following result, which was suggested by a question of Gordon Slade (personal communication).

Theorem 1 : There exists a constant $\lambda_0 \in (1, \infty)$ such that

$$E(\exp \lambda \gamma) \begin{cases} < \infty & \text{if } \lambda < \lambda_0, \\ = \infty & \text{if } \lambda > \lambda_0. \end{cases}$$

Remark : Our proof will show that

$$4 \prod_{j=1}^{\infty} (1 - 2^{-j}) \leq \lambda_0 \leq 16 \pi e^5 / (\log 2)^2.$$

Both these bounds can be improved rather easily.

After the first version of this work had been completed we learnt of an unpublished work of M. Yor [9], who uses a different method based on his approach in [8] to check that $E[\exp \lambda \gamma] < \infty$ for $\lambda > 0$ small enough.

Before proving Theorem 1, let us briefly recall the construction of γ given in [4]. First consider another planar Brownian motion B' with $B'_0 = 0$, independent of B . The random variable

$$\alpha_0 := \int_0^1 \int_0^1 \delta_0(B_s - B'_t) \, ds \, dt$$

can be defined as the value at 0 of the continuous density of the random measure on \mathbb{R}^2

$$\mu(g) = \int_0^1 \int_0^1 g(B_s - B'_t) \, ds \, dt \quad (4)$$

(see e.g. [3]). Moreover $\alpha_0 \in L^p$ for every $p < \infty$.

Then, for every integer $n \geq 1$ and for every $k \in \{1, \dots, 2^{n-1}\}$, set

$$A_k^n = [(2k-2)2^{-n}, (2k-1)2^{-n}) \times ((2k-1)2^{-n}, 2k2^{-n}].$$

From the case of two independent Brownian motions, it is straightforward to define

$$\alpha(A_k^n) = \iint_{A_k^n} \delta_0(B_s - B_t) \, ds \, dt.$$

The following facts are immediate from the standard properties of Brownian motion.

(i) For every $n \geq 1$, the variables $\alpha(A_1^n), \dots, \alpha(A_{2^{n-1}}^n)$ are independent.

(ii) $\alpha(A_k^n) \stackrel{(d)}{=} 2^{-n} \alpha_0$.

One can then define γ as

$$\gamma := \sum_{n=1}^{\infty} \left(\sum_{k=1}^{2^{n-1}} (\alpha(A_k^n) - E(\alpha(A_k^n))) \right) \quad (5)$$

and, from (i) and (ii), it is easy to verify that the series converges a.s. and in L^2 .

Lemma 2 : Set $a_1 = 1/2$, $a_2 = e^{-5}(\log 2)^2/(8\pi)$. There exist two positive constants C_1 , C_2 such that for every $p \geq 1$,

$$C_2 a_2^p p! \leq E((\alpha_0)^p) \leq C_1 a_1^p p!.$$

Proof : The upper bound is essentially contained in Rosen [6], formula (2.15). We give the argument for the sake of completeness and also to get an explicit constant. We start from the following identity, which is a special case of formula (2.5) of [3] :

$$\begin{aligned} E[(\alpha_0)^p] &= (2\pi)^{-2p} \int_{(\mathbb{R}^2)^p} d\xi_1 \dots d\xi_p \int_{[0,1]^{2p}} ds_1 \dots ds_p dt_1 \dots dt_p \\ &\quad \times \exp - \frac{1}{2} \text{var} \left(\sum_{j=1}^p \xi_j \cdot (B_{s_j} - B'_{t_j}) \right) \end{aligned}$$

(to verify that $E[(\alpha_0)^p]$ is bounded above by the right side, which is all that we need for the upper bound, write

$$\alpha_0 = \lim_{\epsilon \downarrow 0} \int_0^1 \int_0^1 ds dt p_{\epsilon}(B_s - B'_t), \quad \text{a.s.}$$

where $p_{\epsilon}(\cdot)$ is the usual Gaussian kernel, express $p_{\epsilon}(\cdot)$ in terms of its Fourier transform and use Fatou's lemma). Let \mathcal{P}_p be the set of all permutations of $\{1, \dots, p\}$ and for $\sigma \in \mathcal{P}_p$ set

$$A_{\sigma} = \{(s_1, \dots, s_p, t_1, \dots, t_p); 0 < s_1 < \dots < s_p \leq 1, 0 < t_{\sigma(1)} < \dots < t_{\sigma(p)} \leq 1\}.$$

Then,

$$\begin{aligned} E[(\alpha_0)^p] &= p! (2\pi)^{-2p} \sum_{\sigma \in \mathcal{P}_p} \int d\xi_1 \dots d\xi_p \int_{A_{\sigma}} ds_1 \dots ds_p dt_1 \dots dt_p \\ &\quad \times \exp - \frac{1}{2} \text{var} \left(\sum_{j=1}^p \xi_j \cdot (B_{s_j} - B'_{t_j}) \right). \end{aligned}$$

For every fixed $\sigma \in \mathcal{P}_p$, set

$$u_j = \sum_{k=j}^p \xi_k, \quad v_j = \sum_{k=j}^p \xi_{\sigma(k)}, \quad j \in \{1, \dots, p\},$$

so that, if $(s_1, \dots, t_p) \in A_\sigma$,

$$\begin{aligned} \text{var} \left(\sum_{j=1}^p \xi_j \cdot (B_{s_j} - B'_{t_j}) \right) &= \text{var} \left(\sum_{j=1}^p u_j \cdot (B_{s_j} - B_{s_{j-1}}) - \sum_{j=1}^p v_j \cdot (B'_{t_{\sigma(j)}} - B'_{t_{\sigma(j-1)}}) \right) \\ &= \sum_{j=1}^p |u_j|^2 (s_j - s_{j-1}) + \sum_{j=1}^p |v_j|^2 (t_{\sigma(j)} - t_{\sigma(j-1)}) \end{aligned}$$

where by convention $s_0 = t_{\sigma(0)} = 0$. However, by the Cauchy-Schwarz inequality, if $(s_1, \dots, t_p) \in A_\sigma$,

$$\begin{aligned} &\int_{(\mathbb{R}^2)^p} d\xi_1 \dots d\xi_p \exp - \frac{1}{2} \left(\sum_{j=1}^p |u_j|^2 (s_j - s_{j-1}) + \sum_{j=1}^p |v_j|^2 (t_{\sigma(j)} - t_{\sigma(j-1)}) \right) \\ &\leq \left(\int d\xi_1 \dots d\xi_p \exp - \sum_{j=1}^p |u_j|^2 (s_j - s_{j-1}) \right)^{\frac{1}{2}} \\ &\quad \times \left(\int d\xi_1 \dots d\xi_p \exp - \sum_{j=1}^p |v_j|^2 (t_{\sigma(j)} - t_{\sigma(j-1)}) \right)^{\frac{1}{2}} \\ &= \pi^p \prod_{j=1}^p \left((s_j - s_{j-1})^{-1/2} (t_{\sigma(j)} - t_{\sigma(j-1)})^{-1/2} \right). \end{aligned}$$

Hence, by coming back to the previous formula for $E[(\alpha_0)^p]$,

$$E[(\alpha_0)^p] \leq 2^{-2p} \pi^{-p} (p!)^2 \left(\int_{0 < s_1 < \dots < s_p \leq 1} \frac{ds_1 \dots ds_p}{\sqrt{s_1(s_2 - s_1) \dots (s_p - s_{p-1})}} \right)^2.$$

Elementary calculations give

$$\begin{aligned} J_p &= \int_{0 < s_1 < \dots < s_p \leq 1} \frac{ds_1 \dots ds_p}{\sqrt{s_1(s_2 - s_1) \dots (s_p - s_{p-1})}} \\ &= \begin{cases} \frac{2^p}{p \times (p-2) \times \dots \times 2} \left(\frac{\pi}{2}\right)^{p/2} & \text{if } p \text{ is even} \\ \frac{2^p}{p \times (p-2) \times \dots \times 3 \times 1} \left(\frac{\pi}{2}\right)^{(p-1)/2} & \text{if } p \text{ is odd,} \end{cases} \end{aligned}$$

which implies

$$J_p \sim_{p \rightarrow \infty} \left(\frac{2}{\pi}\right)^{1/4} p^{-1/4} (2\pi)^{p/2} (p!)^{-1/2}.$$

This gives the upper bound of Lemma 2.

For the lower bound, we use another equivalent formula for $E[(\alpha_o)^p]$ (see Proposition 2.1 of [5]). If $\Delta_p = \{(s_1, \dots, s_p) \in (0, \infty)^p; s_1 + \dots + s_p \leq 1\}$ we have

$$\begin{aligned} E[(\alpha_o^p)] &= (2\pi)^{-2p} \int_{(\mathbb{R}^2)^p} dy_1 \dots dy_p \left(\sum_{\sigma \in \mathcal{S}_p} \int_{\Delta_p} \frac{ds_1 \dots ds_p}{s_1 \dots s_p} \exp - \sum_{j=1}^p \frac{|y_{\sigma(j)} - y_{\sigma(j-1)}|^2}{2s_j} \right)^2 \\ &\geq (2\pi)^{-2p} \int_{(\mathbb{R}^2)^p} dy_1 \dots dy_p \left(\sum_{\sigma \in \mathcal{S}_p} \prod_{j=1}^p \int_0^{1/p} \frac{ds}{s} \exp - \frac{|y_{\sigma(j)} - y_{\sigma(j-1)}|^2}{2s} \right)^2 \\ &= (2\pi)^{-2p} p^{-p} \\ &\quad \times \left(\sum_{\sigma, \tau \in \mathcal{S}_p} \int_{(\mathbb{R}^2)^p} dz_1 \dots dz_p \prod_{j=1}^p \left(\psi \left(\frac{|z_{\sigma(j)} - z_{\sigma(j-1)}|^2}{2} \right) \psi \left(\frac{|z_{\tau(j)} - z_{\tau(j-1)}|^2}{2} \right) \right) \right) \end{aligned}$$

where

$$\psi(r) = \int_0^1 \frac{ds}{s} e^{-r/s} = \int_1^\infty \frac{du}{u} e^{-ru}.$$

We then use the crude bound $\psi(r) \geq \psi(1) > e^{-2} \log 2$ for $r \in (0, 1]$ and by integrating over $\{|z_j| \leq 1/\sqrt{2}\}$ in the previous inequality, we get the lower bound of Lemma 2. \square

Proof of Theorem 1 : For simplicity, write $\alpha_{n,k} = \alpha(A_k^n)$ and $\bar{\alpha}_{n,k} = \alpha_{n,k} - E(\alpha_{n,k})$, $\bar{\alpha}_o = \alpha_o - E(\alpha_o)$. For $\lambda > 0$, set

$$\varphi(\lambda) = E[\exp \lambda \bar{\alpha}_o].$$

By Lemma 2, $\varphi(\lambda) < \infty$ for $\lambda < 2$. Since $\varphi'(0) = 0$ we may for every $\lambda_1 \in (0, 2)$ find a positive constant c such that

$$\varphi(\lambda) \leq 1 + c \lambda^2, \quad \forall \lambda \in [0, \lambda_1].$$

Fix $\lambda_1 \in (0, 2)$ and $a \in (0, 1)$. For every $N \geq 1$ set

$$b_N = 2\lambda_1 \prod_{j=2}^N (1 - 2^{-a(j-1)})$$

($b_1 = 2\lambda_1$). Then, by the Hölder inequality, and properties (i), (ii) above, we have for $N \geq 2$,

$$\begin{aligned}
 & E \left[\exp b_N \sum_{n=1}^N \sum_{k=1}^{2^{n-1}} \bar{\alpha}_{n,k} \right] \\
 & \leq E \left[\exp \frac{b_N}{1-2^{-a(N-1)}} \sum_{n=1}^{N-1} \sum_{k=1}^{2^{n-1}} \bar{\alpha}_{n,k} \right]^{1-2^{-a(N-1)}} \\
 & \quad \times E \left[\exp 2^{a(N-1)} b_N \sum_{k=1}^{2^{N-1}} \bar{\alpha}_{n,k} \right]^{2^{-a(N-1)}} \\
 & \leq E \left[\exp b_{N-1} \sum_{n=1}^{N-1} \sum_{k=1}^{2^{n-1}} \bar{\alpha}_{n,k} \right] \varphi \left(b_N 2^{a(N-1)-N} \right)^{2^{(1-a)(N-1)}}
 \end{aligned}$$

Notice that $b_N 2^{a(N-1)-N} \leq \lambda_1$. It follows that

$$\begin{aligned}
 \varphi \left(b_N 2^{a(N-1)-N} \right)^{2^{(1-a)(N-1)}} & \leq \left(1 + c b_N^2 2^{2((a-1)N-a)} \right)^{2^{(1-a)(N-1)}} \\
 & \leq \exp(c' 2^{(a-1)N}),
 \end{aligned}$$

for a constant c' independent of N . By induction we get

$$\begin{aligned}
 E \left[\exp b_N \sum_{n=1}^N \sum_{k=1}^{2^{n-1}} \bar{\alpha}_{n,k} \right] & \leq \exp \left(c' \sum_{n=2}^N 2^{(a-1)n} \right) E \left(\exp b_1 \bar{\alpha}_{1,1} \right) \\
 & \leq \exp \left(c' (1-2^{a-1})^{-1} \right) \varphi(\lambda_1).
 \end{aligned}$$

Letting N tend to ∞ and using Fatou's lemma, we obtain $E[\exp b_\infty \gamma] < \infty$ for $b_\infty = 2\lambda_1 \prod_{j=1}^{\infty} (1-2^{-aj})$. Since $a \in (0,1)$ and $\lambda_1 \in (0,2)$ were arbitrary, we conclude that $E[\exp \lambda \gamma] < \infty$, for $\lambda < 4 \prod_{j=1}^{\infty} (1-2^{-j})$.

Let us now check that $E[\exp \lambda \gamma] = \infty$ for λ large enough. From the definition of γ we have

$$\gamma = \bar{\alpha}_{1,1} + \bar{\alpha}_{1,2} + \tilde{\gamma}$$

where $\alpha_{1,1}$, $\alpha_{1,2}$ are independent and distributed as $\alpha_0/2$, and $\tilde{\gamma}$ is distributed as $\gamma/2$. Using (2), it follows that if $E[\exp a \gamma] < \infty$ for some $a > 0$ then $E[\exp b \alpha_0] < \infty$ for $b < a/2$. By Lemma 2 we have

$$E[\exp b \alpha_0] = \infty, \quad \text{if } b > \frac{1}{a_2}.$$

It follows that $E[\exp \lambda \gamma] = \infty$ for $\lambda > \frac{2}{a_2}$. \square

Remarks : (a) The first part of the proof of Theorem 1 is easily adapted to give a short proof of (2). We have trivially $E[\exp -\lambda \bar{\alpha}_0] < \infty$ for every $\lambda > 0$ so that for every $K > 0$ there exists a constant c such that

$$E[\exp -\lambda \bar{\alpha}_0] \leq 1 + c \lambda^2, \quad \forall \lambda \in [0, K].$$

We then fix $\lambda > 0$ and take :

$$b_N = -2\lambda \prod_{j=2}^N (1-2^{-a(j-1)}) , \quad b_\infty = -2\lambda \prod_{j=1}^{\infty} (1-2^{-aj})$$

and the same calculations as in the previous proof yield $E[\exp b_\infty \gamma] < \infty$. This gives (2) since λ was arbitrary.

(b) In the one-dimensional case, the analogue of the variable γ is the integral

$$\int_{\mathbb{R}} dx (L_1^x)^2$$

where L_1^x denotes the local time at level x , at time 1 of the linear Brownian motion B started at 0 (there is no need for renormalization in dimension 1). It is easy to check that for every $\lambda > 0$

$$E\left(\exp \lambda \int_{\mathbb{R}} dx (L_1^x)^2\right) < \infty.$$

One may argue as follows. By Jensen's inequality,

$$\exp\left(\lambda \int dx (L_1^x)^2\right) \leq \int dx L_1^x \exp \lambda L_1^x.$$

However, if $T_x = \inf\{t, B_t = x\}$,

$$E[L_1^x \exp \lambda L_1^x] = E\left[1_{\{T_x \leq 1\}} L_1^x \exp \lambda L_1^x\right] \leq P(T_x \leq 1) E[L_1^0 \exp \lambda L_1^0].$$

Hence,

$$E[\exp(\lambda \int dx (L_1^x)^2)] \leq \left(\int dx P[T_x < 1] \right) E[L_1^0 \exp \lambda L_1^0] = C E[L_1^0 \exp \lambda L_1^0].$$

By a classical result of Lévy, L_1^0 has the same distribution as $|B_1|$. Therefore, $E[L_1^0 \exp \lambda L_1^0] < \infty$, which gives the desired result.

Another approach to (6), suggested by M. Yor, would be to bound

$$\int dx (L_1^x)^2 \leq L_1^* := \sup_{x \in \mathbb{R}} L_1^x,$$

and then to use the fact that L_x^* has exponential moments (see Borodin [2], Theorem 1.7, it is even true that $E(\exp \lambda (L_x^*)^2) < \infty$ for $\lambda > 0$ small).

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