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Asymptotic windings of planar Brownian motion revisited via the Ornstein-Uhlenbeck process

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ABSTRACT. A celebrated theorem of Spitzer suggests that the number of windings made by a planar Brownian motion Z around the origin and taken in the logarithmic time-scale, is asymptotically close to a Cauchy process. The purpose of this paper is to show that this informal consideration can be made precise by introducing the Ornstein-Uhlenbeck process $X(t) = e^{-t/2}Z(e^t)$. This yields short proofs of known results as well as some new features on the asymptotic behaviour of the winding number (in distribution and pathwise).

KEY WORDS. Planar Brownian motion, winding number, Ornstein-Uhlenbeck process.

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Introduction

Let $Z = (Z(t), t \geq 0)$ be a complex Brownian motion started away from 0 and $\theta = (\theta(t), t \geq 0)$ a continuous version of its argument. The celebrated Theorem of Spitzer [21], which states the convergence in law of $2\theta(t)/\log t$ as $t \to \infty$ towards the standard Cauchy distribution, is at the origin of numerous works on Brownian winding numbers. See in particular Le Gall-Yor [12,13], Pitman-Yor [16], Yor [23] and the references therein for multivariate extensions of Spitzer's Theorem. The almost sure asymptotic behaviour of θ has recently received attention from Bertoin-Werner [1] and Shi [20]. See also Lyons-McKean [14] and Gruet-Mountford [10] for related almost sure results.

Loosely speaking, Spitzer's Theorem suggests that the winding number taken in the logarithmic time-scale is asymptotically close to a Cauchy process. The purpose of this paper is to show that this informal consideration can be made precise and then yields elementary proofs of known results as well as some new information on θ. Specifically, the idea consists of working not directly with Z but rather with the Ornstein-Uhlenbeck process

$$X(t) = e^{-t/2}Z(e^t) \quad (t \geq 0).$$

Plainly, a continuous version of the argument of X is given by

$$\alpha(t) = \theta(e^t) \quad (t \geq 0),$$

which is precisely θ taken in the logarithmic time-scale. The key point is that the Ornstein-Uhlenbeck process is positive recurrent, that is X has an invariant probability measure. So, Limit Theorems are simpler for X than for Z, which is null recurrent. This allows us to replace the time-scale t for X by L_t, where $L = (L_t, t \geq 0)$ is the local time process of the linear diffusion $|X|^2$ at level 1. More precisely, the Ergodic Theorem implies that $t \sim L_t$ almost surely as $t \to \infty$. Hence the asymptotic study of θ essentially reduces to that of the time-changed process $\alpha \circ \tau$, where τ denotes the right-continuous inverse of L. Finally, it is easy to see that $2\alpha \circ \tau$ is a Lévy process fairly close to a standard Cauchy process, and relevant informations on its asymptotic behaviour can be deduced from the literature (see in particular the survey by Fristedt [8]).

This approach should be compared to the elegant proofs of Spitzer's Theorem by Williams [22], Durrett [4] and Messulam-Yor [15] which nevertheless are not suited for studying the almost-sure behaviour of θ. The idea of introducing a positive recurrent Markov process to simplify the study of Brownian windings appears in Franchi [7] where a Brownian motion $\tilde{Z}$ on a sphere is used. However the time-substitution related to the transformation from Z to $\tilde{Z}$ is random and requires a careful analysis, whereas ours is deterministic.

We mention that, although the present approach is one of the simplest and most natural to elucidate the asymptotic behaviour of θ, it does not seem suited for the study of windings around several points (e.g. Pitman-Yor [16], Franchi [7] and Gruet [9]). We also point out that time-inversion reduces the asymptotic study as $t \to 0+$ of a complex Brownian motion $Z' = (Z'_t, t \geq 0)$ started at $Z'_0 = 0$, to that of Z at infinity. In particular, all the results of this paper have analogs for small times; precise statements are left to the reader. Observe that time-inversion for Brownian motion simply corresponds to time-reversal for the Ornstein-Uhlenbeck process.

This paper is organized as follows. Section 1 is devoted to preliminaries on the Ornstein-Uhlenbeck process X. Section 2 contains new proofs of known results and some new features on the asymptotic behaviour of θ and related processes.

1. The Ornstein-Uhlenbeck process

Let us first set down some notations. We consider $Z = (Z(t), t \geq 1)$ a complex Brownian motion started at time 1 from $Z_1 = 1$. The continuous specification of its argument which is null at time 1 is denoted by $\theta = (\theta(t), t \geq 1)$. Recall that θ can be expressed as

$$\theta(t) = \beta_{A(t)} \quad (t \geq 1)$$

where $A(t) = \int_1^t |Z_s|^{-2} ds$ and $\beta = (\beta_s, s \geq 0)$ is a linear Brownian motion started from $\beta_0 = 0$ which is independent of the radial component $|Z|$; see for instance Revuz-Yor [18] on page 181. The increasing process $A = (A(t), t \geq 0)$ is often refered to as the clock of θ.

Next we put

$$X(t) = e^{-t/2} Z(e^t) \quad (t \geq 0)$$

so that $X = (X(t), t \geq 0)$ is a complex Ornstein-Uhlenbeck process started at $X(0) = 1$; see for instance Revuz-Yor [18] on page 35-36. It is a positive recurrent diffusion process in the sense of Harris. More precisely, the law of $\mathcal{N} + i\mathcal{N}'$ is its unique invariant probability measure, where $\mathcal{N}$ and $\mathcal{N}'$ are two independent real-valued standard normal variables. We also consider for all $t \geq 0$, its radial component

$$R(t) = |X(t)| = e^{-t/2}|Z(e^t)|$$

and the continuous specification of its argument which is null at the origin

$$\alpha(t) = \theta(e^t).$$

The skew-product decomposition of Z yields readily the skew-product decomposition of X:

$$\alpha(t) = \beta_{H(t)} \text{ where } H(t) = A(e^t) = \int_0^t R(s)^{-2} ds \tag{1}$$

for all $t \geq 0$; note that the linear Brownian motion β is independent of the radial component R (since it is independent of $|Z|$).

Our next purpose is to describe H as a functional of a Brownian motion, using Feller's representation of one-dimensional diffusions (see for instance Rogers-Williams [19], chapter V-28). It is easy to check using Itô's formula and Lévy's characterization of Brownian motion, that R^2 is a diffusion process valued in $(0, \infty)$ with infinitesimal generator

$$\mathcal{A}f(x) = 2xf''(x) + (2 - x)f'(x)$$

where $f \in \mathcal{C}^2(0, \infty)$. Again R^2 is a positive recurrent process and its invariant probability measure is $\mathfrak{m}(dx) = \frac{1}{2}1_{\{x \geq 0\}} e^{-x/2} dx$ (that is the law of $\mathcal{N}^2 + \mathcal{N}'^2$). Now $\mathfrak{m}$ is the natural choice for the speed measure and this implies that the scale function is

$$\mathfrak{s}(x) = \frac{1}{2} \int_1^x t^{-1} e^{t/2} dt \qquad (x > 0),$$

so that $\mathcal{A} = \frac{1}{2}(d/d\mathfrak{m})(d/d\mathfrak{s})$. Then the process $M = \mathfrak{s}(R^2)$ is a local martingale with bracket

$$\langle M \rangle_t = \int_0^t R_s^{-2} \exp(R_s^2) ds \qquad (t \geq 0).$$

We denote by $B = (B_t, t \geq 0)$ the Brownian motion of Dubins-Schwarz associated with M, so that $M(t) = B_{\langle M \rangle_t}$. The preceding equation can be rewritten as

$$d\langle M \rangle_t = \mathfrak{r}(B_{\langle M \rangle_t})^{-1} \exp(\mathfrak{r}(B_{\langle M \rangle_t})) dt, \tag{2}$$

where $\mathfrak{r} : (-\infty, \infty) \to (0, \infty)$ is the inverse function of $\mathfrak{s}$.

Finally, we consider $\ell = (\ell_t, t \geq 0)$, the local time process of B at level 0, so that

$$L_t = \ell_{\langle M \rangle_t} \qquad (t \geq 0)$$

is the local time process of the diffusion R^2 at level 1. The right-continuous inverse of L

$$\tau_t = \inf\{u, L_u > t\}$$

satisfies

$$\sigma(t) = \langle M \rangle_{\tau(t)} \tag{3}$$

where $\sigma(t) = \inf\{u, \ell_u > t\}$ is the right-continuous inverse of ℓ. So, using (1), (2) and (3), we get the key-identity

$$H(\tau_t) = \int_0^{\sigma(t)} e^{-\mathfrak{r}(B_s)} ds \ . \tag{4}$$

The total time spent by the Brownian motion B in $(-\infty, 0)$ on the time-interval $[0, \sigma(t)]$,

$$S(t) = \int_0^{\sigma(t)} 1_{\{B_s \le 0\}} ds \ ,$$

will also play a major rôle in our approach. We recall that H denotes the "clock" of α (see (1)) and claim the following Lemma that we will use throughout the paper.

Lemma 1. *(i) $S = (S(t), t \ge 0)$ is a stable subordinator of index 1/2. More precisely, for every $\lambda \ge 0$,*

$$E\left(\exp\{-\lambda S(t)\}\right) = \exp\left\{-t(\lambda/2)^{1/2}\right\}.$$

(ii) For every $\varepsilon > 0$, almost surely, for all large enough t

$$(1-\varepsilon)S(t) \le H(\tau_t) \le (1+\varepsilon)S(t)$$

and

$$(1-\varepsilon)S((1-\varepsilon)t) \le H(t) \le (1+\varepsilon)S((1+\varepsilon)t).$$

Proof: (i) is well-known (see e.g. exercice 2.17 on page 449 in Revuz-Yor [18]). Let us rewrite (4) as

$$H(\tau_t) = S(t) + \Delta^+(t) - \Delta^-(t),$$

where for all $t \ge 0$

$$\Delta^+(t) = \int_0^{\sigma(t)} 1_{\{B_s > 0\}} \exp\{-\mathfrak{r}(B_s)\} ds,$$

$$\Delta^-(t) = \int_0^{\sigma(t)} 1_{\{B_s \le 0\}} (1 - \exp\{-\mathfrak{r}(B_s)\}) ds.$$

Then Δ^+ and Δ^- are two subordinators (e.g. Proposition 2.7 on page 445 in Revuz-Yor [18]) with finite mean since

$$E(\Delta^+(1)) = \int_0^{\infty} e^{-\mathfrak{r}(x)} dx = \frac{1}{2}\int_1^{\infty} e^{-t} e^{t/2} t^{-1} dt < \infty$$

and

$$E(\Delta^-(1)) = \int_{-\infty}^0 (1 - e^{-\mathfrak{t}(x)})dx = \frac{1}{2}\int_0^1 (1 - e^{-t})e^{t/2}t^{-1}dt < \infty$$

(these calculations follow e.g. from the Ray-Knight Theorem on Brownian local times, see Revuz-Yor [18] on page 422). Now, the Strong Law of Large Numbers gives

$$\lim_{t\to\infty} t^{-1}\Delta^{+/-}(t) = E(\Delta^{+/-}(1)) \quad \text{a.s.},$$

and

$$\lim_{t\to\infty} t^{-1}S(t) = \infty \quad \text{a.s.}$$

This implies the first assertion of (ii) since $S - \Delta^- \leq H \circ \tau \leq S + \Delta^+$.

Finally, the Ergodic Theorem (or again the Strong Law of Large Numbers) gives

$$\lim_{t\to\infty} t^{-1}\tau(t) = E(\tau(1)) = 1 \quad \text{a.s.},$$

and the second assertion of (ii) follows from the first. ◇

In subsection 2.3, we will also need the following technical Lemma.

Lemma 2. *There is a constant $k > 0$ such that*

$$E(|\tau(t) - t|^{3/2}) \leq kt^{3/4}$$

for every $t > 0$.

Proof. Let $(\ell_t^x, t \geq 0)$ denote the local time at the level x of B, so $\ell_{\langle M\rangle_t}^{\mathfrak{s}(x)}$ is the local time of R^2 at level $x > 0$ and time t. Therefore

$$\tau(t) = \frac{1}{2}\int_0^\infty \ell_{\sigma(t)}^{\mathfrak{s}(x)} e^{-x/2}dx$$

(recall that σ is the right-continuous inverse of $\ell = \ell^0$ and that $\mathfrak{m}(dx) = \frac{1}{2}1_{\{x\geq 0\}}e^{-x/2}dx$ is the speed measure of R^2). Applying Hölder's inequality, we get

$$E(|\tau(t) - t|^{3/2}) \leq \frac{1}{2}\int_0^\infty e^{-x/2}E(|\ell_{\sigma(t)}^{\mathfrak{s}(x)} - t|^{3/2})dx.$$

But we deduce from the Ray-Knight Theorem (see Revuz-Yor [18] on page 422) that

$$E(|\ell_{\sigma(t)}^{\mathfrak{s}(x)} - t|^{3/2}) \leq (4t\mathfrak{s}(x))^{3/4},$$

and since $\mathfrak{s}(x) \leq \sup(e^{x/2}, |\log x|)$,

$$E(|\tau(t) - t|^{3/2}) \leq kt^{3/4}$$

for some positive constant k. ◇

2. Asymptotic results

We will now use the material developed in the preceding section to deduce informations on the asymptotic behaviour of the winding number θ and the clock A. Typically, Lemma 1 shows that the clock H is asymptotically close to the stable subordinator S. It yields useful bounds for θ and allows us to reduce most studies to known results on the asymptotic behaviour of the Cauchy process $C = 2\beta \circ S$ (however, this approach is not completely successful for the pathwise liminf study, see the remark at the end of subsection 2.3).

This method applies as well to other functionals such as the number of "very big" windings Θ which we introduce by analogy with the number of big windings (see Messulam-Yor [15], Pitman-Yor [16]). Specifically, we put

$$\Theta(t) = \int_1^t 1_{\{|Z(s)|>\sqrt{s}\}} d\theta(s), \quad (t \geq 1)$$

where the integral is taken in the sense of stochastic integration with respect to the martingale θ. Plainly,

$$\Theta(e^t) = \int_0^t 1_{\{R(s)>1\}} d\alpha(s) \ , \tag{5}$$

so that Θ taken in the logarithmic time-scale is the number of big windings made by the Ornstein-Uhlenbeck process. See also the Appendix.

2.1. Convergence in distribution

First, we study convergence in distribution as t goes to infinity, for which we will use the symbol $\stackrel{(\mathrm{d})}{\to}$.

Theorem 1. *(i) Recall that $S(1)$ has stable $(1/2)$ distribution, and more precisely, $E(\exp\{-\lambda S(1)\}) = \exp -\{(\lambda/2)^{1/2}\}$. We have*

$$\frac{A(t)}{(\log t)^2} \stackrel{(\mathrm{d})}{\longrightarrow} S(1).$$

(ii) Let C_1 denote a standard Cauchy variable, i.e. $P(C_1 \in dx) = (\pi(1+x^2))^{-1}dx$. We have

$$\frac{2\theta(t)}{\log t} \stackrel{(\mathrm{d})}{\longrightarrow} C_1.$$

(iii) Let $\mathcal{N}$ be a standard normal variable. We have

$$\frac{\Theta(t)}{(\log t)^{1/2}} \stackrel{(\mathrm{d})}{\longrightarrow} \kappa\mathcal{N}.$$

where $\kappa^2 = \frac{1}{2}\int_1^\infty u^{-1}e^{-u/2}du$.

The first statement is a classical step in the proof of Spitzer's Theorem (see e.g. Durrett [4]), the second is Spitzer's Theorem and the third should be compared with results in Messulam-Yor [15]. We also point out that the argument of the proof can easily be modified to establish a result of convergence in the sense of finite-dimensional distributions; recall however that there is no result of convergence in the sense of Skorokhod of a suitable renormalization of $(\theta(e^t), t \geq 0)$ to a Cauchy process (see Durrett [5] on page 137).

Proof: (i) is an immediate consequence of Lemma 1 and the identity $A(e^t) = H(t)$. (ii) follows since $\theta(t) \overset{(d)}{=} A(t)^{1/2}\beta_1$ and $2S(1)^{1/2}\beta_1 \overset{(d)}{=} C_1$.

(iii) The skew-product representation (1) and (5) yield

$$\Theta(e^t) \overset{(d)}{=} \left(\int_0^t 1_{\{R(s)>1\}} R(s)^{-2} ds\right)^{1/2} \mathcal{N}.$$

Finally, the Ergodic Theorem implies

$$\lim_{t\to\infty} t^{-1} \int_0^t 1_{\{R(s)>1\}} R(s)^{-2} ds = \frac{1}{2}\int_1^\infty u^{-1} e^{-u/2} du \quad \text{a.s.} \tag{6}$$

(recall that $\frac{1}{2} 1_{\{u\geq 0\}} e^{-u/2} du$ is the invariant probability measure of R^2). ◇

2.2. "Limsup" results

Now, we turn our attention to the sample path "limsup" results.

Theorem 2. *Consider $f : (0,\infty) \to (0,\infty)$, an increasing function. We have:*

(*i*)
$$\limsup_{t\to\infty} \frac{A(t)}{f(t)^2} = 0 \text{ or } \infty \quad a.s.$$

according as the integral $\int^\infty (tf(t))^{-1} dt$ converges or diverges.

(*ii*)
$$\limsup_{t\to\infty} \frac{\theta(t)}{f(t)} = 0 \text{ or } \infty \quad a.s.$$

according as the integral $\int^\infty (tf(t))^{-1} dt$ converges or diverges.

(*iii*)
$$\limsup_{t\to\infty} \frac{\Theta(t)}{(2\log t \log_3 t)^{1/2}} = \kappa \quad a.s.$$

where $\log_3 = \log\log\log$ and $\kappa^2 = \frac{1}{2}\int_1^\infty e^{-u/2} u^{-1} du$.

The first two statements rephrase respectively Theorem 3 and 1 of Bertoin-Werner [1]. The third can be viewed as Khintchine's Law of the iterated logarithm for the very big windings number; more precisely, one can also prove an analogue of Kolmogorov's test for Θ by the same method.

Proof of Theorem 2-(i): Recall that S is a stable subordinator with index $1/2$. Then it is known that if $g : (0, \infty) \to (0, \infty)$ is an increasing function,

$$\limsup_{t\to\infty} \frac{S(t)}{g(t)^2} = 0 \text{ or } \infty \quad \text{a.s.}$$

according as the integral $\int^{\infty} g(t)^{-1} dt$ converges or diverges (see Theorem 11.2 in Fristedt [8] or Feller [6]). We conclude by taking $g(t) = f(e^t)$ and applying Lemma 1-(ii) and (1). ◇

Proof of Theorem 2-(iii): Since R and β are independent, we have, by (5) and (1)

$$\Theta(e^t) = \tilde{\beta}\left(\int_0^t 1_{\{R(s)>1\}} R(s)^{-2} ds\right), \tag{7}$$

where $\tilde{\beta} = (\tilde{\beta}(t), t \geq 0)$ is a linear Brownian motion independent of R. Now (iii) follows from (6) and the standard law of the iterated logarithm for $\tilde{\beta}$. ◇

The first part of the next Lemma is the key to Theorem 2-(ii). The second part will be used to study the "liminf" behaviour. Recall that β is a Brownian motion independent of the stable process S, and put $\bar{\beta}(t) = \sup\{\beta_s, 0 \leq s \leq t\}$, $S(t-) = \lim_{s\to t, s<t} S(s)$.

Lemma 3. *Let $g : (0, \infty) \to (0, \infty)$ be an increasing function. We have*

(*i*)
$$\limsup_{t\to\infty} \frac{\bar{\beta}(S(t))}{g(t)} = \limsup_{t\to\infty} \frac{\bar{\beta}(S(t-))}{g(t)} = 0 \text{ or } \infty \quad \text{a.s.}$$

according as the integral $\int^{\infty} dt/g(t)$ converges or diverges.

(*ii*)
$$\liminf_{t\to\infty} \frac{\bar{\beta}(S(t))}{g(t)} = \liminf_{t\to\infty} \frac{\bar{\beta}(S(t-))}{g(t)} = 0 \text{ or } \infty \quad \text{a.s.}$$

according as the integral $\int^{\infty} t^{-2} g(t) dt$ diverges or converges.

Proof: (i) The subordinated process $C_t = 2\beta(S(t))$, $(t \geq 0)$ is a standard Cauchy process. According to Theorem 11.2 in Fristedt [8],

$$\limsup_{t\to\infty} \frac{C_t}{g(t)} = \limsup_{t\to\infty} \frac{C_{t-}}{g(t)} = 0 \text{ or } \infty \quad \text{a.s.}$$

according as the integral $\int^{\infty} dt/g(t)$ converges or diverges. So, if the integral diverges, then $\limsup_{t\to\infty} \bar{\beta}(S(t))/g(t) = \limsup_{t\to\infty} \bar{\beta}(S(t-))/g(t) = \infty$ a.s., since obviously $\sup\{C_s, 0 \leq s \leq t\} \leq 2\bar{\beta}(S(t))$ for all t.

Assume now that the integral converges, so $\limsup_{t\to\infty} C_t/g(t) = 0$ a.s. Consider the point process $\gamma = (\gamma_s, s \geq 0)$

$$\gamma_s = 2 \sup\{\beta_u - \beta_{S(s-)}, S(s-) \leq u \leq S(s)\}.$$

Using the property that the process of the jumps of S is a Poisson point process and the independence of β and S, we see that γ is a Poisson point process. It then follows from Lévy's identity and the property that $2\beta \circ S$ is a Cauchy process, that γ has the same distribution as the process of the absolute value of the jumps of a Cauchy process. Therefore the charasteristic measure of γ is

$$n(dx) = c1_{\{x>0\}}x^{-2}dx$$

where $c > 0$ is some positive constant. We deduce that for every $\varepsilon > 0$

$$\int^{\infty} n((\varepsilon g(t), \infty))dt = \frac{c}{\varepsilon}\int^{\infty}\frac{dt}{g(t)} < \infty$$

and thus, a.s., $\gamma_s \leq \varepsilon g(s)$ for all large enough s. Since

$$\bar{\beta}(S(t)) \leq \sup\{C_s, 0 \leq s \leq t\} + \sup\{\gamma_s, 0 \leq s \leq t\},$$

it follows that a.s.

$$\liminf_{t\to\infty}\frac{\bar{\beta}(S(t))}{g(t)} = \liminf_{t\to\infty}\frac{\bar{\beta}(S(t-))}{g(t)} = 0.$$

(ii) follows readily from (i) and the observation that the right-continuous inverse of $\bar{\beta}$ (respectively of S) has the same law as S (respectively as $\bar{\beta}$). ◇

Proof of Theorem 2-(ii): According to (1) and Lemma 1-(ii), we have almost surely

$$\bar{\beta}_{(1-\varepsilon)S((1-\varepsilon)t)} \leq \sup\{\alpha(s), 0 \leq s \leq t\} \leq \bar{\beta}_{(1+\varepsilon)S((1+\varepsilon)t)}$$

for all large enough t. We conclude applying Lemma 3-(i). ◇

2.3. "Liminf" results

Finally, we study the "liminf" asymptotic behaviours of the clock A, the supremas of the winding number θ and the very big winding number Θ. We will first turn our attention to the study of unilateral supremas, which is the easiest part. Recall the notation $\log_3 = \log\log\log$.

Theorem 3. *We have*

(*i*)
$$\liminf_{t\to\infty}\frac{\log_3 t}{(\log t)^2}A(t) = 1/8 \quad a.s.$$

(ii) Let $f : (0,\infty) \to (0,\infty)$ be an increasing function. Then

$$\liminf_{t\to\infty}\frac{1}{f(t)}\sup\{\theta(s), 1 \leq s \leq t\} = 0 \text{ or } \infty \quad a.s.$$

according as the integral $\int^{\infty} f(t)t^{-1}(\log t)^{-2}dt$ diverges or converges.

(iii) Let f: $(0,\infty) \to (0,\infty)$ be an increasing function. Then

$$\liminf_{t\to\infty} \frac{1}{f(t)} \sup\{\Theta(s), 1 \le s \le t\} = 0 \text{ or } \infty \text{ a.s.}$$

according as the integral $\int^\infty f(t)t^{-1}(\log t)^{-3/2}dt$ diverges or converges.

Proof: Recall from Lemma 1-(i) that S is a stable subordinator with Laplace exponent $\lambda \to (\lambda/2)^{1/2}$. According to Breiman [2], we have

$$\liminf_{t\to\infty} t^{-2} \log_2 t\ S(t) = 1/8 \quad \text{a.s.}$$

where $\log_2 = \log\log$. Thus (i) follows from Lemma 1-(ii) and from the identity $H(t) = A(e^t)$. On the other hand (ii) follows from Lemma 3-(ii) by the same argument as in the proof of Theorem 2-(ii). Finally, we deduce (iii) from (6), (7) and the integral test of Hirsch [11]. ◇

The "liminf" behaviour for the maximum of the modulus is specified in Theorem 4. The first part is due to Shi [20] and requires a delicate analysis (which however is simpler than the calculations of Shi). The second statement is an immediate consequence of Chung's law of the iterated logarithm (see Chung [3]) and equations (6) and (7); we omit the details of its proof.

Theorem 4. *We have*

(i)
$$\liminf_{t\to\infty} \frac{\log_3 t}{\log t} \sup\{|\theta(s)|, 1 \le s \le t\} = \pi/4 \quad a.s.$$

(ii)
$$\liminf_{t\to\infty} \sqrt{\frac{\log_3 t}{\log t}} \sup\{|\Theta(s)|, 1 \le s \le t\} = \kappa\pi^2/8 \quad a.s.$$

where $\kappa^2 = \frac{1}{2}\int_1^\infty e^{-u/2}u^{-1}du$.

Proof of the lower bound in Theorem 4-(i): It follows from Chung [3] on page 206 that for any continuous increasing process $D = (D_t, t \ge 0)$ independent of the Brownian motion β, and for every $\lambda > 0$ and $t > 0$

$$\frac{2}{\pi} E(\exp\{\frac{-\pi^2}{8\lambda^2} D_t\}) \le P(\sup_{0\le s\le D_t} |\beta_s| \le \lambda) \le \frac{4}{\pi} E(\exp\{\frac{-\pi^2}{8\lambda^2} D_t\}) , \tag{8}$$

see also Lemma 1 in Shi [20]. In particular, for $D = H$,

$$P(\sup_{0\le s\le t} |\beta_{H(s)}| \le \lambda) \le \frac{4}{\pi} E(\exp\{-\frac{\pi^2}{8\lambda^2} H(t)\}). \tag{9}$$

Recall that τ denotes the inverse function of the local time of R^2 at level 1 and that $H(\tau_t) = S(t) + \Delta^+(t) - \Delta^-(t)$. Introduce for every $\varepsilon > 0$,

$$a_\varepsilon(t) = P(|\tau(t)/t - 1| > \varepsilon), \; b_\varepsilon(t) = P(\varepsilon S(t) < \Delta^+(t) + \Delta^-(t)), \tag{10}$$

and put, for every integer $n > 0$,

$$t_n = (1+\varepsilon)^n \text{ and } \lambda_n = \frac{\pi t_n}{4 \log_2 t_n}(1-\varepsilon)^2.$$

We deduce from (9) that

$$\begin{aligned} &\frac{\pi}{4} P(\sup_{0 \le s \le t_n} |\beta_{H(s)}| \le \lambda_n) \\ &\le a_\varepsilon(t_n) + b_\varepsilon(t_n) + E\left(\exp\left\{\frac{-\pi^2}{8\lambda_n^2}(1-\varepsilon)S((1-\varepsilon)t_n)\right\}\right). \end{aligned}$$

First, we observe that the series $\sum_n a_\varepsilon(t_n)$ converges. Specifically, Lemma 2 and Chebyshev's inequality yield

$$a_\varepsilon(t_n) = P(|t_n^{-1}\tau(t_n) - 1| > \varepsilon) \le k\varepsilon^{-3/2}(1+\varepsilon)^{-3n/4},$$

which proves our assertion.

Then, we check that $\sum_n b_\varepsilon(t_n)$ converges. Using the existence of continuous densities for the stable distribution on the one hand, and Chebyshev's inequality on the other hand, we get

$$b_\varepsilon(t) \le P(S(t) < t^{3/2}\varepsilon^{-1}) + P(\Delta^+(t) + \Delta^-(t) \ge t^{3/2}) \le k' t^{-1/2} \tag{11}$$

for some positive constant k' depending on ε, which entails our claim.

Finally, we deduce from Lemma 1-(i) that

$$\begin{aligned} E\left(\exp\left\{\frac{-\pi^2}{8\lambda_n^2}(1-\varepsilon)S((1-\varepsilon)t_n)\right\}\right) &= \exp\left\{-(1-\varepsilon)t_n \frac{\pi}{4\lambda_n}(1-\varepsilon)^{1/2}\right\} \\ &= O(n^{-1/\sqrt{(1-\varepsilon)}}). \end{aligned}$$

In conclusion, the series $\sum_n P(\sup_{0 \le s \le t_n} |\beta_{H(s)}| \le \lambda_n)$ converges. By Borel-Cantelli's Lemma and an immediate argument of monotonicity, we obtain

$$\liminf_{t \to \infty} \frac{\log_2 t}{t} \sup\{|\beta_{H(s)}|, 0 \le s \le t\} \ge \frac{\pi(1-\varepsilon)^2}{4(1+\varepsilon)} \quad \text{a.s.} \quad \diamond$$

Proof of the upper bound in Theorem 4-(i): We put now, for every integer $n > 0$ and $\varepsilon > 0$

$$t_n = \exp(n^{1+\varepsilon}), \; \lambda_n = \frac{\pi t_n}{4 \log_2 t_n}(1+\varepsilon)^2,$$

and we consider the events

$$U_n = \left\{ \sup\{|\beta_{H(s)} - \beta_{H(\tau(t_{n-1}))}| \ , \ \tau_{t_{n-1}} \le s < \tau_{t_n}\} < \lambda_n \right\} \ .$$

Since β and $H \circ \tau$ are two independent processes with independent increments, the events U_n are independent. Moreover, (9) implies that

$$\begin{aligned} P(U_n) &= P(\sup\{|\beta_s|, 0 \le s \le H(\tau_{t_n}) - H(\tau_{t_{n-1}})\} < \lambda_n) \\ &\ge P(\sup\{|\beta_s|, 0 \le s \le H(\tau_{t_n})\} < \lambda_n) \\ &\ge \frac{2}{\pi} E\left(\exp\{\frac{\pi^2}{8\lambda_n^2} H(\tau_{t_n})\}\right) . \end{aligned}$$

Now recall the notation $b_\varepsilon(t)$ in (10) and Lemma 1-(i). We deduce

$$\begin{aligned} \frac{\pi}{2} P(U_n) &\ge E\left(\exp\{\frac{\pi^2}{8\lambda_n^2} S(t_n)\}\right) - b_\varepsilon(t_n) \\ &= \exp\{-(1+\varepsilon)^{-3/2} \log_2 t_n\} - b_\varepsilon(t_n). \end{aligned}$$

On the one hand,

$$\sum_{n>0} \exp\{-(1+\varepsilon)^{-3/2} \log_2 t_n\} = \sum_{n>0} n^{-1/\sqrt{(1+\varepsilon)}} = \infty.$$

On the other hand, we see by (11) that the series $\sum b_\varepsilon(t_n)$ converges. Hence $\sum P(U_n) = \infty$, and since the U_n's are independent, the events U_n occur for infinitely many n's, almost surely.

All that is needed now is to check that

$$\lim_{n\to\infty} \lambda_n^{-1} \beta_{H(\tau(t_{n-1}))} = 0 \quad \text{a.s.} \tag{12}$$

According to the law of the iterated logarithm for β (remind that β and $H \circ \tau$ are independent) and to Lemma 1-(ii), we have a.s., for all large enough t

$$\begin{aligned} |\beta_{H(\tau_t)}| &\le 2\,(H(\tau_t) \log_2 H(\tau_t))^{1/2} \\ &\le 4\,((1+\varepsilon) S(t) \log_2 S(t))^{1/2} . \end{aligned}$$

On the other hand, we know that a.s. $S_t \le t^2 (\log t)^4$ for all large enough t (see e.g. Theorem 6.1 in Fristedt [8]). In conclusion, a.s.

$$|\beta_{H(\tau_t)}| \le 8t(\log t)^3$$

for all large enough t; (12) follows. ◇

Remark. Pruitt-Taylor [17] proved that

$$\liminf_{t\to\infty} \frac{\log_2 t}{t} \sup_{s\le t} |C_s| = c \quad \text{a.s.}$$

for some positive constant c, which does not seem to be known explicitely. It is easy to deduce that

$$\liminf_{t\to\infty} \frac{\log_3 t}{\log t} \sup_{0\le s\le t} |\theta_s| = c' \quad \text{a.s.}$$

for some constant $c' \geq c$. But showing the result of Shi [20] is more delicate.

Appendix

In this section, we present some comments on the so-called very big winding numbers, communicated to us by Marc Yor.

In a more general setting, one can consider for every $\nu \geq 0$, the number of ν–big windings

$$\theta_t^{(\nu)} = \int_1^t 1_{\{|Z_s|\geq s^\nu\}} d\theta_s \qquad (t \geq 1).$$

In particular, $\theta^{(0)}$ coincides with the number of big windings in the sense e.g. of Messulam-Yor [15]. The case $\nu > 1/2$ is degenerate, because the Law of the Iterated Logarithm implies that $\theta_t^{(\nu)}$ then stays constant for all large enough t, a.s. In the case $0 \leq \nu < 1/2$, $2\theta_t^{(\nu)}/\log t$ converges in distribution as $t \to \infty$ to the law with characteristic function

$$\lambda \to \left(\cosh\{(1-2\nu)\lambda\}\right)^{1/(2\nu-1)} .$$

This can be deduced from results in Le Gall-Yor [13], see in particular equation (5.f) and section 6 there. Similar arguments apply to the study of the number of ν–small windings defined by

$$\theta_t^{(-\nu)} = \int_1^t 1_{\{|Z_s|\leq s^{-\nu}\}} d\theta_s \qquad (t \geq 1).$$

The number of very big windings $\theta^{(1/2)} = \Theta$ appears therefore as a critical case. Our Limit Theorem 1-(iii) can be viewed as a consequence of a general Ergodic Theorem for Brownian motion which we now state. Let $W = (W_u, u \geq 0)$ be a d–dimensional Brownian motion started at 0, and introduce for every $s \geq 1$ the re-scaled process

$$W_u^{(s)} = s^{-1/2} W_{su} \qquad (u \geq 0).$$

The Wiener measure is invariant for the ergodic shift $W \to W^{(s)}$, and it follows that for every functional $F \geq 0$ on Wiener space,

$$\lim_{t\to\infty} \frac{1}{\log t} \int_1^t \frac{ds}{s} F(W^{(s)}) = E(F(W)) \quad a.s. \tag{13}$$

This fact has been noticed and used by many authors, amongst whom O. Adelman, J. Neveu... Applying this result to

$$F(W) = |W_1|^{-2} 1_{\{|W_1|\geq 1\}}$$

yields readily Theorem 1-(iii). Theorem 1-(iii) is related to Proposition 1 in [12] which follows in this setting from (13) applied to

$$F(W) = |W_1|^{-2} 1_{\{|W_1| \geq \epsilon |\xi_1|\}} ,$$

where ξ is a real-valued Brownian motion independent of W. See also exercise (3.20) on page 400 in [18] for further applications of (13). Finally, we point out that in our framework, (13) can be rephrased in a more "usual" form using the stationary Ornstein-Uhlenbeck process $Y_u = e^{-u/2} W(e^u)$ $(-\infty < u < \infty)$. More precisely, we have the standard Ergodic Theorem

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t G(Y \circ T_s) ds \;=\; E(G(Y)) \quad a.s.,$$

where the shift T_s is the translation operator and the functional G is specified by the relation $G(Y) = F(W)$.

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References

1. J. Bertoin and W. Werner, 'Comportement asymptotique du nombre de tours effectués par la trajectoire brownienne plane', *Séminaire de Probabilités* XXVIII, Lect. Notes in Math., Springer (1994).

2. L. Breiman, 'A delicate law of the iterated logarithm for non-decreasing stable processes', *Ann. Math. Statist.* **39** (1968) 1818-1824. [correction id. **41** (1970) 1126-1127].

3. K.L. Chung, 'On the maximum partial sums of sequences of independent random variables', *Trans. Amer. Math. Soc.* **64** (1948) 205-233.

4. R. Durrett, 'A new proof of Spitzer's result on the winding of two-dimensional Brownian motion', *Ann. Probab.* **10** (1982) 244-246.

5. R. Durrett, 'Brownian motion and martingales in analysis', Wadsworth (1984).

6. W.E. Feller, 'A limit theorem for random variables with infinite moments', *Amer. J. Math.* **68** (1946) 257-262.

7. J. Franchi, 'Théorème des résidus stochastique et asymptotique pour 1-formes sur S^2', *Prépublication du Laboratoire de Probabilités* **16**, Université de Paris VI (1989).

8. B.E. Fristedt, 'Sample functions of stochastic processes with stationary independent increments', *Advances in Probability* III, P. Ney, S. Port (eds) 241-396, Dekker (1974).

9. J.C. Gruet, 'Enroulement du mouvement brownien plan autour de deux points', *Thèse de Doctorat*, Université de Paris VI (1990).