SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

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Séminaire de probabilités (Strasbourg), tome 27 (1993), p. 173-176
<http://www.numdam.org/item?id=SPS_1993__27__173_0>

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1 Introduction

Let $\{B_t : t \geq 0\}$ be a standard Brownian motion, let $L(t,x)$ be its local time and let $A(t,x) = \int_{-\infty}^{x} L(t,y)\,dy$. The process $\{A(t,B_t), t \geq 0\}$ comes up naturally in the study of the local time sheet, and was studied in some detail in joint work with L. C. G. Rogers [1, 2, 3]. It was shown there that it is a Dirichlet process but not a semimartingale, at least relative to the Brownian filtration: it is the sum of a stochastic integral plus a continuous process X; X has zero quadratic variation (so $A(t,B_t)$ is a Dirichlet process) but it has non-trivial $\frac{4}{3}$-power variation [2] (and hence infinite variation, which is why $A(t,B_t)$ is not a semimartingale).

This note is a byproduct of [3], where the exact $\frac{4}{3}$-variation of X was determined. We will give a decomposition different from the one used there, one which puts things in a rather different context and leads to some heuristic remarks on a formal connection with distributions.

2 The Decomposition

Theorem 1 *$A(t,B_t)$ has the decomposition*

$$A(t,B_t) = \int_0^t L(s,B_s)\,dB_s + X_t \tag{1}$$

where

$$\begin{aligned} X_t &= \lim_{\varepsilon\to 0+} \frac{1}{2}\int_0^t \frac{L(s,B_s) - L(s,B_s-\varepsilon)}{\varepsilon}\,ds \\ &= t + \lim_{\varepsilon\to 0+} \frac{1}{2}\int_0^t \frac{L(s,B_s+\varepsilon) - L(s,B_s)}{\varepsilon}\,ds\,. \end{aligned} \tag{2}$$

The limits exist in probability, uniformly for t in compact sets.

PROOF. Let ϕ_ε be an approximate identity and let

$$\psi_\varepsilon(x) = \int_{-\infty}^{x} \phi_\varepsilon(y)\,dy\,. \tag{3}$$

Then

$$\begin{aligned} A(t,B_t) &= \int_0^t I_{\{B_t - B_s \geq 0\}}\,ds \\ &= \lim_{\varepsilon\to 0+} \int_0^t \psi_\varepsilon(B_t - B_s)\,ds\,. \end{aligned} \tag{4}$$

This holds for all t since $\{s : B_s = B_t\}$ has Lebesgue measure zero. Let us write ϕ and ψ in place of ϕ_ϵ and ψ_ϵ respectively, and expand the integral in (4) by Ito's lemma:

$$(5)\int_0^t \psi(B_t - B_s)\,ds = t\psi(0) + \int_0^t\int_s^t \phi(B_u - B_s)\,dB_u\,ds + \frac{1}{2}\int_0^t\int_s^t \phi'(B_u - B_s)\,du\,ds\,.$$

Assuming that ϕ is Lipschitz, it is a simple matter to interchange the order of integration in the stochastic integral term on the right hand side of (5).

We would like to extend (5) to some discontinuous ϕ. Suppose that ν is a finite signed measure of zero total mass and compact support, and let $\phi(x) = \nu(-\infty, x]$. Let ϕ_n be a sequence of uniformly bounded C_K^∞ functions such that the measures $\phi_n'(x)\,dx$ converge weakly to ν, and such that there exists C such that $\int |\phi_n'(x)|\,dx \le C$ for all n. Let $\psi_n(x) = \int_{-\infty}^x \phi_n(y)\,dy$. Notice that we can choose the ϕ_n to all be supported in the same compact interval, so that the ψ_n will be uniformly bounded. For each n, write the left hand side of (5) in the form

$$\int_0^t \psi_n(B_t - B_s)\,ds = \int_{-\infty}^\infty \psi_n(B_t - x)L(t,x)\,dx\,.$$

Since the ψ_n are uniformly bounded and $\psi_n(x) \to \psi(x)$ for all x, the left-hand side of (5) converges to

$$\int_{-\infty}^\infty \psi(B_t - x)L(t,x)\,dx\,.$$

On the right-hand side of (5), $\psi_n(0) \to \psi(0)$, while the second term is

$$\int_0^t \int_0^u \phi_n(B_u - B_s)\,ds\,dB_u = \int_0^t \left[\int_{-\infty}^\infty \phi_n(B_u - x)L(u,x)\,dx\right] dB_u\,.$$

Using the uniform bound on the ϕ_n and the fact that $\phi_n \to \phi$ at all continuity points, and hence a.e., it is easy to see that this converges to

$$\int_0^t \left[\int_{-\infty}^\infty L(u,x)\phi(B_u - x)\,dx\right] dB_u\,.$$

To handle the final integral in (5), first change order, then introduce local time:

$$\frac{1}{2}\int_0^t \int_0^u \phi_n'(B_u - B_s)\,ds\,du = \frac{1}{2}\int_{-\infty}^\infty \int_{-\infty}^\infty \int_0^t \phi_n'(y - x)L(u,x)L(du,y)\,dy\,dx\,.$$

Integrate first over x, then let $n \to \infty$ and use thc fact that the ϕ_n' converge weakly to ν and are uniformly bounded in L^1 to see that this is

$$\begin{aligned} &= \frac{1}{2}\int_{-\infty}^\infty \int_0^t \left[\int_{-\infty}^\infty \phi_n'(z)L(u,y-z)\,dz\right] L(du,y)\,dy \\ &\to \frac{1}{2}\int_{-\infty}^\infty \int_0^t \left[\int_{-\infty}^\infty L(u,y-z)\nu(dz)\right] L(du,y)\,dy\,, \end{aligned}$$

giving

$$\begin{aligned} \int_0^t \psi(B_t - x)L(t,x)\,dx &= t\psi(0) + \int_0^t \left[\int_{-\infty}^\infty L(u,x)\nu(-\infty,x]\,dx\right] dB_u \\ (6) \qquad & \quad + \frac{1}{2}\int_{-\infty}^\infty \int_0^t \left[\int_{-\infty}^\infty L(u,y-z)\nu(dz)\right] L(du,y)\,dy\,. \end{aligned}$$

If $\nu = \varepsilon_{-1}(\delta_0 - \delta^\varepsilon)$ then $\psi(0) = 0$, so this is

$$\begin{aligned}
\int_0^t \psi(B_t - x)L(t,x)\,dx &= \int_0^t \left[\int_{-\infty}^{\infty} L(u,x)\nu((-\infty,x])\,dx\right] dB_u \\
&\quad +\frac{1}{2}\int_{-\infty}^{\infty}\int_0^t \varepsilon^{-1}(L(u,y) - L(u,y-\varepsilon))L(du,y)\,dy \qquad (7)\\
&= \int_0^t \left[\int_{-\infty}^{\infty} L(u,x)\nu((-\infty,x])\,dx\right] dB_u \\
&\quad +\frac{1}{2}\int_0^t \varepsilon^{-1}(L(u,B_u) - L(u,B_u-\varepsilon))\,du\,.
\end{aligned}$$

Now we can let $\varepsilon \to 0^+$. The first two terms converge in L^2, hence so does the third, giving

$$(8)\qquad A(t,B_t) = \int_0^t L(u,B_u)\,du + \lim_{\varepsilon\to 0^+}\frac{1}{2}\int_0^t \varepsilon^{-1}(L(u,B_u) - L(u,B_u-\varepsilon))\,du\,.$$

This proves the first half of (2). To get the second half, apply the same argument to $\nu_n \equiv \varepsilon^{-1}(\delta_{-\varepsilon} - \delta_0)$ and note that this time $\psi(0) = 1$ for all n, so that

$$(9)\quad A(t,B_t) = t + \int_0^t L(u,B_u)\,dB_u + \lim_{\varepsilon\to 0^+}\frac{1}{2}\int_0^t \varepsilon^{-1}(L(u,B_u+\varepsilon) - L(u,B_u))\,du\,.$$

To see the limits in (2) are uniform, notice that all three terms in (7) are continuous in t. The left-hand side converges uniformly in t for t in compacts, and the stochastic integral converges in L^2, again uniformly in t, hence the final integrals in (8) and (9) also converge. ♣

3 Some Remarks

Remark 1 The equation (2) can be interpreted in terms of Schwartz distributions. Let $\frac{\partial^+}{\partial x}$ and $\frac{\partial^-}{\partial x}$ represent the right-hand and left-hand partial derivatives. Consider $x \mapsto L(t, B_t + x)$. The limits in (2) just involve $\frac{\partial^+ L}{\partial x}$ and $\frac{\partial^- L}{\partial x}$, which evidently exist in some distributional sense, so we can formally rewrite (2) as

$$\begin{aligned}
(10)\qquad X_t &= \frac{1}{2}\int_0^t \frac{\partial^- L}{\partial x}(s,B_s)\,ds \\
&= \frac{1}{2}\int_0^t \left(2 + \frac{\partial^+ L}{\partial x}(s,B_s)\right) ds\,.
\end{aligned}$$

The partials are not functions, for if they were, X would be of bounded variation, whereas it is known [2, 3] to have nontrivial $\frac{4}{3}$-variation, and hence infinite variation.

Remark 2 Here is a quick formal but non-rigorous argument which shows that Theorem 1 is a disguised version of Ito's lemma. Notice that $A(t,x)$ is continuously differentiable in t as long as $B_t \neq x$ ($A(t,x) = \int_0^t I_{\{B_s \leq x\}}\,ds$ so $\frac{\partial A}{\partial t}(t,x) = I_{\{B_t \leq x\}}$) and it is continuously differentiable in x ($\frac{\partial A}{\partial x} = L(t,x)$), but the second derivative fails to exist.

Thus one can almost, but not quite, apply the classical version of Ito's lemma. If we ignore this inconvenience and apply it purely formally to $A(t, B_t + \varepsilon)$ and $A(t, B_t - \varepsilon)$, noting that $I_{\{B_t \leq B_t + \varepsilon\}} = 1$ and $I_{\{B_t \leq B_t - \varepsilon\}} = 0$, we get

$$A(t, B_t + \varepsilon) = t + \int_0^t L(s, B_s + \varepsilon)\, dB_s + \frac{1}{2}\int_0^t \frac{\partial L}{\partial x}(s, B_s + \varepsilon)\, ds$$

and

$$A(t, B_t - \varepsilon) = \int_0^t L(s, B_s - \varepsilon)\, dB_s + \int_0^t \frac{1}{2}\frac{\partial L}{\partial x}(s, B_s - \varepsilon)\, ds \,.$$

Now just let $\varepsilon \downarrow 0$ to get (10).

Remark 3 If we subtract the two expressions for X, we see that

$$\int_0^t \left(\frac{\partial^+ L}{\partial x}(s, B_s) - \frac{\partial^- L}{\partial x}(s, B_s) \right) ds = -2t \,.$$

for all t, which leads to the conclusion that, in some distribution sense

$$\frac{\partial^+ L}{\partial x}(s, B_s) - \frac{\partial^- L}{\partial x}(s, B_s) \equiv -2 \tag{11}$$

for a.e. s. A similar phenomenon occurs with expectations:

$$\frac{\partial^+}{\partial x} E^y\{L(t,x)\}\bigg|_{x=y} - \frac{\partial^-}{\partial x} E^y\{L(t,x)\}\bigg|_{x=y} = -2 \,. \tag{12}$$

In some sense, then, (10) is an almost-everywhere form of (12).

References

[1] Rogers, L. C. G., and Walsh, John B., Local time and stochastic area integrals, *Ann. Probability* **19**, 457–482, 1991.

[2] Rogers, L. C. G., and Walsh, John B., The intrinsic local time sheet of Brownian motion. *Prob. Th. Rel. Fields* **88**, 363–379, 1991.

[3] Rogers, L. C. G., and Walsh, John B., The exact $\frac{4}{3}$-variation of a process arising from Brownian motion. *Preprint.*