SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

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Séminaire de probabilités (Strasbourg), tome 28 (1994), p. 122-137
<http://www.numdam.org/item?id=SPS_1994__28__122_0>

Liminf behaviours of the windings and Lévy's stochastic areas of planar Brownian motion

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1. Introduction

Let $\{X(t)+iY(t); t \geq 0\}$ be a planar Brownian motion (two-dimensional Wiener process), starting at a point z_0 away from 0. Since it almost surely never hits 0, there exists a continuous determination of $\theta(t)$, the total angle wound by the Brownian motion around 0 up to time t. Spitzer (1958) showed the weak convergence of θ:

$$\frac{2}{\log t}\theta(t) \xrightarrow{(d)} C, \tag{1.1}$$

where C is a random variable having a symmetric Cauchy distribution of parameter 1. The last twenty years or so have seen rather spectacular developments on the asymptotic law of winding numbers of Brownian motion. See for example Williams (1974), Durrett (1982), Messulam & Yor (1982), Lyons & McKean (1984), Pitman & Yor (1986 & 1989), and the book of Yor (1992, Chapters 5 & 7) for a detailed survey and up-to-date references. Recently Bertoin & Werner (1994a) were interested in the *almost sure* asymptotic behaviour of θ. By making use of an exact distribution for θ given in Spitzer (1958) and by studying level crossings of the radial part of the Brownian motion, they proved the following

THEOREM A (Bertoin & Werner 1994a). *For every non-decreasing function $f > 0$,*

$$\limsup_{t\to\infty} \frac{\theta(t)}{f(t)\log t} = \begin{cases} 0 \\ \infty \end{cases}, \text{ a.s.} \iff \int^{\infty} \frac{dt}{tf(t)\log t} \begin{cases} < \infty \\ = \infty \end{cases}.$$

So, in particular, $\limsup_{t\to\infty}(\log t)^{-1}(\log\log t)^{-a}\theta(t)$ is equal to 0 when $a > 1$, and to ∞ otherwise. See also Franchi (1993) and Gruet & Mountford (1993) for Brownian motion valued in a compact space.

To provide further insight on the path properties of θ, it is of interest to investigate its liminf behaviour as well. Thanks to the Brownian scaling and rotational invariance properties, we only need to treat the case when $z_0 = 1$. Let $\theta^*(t) = \sup_{0\leq u\leq t} |\theta(u)|$ for $t \geq 0$. In Section 2, we present a liminf integral test for θ which states as follows:

THEOREM 1. *Let $f > 0$ be a non-increasing function such that $f(t)\log t$ is non-decreasing, then*

$$\mathbb{P}\Big[\, \theta^*(t) < f(t)\log t \text{ i.o.} \Big] = 0 \quad \text{or} \quad 1, \quad \text{a.s.}$$

according as

$$\int^{\infty} \frac{dt}{t\, f(t) \log t} \exp\big(-\frac{\pi}{4f(t)}\big) \tag{1.2}$$

converges or diverges. Here, "i.o." stands for "infinitely often" as t tends to ∞.

An immediate consequence of the above theorem is:

COROLLARY 1. *We have*

$$\liminf_{t\to\infty} \frac{\log\log\log t}{\log t}\theta^*(t) = \frac{\pi}{4}, \qquad \text{a.s.} \tag{1.3}$$

The triple logarithm figuring in (1.3) is of no surprise. Indeed, as Spitzer's result (1.1) suggests, the right clock for θ is rather $\log t$ than the usual time t. Corollary 1 is thus a version of Chung's celebrated liminf law of the iterated logarithm (LIL).

Another interesting Brownian functional, which bears some relation with the winding number θ, is Paul Lévy's stochastic area σ defined as the stochastic integral

$$\sigma(t) = \int_0^t X_u dY_u - Y_u dX_u, \quad t \geq 0. \tag{1.4}$$

(Strictly speaking, σ is twice the stochastic area of Brownian motion studied by Lévy (1951), who obtained the exact distribution for each random variable $\sigma(t)$, by exploiting the series representation of Brownian motion with respect to a complete orthonormal system.) The following LIL was due to Berthuet (1981):

THEOREM B (Berthuet 1981). *We have, almost surely,*

$$\limsup_{t\to\infty} \frac{\sigma(t)}{t \log\log t} = \frac{2}{\pi}.$$

See also Baldi (1986), Helmes (1985 & 1986), and Berthuet (1986). Let $\sigma^*(t) = \sup_{0\leq u\leq t} |\sigma(u)|$. Our main result concerning the liminf behaviour of σ is the following integral test:

THEOREM 2. *Let $g > 0$ be a non-increasing function such that $tg(t)$ is non-decreasing, then*

$$\mathbf{P}\Big[\,\sigma^*(t) < tg(t),\ \text{i.o.}\,\Big] = \begin{cases} 0 \\ 1 \end{cases} \iff \int^{\infty} \frac{dt}{tg(t)} e^{-\pi/2g(t)} \begin{cases} < \infty \\ = \infty \end{cases}$$

COROLLARY 2. *The following Chung-type LIL holds:*

$$\liminf_{t\to\infty} \frac{\log\log t}{t}\sigma^*(t) = \frac{\pi}{2}, \qquad \text{a.s.}$$

The plan of the rest of this paper is as follows. In Section 2, we focus on the windings and present a proof of Theorem 1. Lévy's stochastic area is studied in Section 3, where Theorem 2 is to be shown. In section 4, we are interested in, and obtain Chung's LIL for, the ranges of θ and σ. Throughout the paper, we will not distinguish $\xi(t)$ from ξ_t for any stochastic process ξ.

2. Brownian windings

Let us keep the notation previously introduced. In this section, the planar Brownian motion $Z = X + iY$ is assumed to start from 1. Let R be the radial part of Z, i.e. $R^2 = X^2 + Y^2$ and let $H(t) = \int_0^t R_u^{-2} du$. The well-known skew-product representation for two-dimensional Brownian motion goes back at least to Itô-McKean (1974) p.270:

(2.1) $\qquad \theta(t) = \beta(H_t)$, with β a linear Brownian motion *independent* of R.

The following simple preliminary result is needed which will be applied for both θ and σ later on:

LEMMA 1. *Let W be a standard Brownian motion, and D a positive non-decreasing continuous process independent of W. Let $W^*(D_t) = \sup_{0\le u\le t} |W(D_u)|$, then for all $0 \le s \le t$ and $0 < x \le y$,*

$$\frac{8}{3\pi}\mathbf{E}\exp\Big[-\frac{\pi^2}{8x^2}D_t\Big] \le \mathbf{P}\big[\,W^*(D_t) < x\,\big] \le \frac{4}{\pi}\mathbf{E}\exp\Big[-\frac{\pi^2}{8x^2}D_t\Big]; \tag{2.2}$$

$$\mathbf{P}\big[\,W^*(D_s) < x,\ W^*(D_t) < y\,\big] \le \frac{16}{\pi^2}\mathbf{E}\exp\Big[-\frac{\pi^2}{8x^2}D_s - \frac{\pi^2}{8y^2}(D_t - D_s)\Big]. \tag{2.3}$$

Proof of Lemma 1. By conditioning on $\{D_u;\ u \ge 0\}$ and using the Brownian scaling property, (2.2) is trivially deduced from the well-known distribution of Brownian motion under the sup-norm (see Chung (1948) p.221). Since W has independent and stationary increments, the probability on the LHS of (2.3) is equal to

$$\mathbb{E}\Big\{\mathbb{1}_{\{W^*(D_s)<x\}}\mathbf{P}\big[\sup_{0\le u\le D_t-D_s}|W(u)+a|<y\,|\,D\big]\Big|_{a=W(D_s)}\Big\}.$$

Using a general property of Gaussian measures (see for example Ledoux & Talagrand (1991) p.73), the above expression is smaller than $\mathbf{P}[\,W^*(D_s)<x\,]\mathbf{P}[\,W^*(D_t-D_s)<y\,]$. Now (2.3) follows using the second part of (2.2). □

The next lemma concerns the Laplace transform of the clock H:

LEMMA 2. *For all $\mu\ge\nu>0$, $s>0$ and $t>0$, we have*

$$\mathbb{E}\exp\Big[-\frac{\mu^2}{2}H_t\Big]\le 2\,t^{-\mu/2}, \tag{2.4}$$

$$\mathbb{E}\exp\Big[-\frac{\mu^2}{2}H_t-\frac{\nu^2}{2}(H_{t+s}-H_t)\Big]\le 4s^{-\nu/2}t^{-(\mu-\nu)/2}. \tag{2.5}$$

If moreover, $0\le\mu\le1$, then

$$\mathbb{E}\exp\Big[-\frac{\mu^2}{2}H_t\Big]\ge\frac{1}{3}t^{-\mu/2}e^{-1/2t}. \tag{2.6}$$

Proof of Lemma 2. Let us recall that the Gamma function is decreasing on $[1,x_0]$ and increasing on $[x_0,\infty)$, with $1<x_0<2$ and $\Gamma(x_0)\approx 0.886$ (see Abramowitz & Stegun (1965) pp.258-259). Thus $\Gamma(1+x)\le 2\Gamma(1+y)$ for $y\ge x\ge 0$, and $\Gamma(1+\mu/2)/2^{\mu/2}\Gamma(1+\mu)\ge 1/3$ for $0\le\mu\le1$. According to (6.20), (6.21) and (6.25) of Yor (1992),

$$\mathbb{E}\exp\Big[-\frac{\mu^2}{2}H_t\Big]=\frac{1}{(2t)^{\mu/2}\Gamma(\mu/2)}\int_0^1 e^{-z/2t}z^{\mu/2-1}(1-z)^{\mu/2}dz.$$

Since $e^{-1/2t}\le e^{-z/2t}\le 1$ for all $0\le z\le 1$, we have

$$\frac{e^{-1/2t}\Gamma(1+\mu/2)}{(2t)^{\mu/2}\Gamma(1+\mu)}\le\mathbb{E}\exp\Big[-\frac{\mu^2}{2}H_t\Big]\le\frac{\Gamma(1+\mu/2)}{t^{\mu/2}\Gamma(1+\mu)},$$

which yields (2.4) and (2.6). Now let $\mu\ge\nu>0$. It follows from the scaling property of R and (2.4) that

$$\mathbf{E}\Big[\exp\Big(-\frac{\nu^2}{2}(H_{t+s}-H_t)\Big)\,\Big|\,R_t=r\Big]=\mathbf{E}\exp\Big[-\frac{\nu^2}{2}H(sr^{-2})\Big]\le 2r^\nu s^{-\nu/2}.$$

Hence,

$$\mathbb{E}\exp\Big[-\frac{\mu^2}{2}H_t - \frac{\nu^2}{2}(H_{t+s}-H_t)\Big] \le 2s^{-\nu/2}\mathbb{E}\Big[R_t^\nu \exp\Big(-\frac{\mu^2}{2}H_t\Big)\Big]$$
$$= \frac{2s^{-\nu/2}}{(2t)^{(\mu-\nu)/2}\Gamma((\mu-\nu)/2)}\int_0^1 e^{-z/2t}z^{(\mu-\nu)/2-1}(1-z)^{(\mu+\nu)/2}dz,$$

using again (6.20), (6.21) and (6.25) of Yor (1992). The above expression is obviously

$$\le \frac{2s^{-\nu/2}}{(2t)^{(\mu-\nu)/2}}\frac{\Gamma(1+(\mu+\nu)/2)}{\Gamma(1+\mu)} \le 4s^{-\nu/2}t^{-(\mu-\nu)/2},$$

as desired. □

Let us turn to the proof of Theorem 1. Suppose that f satisfies the condition in Theorem 1. Pick a t_0 sufficiently large and define a sequence $\{t_i\}_{i\ge 0}$ by

$$\log t_{i+1} = (1+f(t_i))\log t_i, \quad i \ge 0. \tag{2.7}$$

Since f is positive, it is easily seen that t_n tends to infinity as $n \to \infty$. Put $f_i = f(t_i)$ for all $i \ge 0$. In the rest of this paper, unimportant finite positive constants are denoted by K, K_0, K_1, K_2, $\cdots$ whose value depend only on f_0 and may vary from line to line.

LEMMA 3. *The series*

$$\int^\infty \frac{dt}{tf(t)\log t}\exp\Big(-\frac{\pi}{4f(t)}\Big) \quad \textit{and} \quad \sum_i \exp\Big(-\frac{\pi}{4f_i}\Big)$$

converge and diverge simultaneously.

Proof of Lemma 3. Since $f(t)\log t$ is non-decreasing, we have

$$\frac{f_i}{f_{i+1}} \le \frac{\log t_{i+1}}{\log t_i} = 1 + f_i, \tag{2.8}$$

which is bounded. Therefore,

$$\int^\infty \frac{dt}{tf(t)\log t}\exp\Big(-\frac{\pi}{4f(t)}\Big) = \sum_i \int_{t_i}^{t_{i+1}} \frac{dt}{tf(t)\log t}\exp\Big(-\frac{\pi}{4f(t)}\Big)$$
$$\le \sum_i \frac{1}{f_{i+1}}\exp\Big(-\frac{\pi}{4f_i}\Big)\log\Big(\frac{\log t_{i+1}}{\log t_i}\Big)$$
$$\le K\sum_i \frac{1}{f_i}\exp\Big(-\frac{\pi}{4f_i}\Big)\log(1+f_i)$$
$$\le K\sum_i \exp\Big(-\frac{\pi}{4f_i}\Big),$$

using (2.7) and (2.8). On the other hand, we have, by (2.7),

$$\int^{\infty} \frac{dt}{tf(t)\log t} \exp(-\frac{\pi}{4f(t)})dt \geq \sum_i \frac{1}{f_i} \exp(-\frac{\pi}{4f_{i+1}}) \log(1+f_i).$$

Since $\log(1+f_i)/f_i \geq K^{-1}$ by boundedness of f, the proof of Lemma 3 is completed. □

Proof of Theorem 1. We begin with the convergent part. Suppose that the integral (1.2) converges, which, by Lemma 3, means that $\sum_i e^{-\pi/4f_i} < \infty$. Put $A_i = \{\theta^*(t_i) < f_{i+1} \log t_{i+1}\}$. Since H is continuous and increasing, it follows from (2.2) and (2.4) that

$$\mathbb{P}(A_i) \leq \frac{4}{\pi} \mathbb{E} \exp\Big[-\frac{\pi^2 H(t_i)}{8f_{i+1}^2(\log t_{i+1})^2}\Big] \leq \frac{8}{\pi} \exp\Big(-\frac{\pi \log t_i}{4f_{i+1}\log t_{i+1}}\Big).$$

By (2.7), $\log t_{i+1}/\log t_i = 1+f_i$. Thus

$$\begin{aligned}\mathbb{P}(A_i) &\leq \frac{8}{\pi} \exp\Big(-\frac{\pi}{4f_{i+1}(1+f_i)}\Big) = \frac{8}{\pi} \exp\Big(-\frac{\pi}{4f_{i+1}} + \frac{\pi}{4}\frac{f_i}{f_{i+1}(1+f_i)}\Big)\\ &\leq K \exp\Big(-\frac{\pi}{4f_{i+1}}\Big),\end{aligned}$$

using (2.8). An application of Borel-Cantelli lemma together with a monotonicity argument yield then the convergent part of Theorem 1. Now suppose that $\sum_i e^{-\pi/4f_i} = \infty$. In view of (1.3), we assume without loss of generality that

$$\frac{1}{2\log\log\log t} \leq f(t) \leq \frac{1}{\log\log\log t}. \tag{2.9}$$

(For rigorous justification, we refer to Lemmas a and d of Lipschutz (1956)). Several lines of elementary calculation using (2.7) and (2.9) imply that

$$\exp(\frac{i}{3\log i}) \leq \log t_i \leq \exp(\frac{2i}{\log i}), \tag{2.10}$$

$$\exp(-\frac{\pi}{4f_i}) \leq (\frac{3\log i}{i})^{\pi/4}. \tag{2.11}$$

Let $B_i = \{\theta^*(t_i) < f_i \log t_i\}$. By (2.2) and (2.6), we have

$$\begin{aligned}\mathbb{P}(B_i) &\geq \frac{8}{3\pi} \mathbb{E} \exp\Big[-\frac{\pi^2 H(t_i)}{8f_i^2(\log t_i)^2}\Big]\\ &\geq \frac{8}{9\pi} \exp\Big(-\frac{\pi}{4f_i}\Big),\end{aligned} \tag{2.12}$$

which implies that $\sum_i \mathbb{P}(B_i) = \infty$. Now consider $\mathbb{P}(B_iB_j)$ for $j > i$. First of all, let us notice that, according to our construction (2.7) of $\{t_i, i \ge 0\}$,

$$\frac{\log t_i}{\log t_j} \le (1+f_j)^{-(j-i)}. \tag{2.13}$$

By (2.3) and (2.5), we have

$$\begin{aligned}
\mathbb{P}(B_iB_j) &\le \frac{16}{\pi^2}\mathbb{E}\exp\Big(-\frac{\pi^2 H(t_i)}{8f_i^2(\log t_i)^2} - \frac{\pi^2(H(t_j)-H(t_i))}{8f_j^2(\log t_j)^2}\Big)\\
&\le \frac{64}{\pi^2}\exp\Big(-\frac{\pi\log(t_j-t_i)}{4f_j\log t_j} - \frac{\pi}{4f_i} + \frac{\pi\log t_i}{4f_j\log t_j}\Big)\\
&= \frac{64}{\pi^2}e^{-\pi/4f_i}\exp\Big(-\frac{\pi\log(t_j/t_i-1)}{4f_j\log t_j}\Big).
\end{aligned} \tag{2.14}$$

Let $\delta > 0$ whose value will be precised in (2.16) below, and let n_0 be sufficiently large. Put for all $n > n_0$

$$\begin{aligned}
\Omega_1 &= \{n_0 \le i < j \le n:\ j-i < 1/f_j\};\\
\Omega_2 &= \{n_0 \le i < j \le n:\ 1/f_j \le j-i < i^\delta\};\\
\Omega_3 &= \{n_0 \le i < j \le n:\ j-i \ge i^\delta\}.
\end{aligned}$$

If $(i,j) \in \Omega_1$, then

$$\log\Big(\frac{t_j}{t_i}\Big) = \Big(1-\frac{\log t_i}{\log t_j}\Big)\log t_j \ge \Big(1-(1+f_j)^{-(j-i)}\Big)\log t_j \ge \frac{j-i}{2}f_j\log t_j,$$

using (2.13) and the inequality $1-(1+x)^{-a} \ge xa/2$ ($\forall 0 \le x \le 1/a \le 1$). Since $f_j\log t_j$ is large (by (2.9)), we have $\log(t_j/t_i - 1) \ge \frac{1}{3}(j-i)f_j\log t_j$, which, with the aid of (2.14), implies that

$$\mathbb{P}(B_iB_j) \le \frac{64}{\pi^2}e^{-\pi/4f_i}e^{-\pi(j-i)/12}.$$

Applying (2.12) gives that

$$\sum\sum\nolimits_{(i,j)\in\Omega_1}\mathbb{P}(B_iB_j) \le K_1\sum_{i=1}^{n}\mathbb{P}(B_i). \tag{2.15}$$

By (2.13), there exists $0 < \delta < 1/3$ (depending on the value of f_0) such that for all $(i,j) \in \Omega_2$,

$$\frac{\log t_i}{\log t_j} \le (1+f_j)^{-1/f_j} \le 1-3\delta. \tag{2.16}$$

Thus $\log(t_j/t_i - 1) \geq \log(t_j^{3\delta} - 1) \geq 2\delta \log t_j$. By (2.14), this yields the following estimate:

$$\mathbf{P}(B_iB_j) \leq Ke^{-\pi/2f_i} \exp\Big(-\frac{\delta\pi}{2f_j}\Big).$$

Thus

$$\sum\sum\nolimits_{(i,j)\in\Omega_2} \mathbf{P}(B_iB_j) \leq K\sum_{i=1}^{n} \mathbf{P}(B_i) \sum_{i<j<i+i^\delta} (e^{-\pi/4f_j})^{2\delta}.$$

By (2.11),

$$\sum_{i<j<i+i^\delta} (e^{-\pi/4f_j})^{2\delta} \leq \sum_{i<j<i+i^\delta} \Big(\frac{3\log j}{j}\Big)^{\pi\delta/2} \leq i^\delta(3\log(i+i^\delta))^{\pi\delta/2} i^{-\pi\delta/2} \leq K_0,$$

which readily yields the desired inequality

$$\sum\sum\nolimits_{(i,j)\in\Omega_2} \mathbf{P}(B_iB_j) \leq K_2\sum_{i=1}^{n} \mathbf{P}(B_i). \tag{2.17}$$

Now, let $(i,j) \in \Omega_3$. In this case, $j - (\log j)^2 \geq i + i^\delta - (\log(i+i^\delta))^2 > i$, which implies that $j - i \geq (\log j)^2$. But from (2.9) and (2.10) it follows that $f_j \geq (\log j)^{-2}$. So $j - i \geq f_j^{-2}$. Using (2.13), this implies that

$$\frac{\log t_i}{f_j \log t_j} \leq \frac{1}{f_j}(1+f_j)^{-(j-i)} \leq \frac{1}{f_j}\exp\big(-f_j^{-2}\log(1+f_j)\big) \leq K.$$

Moreover,

$$\frac{t_i}{t_j} \leq \frac{\log t_i}{\log t_j} \leq \exp\big(-f_j^{-2}\log(1+f_j)\big) \leq 1 - 1/K_0,$$

for some $K_0 > 1$. Thus

$$-\frac{\log(1-t_i/t_j)}{f_j\log t_j} \leq \frac{\log K_0}{f_j \log t_j} \leq K.$$

By writing

$$-\frac{\pi\log(t_j/t_i - 1)}{4f_j\log t_j} = -\frac{\pi}{4f_j} - \frac{\pi\log(1-t_i/t_j)}{4f_j\log t_j} + \frac{\pi\log t_i}{4f_j\log t_j} \leq -\frac{\pi}{4f_j} + \frac{\pi K}{2},$$

we have, by (2.14),

$$\mathbf{P}(B_iB_j) \leq K_0 e^{-\pi/4f_i - \pi/4f_j},$$

which in turn implies that

$$\sum\sum_{(i,j)\in\Omega_3} \mathbb{P}(B_iB_j) \le K_3\Big(\sum_{i=1}^{n} \mathbb{P}(B_i)\Big)^2. \tag{2.18}$$

Finally, assembling (2.15), (2.17) and (2.18) gives that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{P}(B_iB_j) \le K\Big(\sum_{i=1}^{n} \mathbb{P}(B_i)\Big)^2.$$

According to Kochen & Stone (1964)'s version of Borel-Cantelli lemma, this together with $\sum_i \mathbb{P}(B_i) = \infty$ yield that

$$\mathbb{P}\Big[\ \theta^*(t) < f(t)\log t \ \text{ i.o.}\ \Big] \ge K^{-1}.$$

It is easy to deduce from the Blumenthal's 0-1 law by time inversion that the above probability equals to 1, which proves the divergent part of Theorem 1. □

3. Lévy's stochastic area

Before studying Lévy's stochastic area process of Brownian motion, we first of all establish a simple preliminary result.

LEMMA 4. *If W is a standard Brownian motion, then for all positive numbers s, t, μ and ν,*

$$\mathbb{E}\exp\Big[-\frac{\mu^2}{2}\int_0^t W_u^2 du - \frac{\nu^2}{2}\int_t^{t+s} W_u^2 du\Big]$$
$$= \Big[\sinh\mu t \cosh\nu s\big(\coth\mu t + \frac{\nu}{\mu}\tanh\nu s\big)\Big]^{-1/2}. \tag{3.1}$$

Proof of Lemma 4. Let us recall two results on the Laplace transform of quadratic functionals of Brownian motion. The first (3.2) is due to Lévy (1951), and the second (3.3) can be found in Pitman & Yor (1982) p.432.

$$\mathbb{E}\Big[\exp\Big(-\frac{\alpha^2}{2}\int_0^1 W_u^2 du\Big)\,\Big|\, W_1 = x\Big] = \Big(\frac{\alpha}{\sinh\alpha}\Big)^{1/2} \exp\big[-\frac{x^2}{2}(\alpha\coth\alpha - 1)\big], \tag{3.2}$$

$$\mathbb{E}\exp\Big[-\frac{\alpha^2}{2}\int_0^1 (W_u + x)^2 du\Big] = (\cosh\alpha)^{-1/2}\exp\big(-\frac{x^2}{2}\alpha\tanh\alpha\big). \tag{3.3}$$

Let p denote the term on the LHS of (3.1). By scaling and Markov properties,

$$p = \mathbb{E}\exp\Big[-\frac{\mu^2t^2}{2}\int_0^1 W_u^2 du - \frac{\nu^2t^2}{2}\int_1^{1+s/t} W_u^2 du\Big]$$
$$= \mathbb{E}\Big\{\mathbb{E}\Big[\exp\Big(-\frac{\mu^2t^2}{2}\int_0^1 W_u^2 du\Big)\,\Big|\, W_1\Big]\, \mathbb{E}\Big[\exp\Big(-\frac{\nu^2t^2}{2}\int_0^{s/t} (W_u + x)^2 du\Big)\Big]_{x\equiv W_1}\Big\}.$$

With the aid of (3.3) and the Brownian scaling property,

$$\mathbb{E}\exp\Big(-\frac{\nu^2t^2}{2}\int_0^{s/t}(W_u+x)^2du\Big) = (\cosh\nu s)^{-1/2}\exp\Big[-\frac{x^2}{2}\nu t\tanh\mu s\Big].$$

Therefore, by (3.2),

$$\begin{aligned} p &= \Big(\frac{\mu t}{\sinh\mu t\,\cosh\nu s}\Big)^{1/2}\mathbb{E}\exp\Big[-\frac{W_1^2}{2}(\mu t\coth\mu t-1)-\frac{W_1^2}{2}\nu t\tanh\mu s\Big] \\ &= \Big(\frac{\mu t}{\sinh\mu t\,\cosh\nu s\,[\mu t\coth\mu t+\nu t\tanh\mu s]}\Big)^{1/2}, \end{aligned}$$

as desired. $\square$

Remark. One can obtain (3.1) directly by solving the associated Sturm-Liouville equation.

Let σ be Lévy's stochastic area defined by (1.4), and $R^2 = X^2+Y^2$ as before. In this section, we assume without loss of generality that the planar Brownian motion Z starts from 0. Since it never returns to the origin (i.e. R does not vanish at any positive time), it follows from Itô's formula that

$$d(R_t^2) = 2R_t d\eta_t + 2dt, \quad R_0 = 0, \tag{3.4}$$

where $\eta_t \equiv \int_0^t (X_u dX_u + Y_u dY_u)/R_u$ is, according to the celebrated Lévy's characterization, a linear Brownian motion. Let $C_t \equiv \int_0^t R_u^2 du$ be the quadratic-variation process of σ. The martingales σ and η being obviously orthogonal, it follows from Knight's theorem (see for example Rogers & Williams (1987) Theorem IV.34.16) that there is a Brownian motion ξ, independent of η, such that

$$\sigma_t = \xi(C_t). \tag{3.5}$$

By Yamada-Watanabe theorem (see Rogers & Williams (1987) Theorem V.40.1), the Bessel process R, determined by equation (3.4), is adapted to the (augmented) filtration generated by η. Consequently, ξ is *independent* of R.

Since $\sinh\mu t\,(\coth\mu t + \nu\mu^{-1}\tanh\nu s) \geq \cosh\mu t$ and $e^x/2 \leq \cosh x \leq e^x$ (for $x \geq 0$), we deduce immediately from Lemma 4 the following estimates for the Laplace transform of the clock C for all positive numbers s, t, μ and ν:

$$e^{-\mu t} \leq \mathbb{E}\exp\Big[-\frac{\mu^2}{2}C_t\Big] \leq 2e^{-\mu t}; \tag{3.6}$$

$$\mathbb{E}\exp\Big[-\frac{\mu^2}{2}C_t - \frac{\nu^2}{2}(C_{t+s}-C_t)\Big] \leq 4e^{-\mu t-\nu s}. \tag{3.7}$$

Proof of Theorem 2. It is very similar to (and easier than) that of Theorem 1 presented in the previous section. So we only state some key steps and omit the details. First of all, choose a sequence $\{t_i;\ i \geq 0\}$ by $t_{i+1} = (1 + g(t_i))t_i$ $(i \geq 0$, with a sufficiently large initial value $t_0)$. As for the windings, we write $g_i = g(t_i)$ for notational simplification. Then in the spirit of Lemma 3, it is seen that the series

$$\int^{\infty} \frac{dt}{tg(t)} \exp(-\frac{\pi}{2g(t)}) \quad \text{and} \quad \sum_i \exp(-\frac{\pi}{2g_i})$$

converge and diverge simultaneously. Let $A_i = \{\sigma^*(t_i) < t_{i+1}g_{i+1}\}$. It follows from (2.2) and the second part of (3.6) that

$$\mathbb{P}(A_i) \leq K \exp(-\frac{\pi}{2g_{i+1}}).$$

Using Borel-Cantelli lemma and a monotonicity argument, this implies the convergent part of Theorem 2. Now suppose that $\sum_i e^{-\pi/2g_i} = \infty$. Let $B_i = \{\sigma^*(t_i) < t_i g_i\}$. Thanks to (2.2) and the first part of (3.6), we get that

$$\mathbb{P}(B_i) \geq K_1 \exp(-\frac{\pi}{2g_i}),$$

while by making use of (2.3) and (3.7) we obtain:

$$\mathbb{P}(B_i B_j) \leq K_2 e^{-\pi/2g_i} \exp\Big(-\frac{\pi}{2g_j} + \frac{\pi t_i}{2g_i t_j}\Big),$$

from which it follows that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{P}(B_i B_j) \leq K\Big(\sum_{i=1}^{n} \mathbb{P}(B_i)\Big)^2.$$

This yields the divergent part of Theorem 2 using Kochen & Stone's version of Borel-Cantelli lemma. □

4. The ranges

In this section, we present a Chung-type LIL concerning the ranges of θ and σ instead of their suprema.

THEOREM 3. *Let θ and σ be defined as in Section 1. Then*

(4.1) $$\liminf_{t\to\infty} \frac{\log\log\log t}{\log t}\Big(\sup_{0\leq u\leq t} \theta(u) - \inf_{0\leq u\leq t} \theta(u)\Big) = \frac{\pi}{2}, \quad a.s.$$

(4.2) $$\liminf_{t\to\infty} \frac{\log\log t}{t}\Big(\sup_{0\leq u\leq t} \sigma(u) - \inf_{0\leq u\leq t} \sigma(u)\Big) = \pi, \quad a.s.$$

LEMMA 5. *Suppose that W is a linear Brownian motion starting from 0, then for every $\delta > 0$ there exists a finite constant $K_\delta > 0$ depending only on δ such that for all $\lambda > 0$,*

$$\text{(4.3)} \qquad \mathbb{P}\Big[\sup_{0\le u\le 1} W(u) - \inf_{0\le u\le 1} W(u) < \lambda\Big] \le K_\delta \exp\Big(-\frac{(1-\delta)\pi^2}{2\lambda^2}\Big).$$

Proof of Lemma 5. Let q be the probability on the LHS of (4.3). Feller (1951) calculated the exact law of the range of Brownian motion:

$$q = \Big(\frac{2}{\pi}\Big)^{1/2} \int_0^\lambda L'\big(\frac{x}{2}\big)\frac{dx}{x} = \Big(\frac{2}{\pi}\Big)^{1/2} \int_0^{\lambda/2} L'(x)\frac{dx}{x},$$

where

$$L(x) = \frac{(2\pi)^{1/2}}{x} \sum_{k=0}^{\infty} \exp\Big(-\frac{(2k+1)^2\pi^2}{8x^2}\Big),$$

is the distribution function of the sup-norm of a standard Brownian bridge. By integration by parts, we obtain:

$$\begin{aligned} q &\le 2\int_0^{\lambda/2} \frac{1}{x^2}\frac{d}{dx}\Big(\sum_{k=0}^{\infty} \exp\big(-\frac{(2k+1)^2\pi^2}{8x^2}\big)\Big) \\ &= \frac{8}{\lambda^2}\sum_{k=0}^{\infty} \exp\Big(-\frac{(2k+1)^2\pi^2}{2\lambda^2}\Big) + \frac{16}{\pi^2}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}\exp\Big(-\frac{(2k+1)^2\pi^2}{2\lambda^2}\Big). \end{aligned}$$

The first infinite series on the RHS in the above inequality is obviously bounded above by $\sum_{k=0}^{\infty} \exp(-(2k+1)\pi^2/2\lambda^2) = (1-e^{-\pi^2/\lambda^2})^{-1}\exp(-\pi^2/2\lambda^2)$, and the second by $\sum_{k=0}^{\infty}(2k+1)^{-2}\exp(-\pi^2/2\lambda^2) = (\pi^2/8)\exp(-\pi^2/2\lambda^2)$. Therefore,

$$q \le \frac{8}{\lambda^2(1-e^{-\pi^2/\lambda^2})}\exp\big(-\frac{\pi^2}{2\lambda^2}\big) + 2\exp\big(-\frac{\pi^2}{2\lambda^2}\big).$$

Thus, to prove Lemma 5 is reduced to showing the existence of a positive finite constant $\tilde{K}_\delta$ such that

$$\text{(4.4)} \qquad \lambda^2(1-e^{-\pi^2/\lambda^2}) \ge \tilde{K}_\delta \exp\big(-\frac{\delta\pi^2}{2\lambda^2}\big).$$

Since $1-e^{-x} \ge x/2$ ($\forall 0 \le x \le \log 2$), it follows that $\lambda^2(1-e^{-\pi^2/\lambda^2}) \ge \pi^2/2$ for all $\lambda \ge \pi/\sqrt{\log 2}$. If $0 < \lambda < \pi/\sqrt{\log 2}$, then $\lambda^2(1-e^{-\pi^2/\lambda^2}) \ge \lambda^2/2$. Therefore for all $\lambda > 0$, we have

$$\lambda^2(1-e^{-\pi^2/\lambda^2}) \ge \min(\pi^2, \lambda^2)/2,$$

which yields (4.4) by choosing

$$\tilde{K}_\delta \equiv \frac{1}{2}\inf_{x>0}\Big[\min(\pi^2, x^2)\exp\big(\frac{\delta\pi^2}{2x^2}\big)\Big] > 0. \qquad \square$$

Proof of Theorem 3. Let $A_\sigma(t) = \sup_{0\le u\le t}\sigma(u) - \inf_{0\le u\le t}\sigma(u)$ be the range of σ. In view of (3.5), by conditioning on R and using Lemma 5 and (3.6) we get that

$$\begin{aligned}\mathbb{P}\Big[\,A_\sigma(t)<\lambda\Big] &\le K_\delta\mathbb{E}\exp\Big[-\frac{(1-\delta)\pi^2}{2\lambda^2}C_t\Big]\\ &\le 2K_\delta\exp\Big(-\frac{(1-\delta)^{1/2}\pi t}{\lambda}\Big).\end{aligned} \tag{4.5}$$

Now pick rational numbers $a>1$ and $\varepsilon>0$. Let $t_n=a^n$ and $\lambda=\pi t_n/(1+\varepsilon)\log\log t_n$ and choose $\delta>0$ sufficiently small such that $(1-\delta)^{1/2}(1+\varepsilon)>1+\varepsilon/2$. It follows from (4.5) that

$$\begin{aligned}\mathbb{P}\Big[\,A_\sigma(t_n)<\frac{\pi}{1+\varepsilon}\frac{t_n}{\log\log t_n}\Big] &\le 2K_\delta\exp(-(1-\delta)^{1/2}(1+\varepsilon)\log\log t_n)\\ &\le \frac{2K_\delta}{(n\log a)^{1+\varepsilon/2}}.\end{aligned}$$

By Borel-Cantelli lemma, we obtain that

$$\liminf_{n\to\infty}\frac{\log\log t_n}{t_n}A_\sigma(t_n)\ge\pi,\quad \text{a.s.}$$

A monotonicity argument yields immediately the lower bound in (4.2). Its upper bound part follows trivially from Corollary 2 and the relation $A_\sigma(t)\le 2\sigma^*(t)$. The proof of the LIL (4.1) is very similar to that of (4.2), by using (2.4) instead of (3.6) and taking the subsequence $\tilde{t}_n=\exp(a^n)$ instead of $t_n=a^n$. The details are omitted. □

5. Remarks

(A) It seems interesting to look for a two-sided Csáki-type integral test for θ or σ. Thanks to (3.6) and (3.7), which are valid for *all* positive numbers s and t, we have:

THEOREM 4. *Let $f>0$ and $g>0$ be non-increasing functions such that $tf(t)$ and $tg(t)$ are non-decreasing. Let $h=f+g$. Then*

$$\mathbb{P}\Big[\sup_{0\le u\le t}\sigma(u)<tf(t),\ -\inf_{0\le u\le t}\sigma(u)<tg(t),\ \text{i.o.}\Big]=0\quad\text{or}\quad 1,\quad \text{a.s.}$$

according as

$$\int^\infty\frac{\min(f(t),g(t))}{t\,h^2(t)}\exp\Big(-\frac{\pi}{h(t)}\Big)\,dt$$

converges or diverges.

An immediate consequence is the following Hirsch-type test for σ.

COROLLARY 3. *If f satisfies the condition in Theorem 4, then*

$$\mathbb{P}\Big[\sup_{0\le u\le t}\sigma(u) < tf(t), \text{ i.o.}\Big] = \begin{cases} 0 \\ 1 \end{cases} \iff \int^{\infty} \frac{f(t)}{t}dt \begin{cases} < \infty \\ = \infty \end{cases}.$$

From (3.6) and (3.7), the proof of Theorem 4 is completed exactly along the lines of that of Csáki (1978)'s Theorem 2.1 (ii). So we feel free to omit the details. On the other hand, the case of the Brownian windings seems more complicated, essentially due to the handicap that (2.5) is valid only for $\mu \ge \nu$. Thus to get a Csáki-type test for θ, sharper estimates on the joint distribution of $(\sup_{0\le u\le t}\theta(u), \inf_{0\le u\le t}\theta(u))$ are needed. A Hirsch-type integral test for θ was obtained by Bertoin & Werner (1994b).

(B) Further path properties of θ have been investigated by Bertoin & Werner (1994b), who gave an elegant proof of Theorem A and (1.3) via Ornstein-Uhlenbeck processes. Another proof of these results are presented in Shi (1994) using a Cauchy-type embedding.

(C) There are many remarkable results on weak convergence of the winding process Θ of a two-dimensional random walk. See Bélisle (1991) for references. For example, for spherically symmetric random walks, Bélisle (1991) obtained a Brownian embedding which shows that Θ behaves (in distribution, though) very much like the so-called Brownian "big winding" process defined as $\int_0^t \mathbf{1}_{\{R_u\ge 1\}}d\theta(u)$ (for weak convergence concerning big windings, see Yor (1992) p.88). Central limit theorems for Θ were established by Dorofeev (1994). Laws of the iterated logarithm can be found in Shi (1994).

(D) Another question needs answered concerning the path properties of θ or σ: how big (or small) are the *increments* of θ or σ? The problem seems to be beyond the scale of this paper.

Acknowledgements. I am grateful to Jean Bertoin and Marc Yor for helpful discussions, and to David Mason for a reference. Thanks are also due to a referee for his careful reading and insightful comments.

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