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# ON THE LÉVY TRANSFORMATION

# OF BROWNIAN MOTIONS AND CONTINUOUS MARTINGALES

L. E. Dubins[1], M. Émery[2], M. Yor[3]

> [...] je vous confie aujourd'hui mes espérances, qui ne reposent encore que sur des calculs de probabilité.
>
> *É. Zola*

## Introduction

If $(B_t)_{t\geqslant 0}$ is a Brownian motion started at 0 and $(L_t)_{t\geqslant 0}$ its local time at 0, Lévy's characterization (see for instance [6] p. 141) implies that

$$\widehat{B} = |B| - L = \int \operatorname{sgn}(B)\, dB$$

is also a Brownian motion. In other words, the *Lévy transformation* $T : B \longrightarrow \widehat{B}$, defined almost everywhere on the Wiener space $W = \mathcal{C}(\mathbb{R}_+, \mathbb{R})$, preserves the Wiener measure $\mu$.

Dubins & Smorodinsky [3] have established that an analogue of $T$ for coin-tossing is ergodic. This increases the plausibility of the following conjecture:

($\mathcal{L}$) *The Lévy transformation is ergodic,*

that is, the $\sigma$-field $\mathcal{J}$ on $W$ of all events a.s. invariant by $T$ is trivial.

Known since the late 70's, the problem of the ergodicity of $T$ is mentioned as an open question in Revuz & Yor [6], page 257.

We shall see that ($\mathcal{L}$) is closely related to the question of knowing which continuous martingales $M = (M_t)_{t\geqslant 0}$ with $M_0 = 0$ have the same law as their Lévy transform $\widehat{M} = \int \operatorname{sgn}(M)\, dM$. (A discussion of this subject is begun in Exercise (2.32) page 231 of Revuz & Yor [6].) Recall that to each continuous martingale $M$ is associated its quadratic variation $\langle M\rangle$; $\langle M\rangle$ is the continuous, non-decreasing, adapted process such that $\langle M\rangle_t = \lim_{n\to\infty} \sum_{k=1}^{2^n} \big(M_{k2^{-n}t} - M_{(k-1)2^{-n}t}\big)^2$. For simplicity, we shall deal only with continuous martingales verifying $M_0 = 0$ and $\langle M\rangle_\infty = \infty$; such processes will be called *divergent martingales*. As is well known

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(see [1] and [2]), each divergent martingale $M$ is obtained by time-changing a Brownian motion, the time-change being given by $\langle M\rangle$. More precisely, to $M$ is associated a (unique) Brownian motion $\beta^M$ such that $M_t = (\beta^M)_{\langle M\rangle_t}$ for all $t$; this defines a map $\beta_M : \Omega \longrightarrow W$ transforming $\mathbb{P}$ into the Wiener measure $\mu$. The law of $M$ is characterized by the joint law of $\beta^M$ and $\langle M\rangle$.

Changing time in the integral $\int \operatorname{sgn}(M)\, dM$ gives $\beta^{\widehat{M}} = \widehat{\beta^M}$; as $\langle\widehat{M}\rangle = \langle M\rangle$, $\widehat{M}$ has the same law as $M$ if and only if

$$(\beta^M)_{\langle M\rangle} \overset{\text{(law)}}{=} T(\beta^M)_{\langle M\rangle} .$$

A sufficient condition is the independence of $\beta^M$ and $\langle M\rangle$. We conjecture that this sufficient condition is also necessary.

Let us give a name to this conjecture:

$(\mathcal{M})$ *A divergent martingale $M$ has the same law as its Lévy transform $\widehat{M}$ (if and) only if the processes $\beta^M$ and $\langle M\rangle$ are independent.*

A reason to believe in this conjecture is Ocone's Theorem A of [5]: The independence of $\beta^M$ and $\langle M\rangle$ is a necessary and sufficient condition for $M$ to have the same law as all integrals $\int H\, dM$, where $H$ ranges either over all deterministic processes of the form $H = \mathbb{1}_{[0,a]} - \mathbb{1}_{(a,\infty)}$, or over all $\{-1,1\}$-valued processes that are predictable for the natural filtration of $M$.

The next section gives some preliminary observations about $(\mathcal{L})$. Then comes our main result, the equivalence of $(\mathcal{L})$ and $(\mathcal{M})$, established, with some further precisions, in the third section. In the last section, we try to understand $(\mathcal{L})$ better, in particular by constructing examples of martingales which are not identical in law with their Lévy transform. The appendix borrows from [5] Ocone's theorem and its proof, with a few remarks.

## Preliminary remarks

The following lemma from ergodic theory is well known.

LEMMA 1. — *Let $(W, \mathcal{G}, \mu)$ be a probability space and $T$ a measurable transformation of $W$ which preserves $\mu$. A random variable $Z \in \mathrm{L}^2(W,\mu)$ is a. s. invariant by $T$ if and only if*

$$\langle Y\circ T, Z\rangle_{\mathrm{L}^2} = \langle Y, Z\rangle_{\mathrm{L}^2}$$

*for all $Y \in \mathrm{L}^2(W,\mu)$.*

PROOF. — If $Z$ is invariant, $Z = Z\circ T$ a. s. and $\langle Y\circ T, Z\rangle = \langle Y\circ T, Z\circ T\rangle = \langle Y, Z\rangle$ by the invariance of $\mu$.

Conversely, if $\langle Y\circ T, Z\rangle = \langle Y, Z\rangle$ for every $Y$, $\langle Y\circ T, Z\rangle = \langle Y\circ T, Z\circ T\rangle$ and $Z\circ T$ is the conditional expectation of $Z$ given $T^{-1}\mathcal{G}$; as $Z$ and its projection $Z\circ T$ on $\mathrm{L}^2(T^{-1}\mathcal{G})$ have the same $\mathrm{L}^2$-norm (invariance of $\mu$), they must be equal. ∎

LEMMA 2. — *With the notations of the introduction, let $S : W \longrightarrow [0,\infty]$ be a stopping time for $B$. The stopped processes $B^S$ and $\widehat{B}^S$ have the same law if and only if $S$ is a. s. invariant.*

PROOF. — If $S$ is invariant, $\widehat{B}^S = (B\circ T)^{S\circ T} = (B^S)\circ T$ has the same law as $B^S$.

Conversely, if $B^S$ and $\widehat{B}^S$ have the same law, the pairs $(B^S, S)$ and $(\widehat{B}^S, S)$ also have the same law, because $S$ is a function of the path of $B^S$ (for instance $S = \sup\{t \in \mathbb{Q} : B_t^S \neq B_\infty^S\}$). Now the Markov property at time $S$ makes it possible to deduce the law of $(B, S)$ from that of $(B^S, S)$ and similarly for $\widehat{B}$; hence $(B, S)$ and $(\widehat{B}, S)$ have the same law. This gives $\langle Y, e^{-S}\rangle = \langle Y\circ T, e^{-S}\rangle$ for every $Y \in L^2(W)$ and $S$ is invariant by Lemma 1. ∎

REMARK. — The Markov property in the above proof cannot be dispensed of: if the random variable $S$ is not a stopping time, it may happen that $B^S$ and $\widehat{B}^S$ have the same law but $S$ is not invariant. Take for instance any $[0,1]$-valued random variable $S$ independent of $\mathcal{F}_1$; $B^S$ and $\widehat{B}^S$ have the same law, namely that of a Brownian motion stopped at some independent time distributed as $S$.

## Equivalence of $(\mathcal{L})$ and $(\mathcal{M})$

THEOREM 1. — *Let $M$ be a divergent martingale. The following three properties are equivalent:*

(i) *$M$ and $\widehat{M}$ have the same law;*

(ii) *$(\beta^M, \langle M\rangle)$ and $(T(\beta^M), \langle M\rangle)$ have the same law;*

(iii) *$\beta^M$ and $\langle M\rangle$ are conditionally independent given the $\sigma$-field $\beta_M^{-1}(\mathcal{J})$.*

Examples of this situation are obtained by taking $\langle M\rangle$ independent of $\beta^M$; if $(\mathcal{L})$ is true, $\beta_M^{-1}(\mathcal{J})$ is trivial and there are no other examples.

PROOF. — Since $\langle M\rangle = \langle \widehat{M}\rangle$ and the law of $(\beta^M, \langle M\rangle)$ depends only on that of $M$, (i) implies (ii). Conversely, (ii) $\Rightarrow$ (i) follows from $M = (\beta^M)_{\langle M\rangle}$ and $\widehat{M} = T(\beta^M)_{\langle M\rangle}$.

(ii) $\Rightarrow$ (iii). Let $F$ be a bounded random variable measurable with respect to $\sigma\{\langle M\rangle_t, t \geqslant 0\}$; there exists a bounded measurable function $f$ on $W$ such that $\mathbb{E}[F|\beta^M] = f(\beta^M)$. For every $g \in L^2(W)$, using the definition of $f$, hypothesis (ii) and again the definition of $f$, one can write

$$\mathbb{E}[f(\beta^M)\, g(\beta^M)] = \mathbb{E}[F\, g(\beta^M)] = \mathbb{E}[F\, g\circ T(\beta^M)] = \mathbb{E}[f(\beta^M)\, g\circ T(\beta^M)]$$

and Lemma 1 gives $f(\beta^M) = f\circ T(\beta^M)$ a.s. So $f$ is $\mathcal{J}$-measurable and coming back to the definition of $f$ one gets $\mathbb{E}[F|\beta^M] = \mathbb{E}[F|\beta_M^{-1}(\mathcal{J})]$, the desired result.

(iii) $\Rightarrow$ (ii). Keep the same notations and call $\mathcal{J}'$ the $\sigma$-field $\beta_M^{-1}(\mathcal{J})$. Hypothesis (iii) gives on the one hand

$$\mathbb{E}[F\,g(\beta^M)] = \mathbb{E}[\mathbb{E}[F|\mathcal{J}']\;\mathbb{E}[g(\beta^M)|\mathcal{J}']]$$

and on the other hand

$$\mathbb{E}[F\,g\circ T(\beta^M)] = \mathbb{E}[\mathbb{E}[F|\mathcal{J}']\;\mathbb{E}[g\circ T(\beta^M)|\mathcal{J}']]\ .$$

Since every $\mathcal{J}'$-measurable random variable has the form $h(\beta^M)$ where $h$ is $\mathcal{J}$-measurable, $\mathbb{E}[g(\beta^M)|\mathcal{J}'] = \mathbb{E}[g\circ T(\beta^M)|\mathcal{J}']$ and we obtain

$$\mathbb{E}[F\,g(\beta^M)] = \mathbb{E}[F\,g\circ T(\beta^M)]\ ,$$

which means precisely that (ii) holds. ∎

Recall that a martingale $M$ is called *pure* if it is divergent and if for each $t \geqslant 0$ its quadratic variation $\langle M\rangle_t$ is measurable for the $\sigma$-field $\sigma\{\beta^M_s\,,\ s \geqslant 0\}$.

A weaker form of Conjecture $(\mathcal{M})$ is obtained by restricting to pure martingales the demand that $M \stackrel{\text{(law)}}{=} \widehat{M}$ if and only if $\beta^M$ and $\langle M\rangle$ are independent; since $\langle M\rangle$ is measurable with respect to $\beta^M$, we get the statement:

$(\mathcal{M}')$ *A pure martingale $M$ has the same law as its Lévy transform $\widehat{M}$ (if and) only if $\langle M\rangle$ is deterministic.*

As will be shown below, $(\mathcal{M}')$ is in fact not weaker than but equivalent to $(\mathcal{M})$.

Similarly, a weaker form of $(\mathcal{L})$ is obtained by restricting to stopping times the statement that all random variables on $W$ invariant by $T$ are constant:

$(\mathcal{L}')$ *On the canonical space $W$, every stopping time invariant by $T$ is constant.*

THEOREM 2. — *The four conjectures $(\mathcal{L})$, $(\mathcal{L}')$, $(\mathcal{M})$ and $(\mathcal{M}')$ are equivalent.*

PROOF. — If $(\mathcal{L})$ is true, $\mathcal{J}$ is trivial and (i) $\Rightarrow$ (iii) in Theorem 1 gives $(\mathcal{M})$. In turn, $(\mathcal{M})$ trivially implies $(\mathcal{M}')$. The theorem will be proved by showing that

$$(\mathcal{L}) \text{ is false} \Longrightarrow (\mathcal{L}') \text{ is false} \Longrightarrow (\mathcal{M}') \text{ is false.}$$

Assume $(\mathcal{L})$ is false. On $W$ endowed with Wiener measure, there exists a non-trivial invariant bounded random variable $F$. Call $B$ the canonical Brownian motion on $W$, $\widehat{B}$ its Lévy transform, $(\mathcal{F}_t)_{t\geqslant 0}$ and $(\widehat{\mathcal{F}}_t)_{t\geqslant 0}$ their natural filtrations. Since $F$ is a functional of $\widehat{B}$, the $(\widehat{\mathcal{F}}_t)$-martingale $M_t = \mathbb{E}[F|\widehat{\mathcal{F}}_t]$ has the form $\mathbb{E}[F] + \int_0^t \widehat{H}_s\,d\widehat{B}_s$ with $\widehat{H}$ predictable in the filtration $(\widehat{\mathcal{F}}_t)_{t\geqslant 0}$; so $M$ is also a martingale for $(\mathcal{F}_t)_{t\geqslant 0}$ and

$$\mathbb{E}[F|\mathcal{F}_t] = \mathbb{E}[F|\widehat{\mathcal{F}}_t] = \mathbb{E}[F\circ T|\widehat{\mathcal{F}}_t] = \mathbb{E}[F|\mathcal{F}_t]\circ T\ :$$

the process $M$ is invariant by $T$.

Using $M$, it is easy to construct a finite, non-constant stopping time $S$ invariant by $T$, for instance $S = t + \mathbb{1}_\Gamma(M_t)$ where $t$ is large enough for $M_t$ to be non-constant and $\Gamma$ is a suitable Borel set. So $(\mathcal{L}')$ is false too.

If $(\mathcal{L}')$ is false, let $S$ be a finite, non-constant, invariant stopping time on Wiener space. For $\alpha > 0$, the increasing process

$$A_t = \int_0^t \left[\mathbb{1}_{[\![0,S]\!]}(s) + \alpha \mathbb{1}_{]\!]S,\infty[\![}(s)\right] ds$$

is not deterministic if $\alpha \neq 1$; it is also invariant, and $(B, A) \stackrel{(\text{law})}{=} (\widehat{B}, A)$; the inverse of $A$, obtained by replacing $\alpha$ with $\alpha^{-1}$, is adapted and each $A_t$ is a stopping time. Consequently, $M_t = B_{A_t}$ is a martingale (for the filtration $\mathcal{G}_t = \mathcal{F}_{A_t}$), satisfying condition (ii) of theorem 1, hence $M \stackrel{(\text{law})}{=} \widehat{M}$. As $\langle M \rangle = A$ is measurable with respect to $\beta^M = B$ but not deterministic, $M$ is a counterexample to $(\mathcal{M}')$ and $(\mathcal{M}')$ does not hold. ∎

## Some Remarks

a) It can be observed that the stopping time $S$ constructed in the first part of the proof takes only two values. This leads to another variant of $(\mathcal{L})$, namely, there are no invariant stopping times taking exactly two values. Of course, this just means that each invariant event belonging to some $\mathcal{F}_t$ is trivial.

b) $(\mathcal{L}')$ amounts to stating that every non constant Brownian stopping time $S$ is not invariant. According to Lemma 2, this means that the stopped processes $B^S$ and $\widehat{B}^S = |B^S| - L^S$ do not have the same law. A sufficient condition is that the random variables $B_S$ and $\widehat{B}_S = |B_S| - L_S$ have different laws. Many stopping times have this property, for instance the first hitting time of a given level by $B$, or by $|B|$ or by $L$...

However, there also exist many stopping times (for the filtration of $B$) such that $B_S$ and $\widehat{B}_S$ are not only identical in law, but a.s. equal; for instance $\inf\{t \geqslant 1 \,:\, B_t = \widehat{B}_t\}$. This stopping time is a.s. finite since the martingale $B - \widehat{B}$ is divergent (for its bracket is $4\int \mathbb{1}_{\{B_t \leqslant 0\}}\, dt$).

But a finite stopping time $S$ such that $B_S = \widehat{B}_S$ cannot be invariant unless it vanishes identically. For in that case $B_S = \widehat{B}_S = \widehat{B}_{S\circ T} = B_S \circ T$ is a function of $\widehat{B}$ only; and, since $B$ and $-B$ have the same conditional law given $\widehat{B}$, $B_S$ must be its own opposite and $B_S = 0$, giving $L_S = |B_S| - \widehat{B}_S = 0$ and $S = 0$. (The same argument shows more generally that a finite, non-negative random variable $S$ measurable for $\widehat{B}$ and verifying $B_S = \phi(\widehat{B}_S)$ must vanish if $\phi$ is a function such that $\phi(x) \neq 0$ for every $x < 0$.)

More precisely, it can be proved that *if $S$ is an invariant, finite, positive stopping time,* $\mathbb{P}[B_S = \widehat{B}_S] \leqslant 1/2$. Let indeed $A$ denote the event $\{B_S = \widehat{B}_S\}$ and suppose $\mathbb{P}[A] > 1/2$. Since $\widehat{B}_S = |B_S| - L_S$ and $L_S > 0$, on $A$ one has $\widehat{B}_S = -B_S - L_S$ and $2B_S = -L_S$; this can be rewritten $2\widehat{B}_S = \inf_{t \leqslant S} \widehat{B}_t$. But $\mathbb{P}[T^{-1}A]$ is also larger than one half, so the event $A \cap T^{-1}A$ is not negligible. On this event, one has on the one hand $2\widehat{B}_S = \widehat{I}_S$ and on the other hand $2\widehat{B}_S = -\widehat{L}_S$

(where $\widehat{I}_t = \inf_{s\leqslant t} \widehat{B}_s$ and $\widehat{L}$ is the local time at zero of $\widehat{B}$). To establish the claim, we shall prove that, if $\widehat{B}$ is a Brownian motion started at 0, the event $\{\exists t>0 : 2\widehat{B}_t = \widehat{I}_t = -\widehat{L}_t\}$ is negligible; replacing $\widehat{B}$ with $-\widehat{B}$, dropping the hats for typographical simplicity and letting $S_t = \sup_{s\leqslant t} B_s$, this amounts to showing that

$$\mathbb{P}[\exists t>0 : 2B_t = S_t = L_t] = 0 .$$

Since both processes $S$ and $L$ are locally constant in the random open set $\{t : B_t \neq S_t \text{ and } B_t \neq 0\}$, if equality $S = L$ holds at some time $t > 0$ such that $B_t = \frac{1}{2}S_t$, it also holds identically in some neighborhood of $t$ and hence at some rational $t$; so we just have to show that $\mathbb{P}[S_t = L_t] = 0$ for each $t > 0$.

From the scaling property of Brownian motion, this probability does not depend on $t$; hence it is also equal to $\mathbb{P}[S_T = L_T]$ where $T$ is an exponential random variable independent of $B$. Now, the joint law of $(S_T, L_T)$ is easily computed from excursion theory arguments; in particular it is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^2_+$, with identical exponential marginals. This proves the claim.

c) Still working in the $\sigma$-field generated by $B$, notice that $(\mathcal{L})$ is true if and only if, for $H$ ranging over a total subset of $L^2$, $\mathbb{E}[H|\mathcal{J}] = \mathbb{E}[H]$, or equivalently by the ergodic theorem

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} H \circ T^k = \mathbb{E}[H] .$$

We shall use the subset consisting of the constant 1 (for which the above equality is trivial) and of all multiple Wiener integrals

$$H = \int_0^\infty dB_{u_1} \int_0^{u_1} dB_{u_2} \dots \int_0^{u_{p-1}} dB_{u_p}\, f(u_1, \dots, u_p)$$

where $p \geqslant 1$ and $f$ satisfies

$$\int_0^\infty du_1 \int_0^{u_1} du_2 \dots \int_0^{u_{p-1}} du_p\, f^2(u_1, \dots, u_p) < \infty .$$

This programme can be carried out successfully in the case $p = 1$. Indeed, writing $B^{(k)} = T^k(B)$ and $\varepsilon^{(k)}_t = \operatorname{sgn} B^{(k)}_t$, Tanaka's formula gives

$$B^{(k)}_t = \int_0^t dB^{(k-1)}_s\, \varepsilon^{(k-1)}_s = \int_0^t dB^{(k-2)}_s\, \varepsilon^{(k-2)}_s \varepsilon^{(k-1)}_s = \int_0^t dB_s\, \varepsilon_s \varepsilon^{(1)}_s \dots \varepsilon^{(k-1)}_s .$$

Now, if $H = \int_0^\infty dB_u\, f(u)$, where $f \in L^2$, we have

$$\frac{1}{n} \sum_{k=1}^{n} H \circ T^k = \int_0^\infty dB_u\, f(u) \Big(\frac{1}{n} \sum_{k=1}^{n} \varepsilon_u \varepsilon^{(1)}_u \dots \varepsilon^{(k-1)}_u\Big)$$

so that

$$\mathbb{E}\Big[\Big(\frac{1}{n} \sum_{k=1}^{n} H \circ T^k\Big)^2\Big] = \int_0^\infty du\, f^2(u)\, \mathbb{E}\Big[\Big(\frac{1}{n} \sum_{k=1}^{n} \varepsilon_u \varepsilon^{(1)}_u \dots \varepsilon^{(k-1)}_u\Big)^2\Big] .$$

For fixed $u$, $\varepsilon_u^{(\ell)}$ is independent of $B^{(\ell+1)}$ and hence also of all the $\varepsilon_u^{(m)}$ for $m > \ell$; so the sequence $\varepsilon_u, \varepsilon_u^{(1)}, \varepsilon_u^{(2)}, \dots$ of Bernoulli variables is independent, whence

$$\mathbb{E}\Big[\Big(\frac{1}{n}\sum_{k=1}^{n}\varepsilon_u\varepsilon_u^{(1)}\dots\varepsilon_u^{(k-1)}\Big)^2\Big] = \frac{1}{n},$$

and $\displaystyle\frac{1}{n}\sum_{k=1}^{n} H\circ T^k$ does tend to zero.

For random variables $H$ belonging to chaoses of higher order, the same method, and the well-known isometry between the $p^{\text{th}}$ chaos and the space of square-integrable symmetric functions of $p$ variables, reduce $(\mathcal{L})$ to an equivalent property.

PROPOSITION. — *$(\mathcal{L})$ is true if and only if for each $p > 1$*

$$\lim_{n\to\infty}\int_0^1 du_1\int_0^{u_1} du_2\dots\int_0^{u_{p-1}} du_p\ \mathbb{E}\Big[\Big(\frac{1}{n}\sum_{k=1}^{n}\prod_{\substack{0\leqslant\ell<k\\ 1\leqslant m\leqslant p}}\varepsilon_{u_m}^{(\ell)}\Big)^2\Big] = 0.$$

Hence, the first step in that direction would be to take $p = 2$ and to get a good estimate of

$$\mathbb{E}\Big[\Big(\frac{1}{n}\sum_{k=1}^{n}\varepsilon_u\varepsilon_v\varepsilon_u^{(1)}\varepsilon_v^{(1)}\dots\varepsilon_u^{(k-1)}\varepsilon_v^{(k-1)}\Big)^2\Big].$$

## APPENDIX: Ocone's Theorem

Ocone's Theorem A of [5] consists of the equivalence between (ii), (iii) and (iv) below. His setting is more general than the following rephrasing: he deals with local martingales (and further extends his results to the càdlàg case).

THEOREM. — *Let $M$ be a continuous, divergent martingale with natural filtration $\mathcal{F} = (\mathcal{F}_t)_{t\geqslant 0}$. The following five statements are equivalent:*

(i) *the processes $\beta^M$ and $\langle M\rangle$ are independent;*

(ii) *for every $\mathcal{F}$-predictable process $H$ taking values in $\{-1, 1\}$, the pairs of processes $(\int H\,dM, \langle M\rangle)$ and $(M, \langle M\rangle)$ have the same law (in particular the martingales $\int H\,dM$ and $M$ have the same law);*

(iii) *for every deterministic function $h$ of the form $\mathbb{1}_{[0,a]} - \mathbb{1}_{(a,\infty)}$, the martingale $\int h\,dM$ has the same law as $M$;*

(iv) *for every $\mathcal{F}$-predictable process $H$ measurable for the product $\sigma$-field $\mathcal{B}(\mathbb{R}_+)\otimes\sigma(\langle M\rangle)$ and such that $\int_0^\infty H_s^2\,d\langle M\rangle_s < \infty$ a. s.,*

$$\mathbb{E}\Big[\exp\Big(i\int_0^\infty H_s\,dM_s\Big)\ \Big|\ \langle M\rangle\Big] = \exp\Big(-\tfrac{1}{2}\int_0^\infty H_s^2\,d\langle M\rangle_s\Big);$$

(v) *for every deterministic function $h$ of the form $\sum_{j=1}^n \lambda_j\,\mathbb{1}_{[0,a_j]}$,*

$$\mathbb{E}\Big[\exp\Big(i\int_0^\infty h(s)\,dM_s\Big)\Big] = \mathbb{E}\Big[\exp\Big(-\tfrac{1}{2}\int_0^\infty h^2(s)\,d\langle M\rangle_s\Big)\Big].$$

Notice that (iii) is a symmetry assumption: given the past of $M$, the conditional law of the future increments is symmetric; (iv) says that, conditionally given $\langle M\rangle$, $M$ is a Gaussian martingale with variance $\langle M\rangle$.

LEMMA 1. — *If a right-continuous process $X$ is a martingale for some (non necessarily right-continuous) filtration $(\mathcal{G}_t)_{t\geqslant 0}$, it is also a martingale for its right-continuous enlargment $(\mathcal{G}_{t+})_{t\geqslant 0}$.*

PROOF. — When $\varepsilon > 0$ tends to 0, $X_{s+\varepsilon}$ tends to $X_s$ in $\mathrm{L}^1$ by uniform integrability; so, for $s < t$, $\mathbb{E}[X_t - X_s|\mathcal{G}_{s+}] = \lim\limits_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \mathbb{E}[X_t - X_{s+\varepsilon}|\mathcal{G}_{s+}] = 0.$ ∎

LEMMA 2. — *Let $\beta$ be a Brownian motion with natural filtration $\mathcal{B}$ and $\mathcal{G}$ a $\sigma$-field independent of $\beta$. If a process $H$ taking values in $\{-1,1\}$ is predictable for the filtration $\mathcal{E}$ defined by $\mathcal{E}_t = \bigcap_{\varepsilon>0}(\mathcal{B}_{t+\varepsilon}\vee\mathcal{G})$, the process $\int H\,d\beta$ is a Brownian motion independent of $\mathcal{G}$.*

PROOF. — Lemma 1 and the independence of $\beta$ and $\mathcal{G}$ imply that $\beta$ is a $\mathcal{E}$-Brownian motion. By Lévy's characterization, $\int H\,d\beta$ is also a $\mathcal{E}$-Brownian motion; so it is independent of $\mathcal{E}_0$, hence of $\mathcal{G}$. ∎

PROOF OF OCONE'S THEOREM. — We shall show that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i). Implications (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v) are trivial.

(i) $\Rightarrow$ (ii) and (iv). We suppose that $\beta = \beta^M$ and $\langle M\rangle$ are independent. Let $A_t = \inf\{s : \langle M\rangle_s > t\}$, so that, with the convention $A_{0-} = 0$, its left-limit is $A_{t-} = \inf\{s : \langle M\rangle_s \geqslant t\}$; denote by $\mathcal{B}$ the natural filtration of $\beta$ and by $\mathcal{E}$ the right-continuous enlargement of the filtration $\mathcal{B}_t\vee\sigma(\langle M\rangle)$. If $T$ is a $\mathcal{F}$-stopping time, $\langle M\rangle_T$ is a $\mathcal{E}$-stopping time since

$$\{\langle M\rangle_T \leqslant t\} = \{T \leqslant A_t\} \in \mathcal{F}_{A_t} \subset \bigcap_{\varepsilon>0}\sigma(M^{A_{t+\varepsilon}}) \subset \bigcap_{\varepsilon>0}[\mathcal{B}_{t+\varepsilon}\vee\sigma(\langle M\rangle)] = \mathcal{E}_t\,.$$

If $H$ is a bounded, $\mathcal{F}$-predictable process, $K_t = H_{A_{t-}}$ is bounded and $\mathcal{E}$-predictable and $\int_0^t H_s\,dM_s = \int_0^{\langle M\rangle_t} K_u\,d\beta_u$ (when $H = \mathbb{1}_{[\![0,T]\!]}$ with $T$ a $\mathcal{F}$-stopping time, $K = \mathbb{1}_{[\![0,\langle M\rangle_T]\!]}$ and both integrals agree since $M_{T\wedge t} = \beta_{\langle M\rangle_{T\wedge t}}$; the general case follows by a monotone class argument).

If furthermore $H$ takes values in $\{-1,1\}$, Lemma 2 with $\mathcal{G} = \sigma(\langle M\rangle)$ says that $\gamma = \int K\,d\beta$ is a Brownian motion independent of $\langle M\rangle$ (as is $\beta$). Consequently, both processes $M = \beta_{\langle M\rangle}$ and $\int H\,dM = \gamma_{\langle M\rangle}$ have the same conditional law given $\langle M\rangle$; this proves (ii).

Taking now $H$ bounded, $\mathcal{F}$-predictable, $[\mathcal{B}(\mathbb{R}_+)\otimes\sigma(\langle M\rangle)]$-measurable and such that $\int_0^\infty H_s^2\,d\langle M\rangle_s < \infty$, $K$ is also $[\mathcal{B}(\mathbb{R}_+)\otimes\sigma(\langle M\rangle)]$-measurable,

$\int_0^\infty K_s^2\,ds < \infty$ and

$$\mathbb{E}\Big[\exp\Big(i\int_0^\infty H_s\,dM_s\Big)\,\Big|\,\langle M\rangle\Big] = \mathbb{E}\Big[\exp\Big(i\int_0^\infty K_s\,d\beta_s\Big)\,\Big|\,\langle M\rangle\Big]$$
$$= \exp\Big(-\tfrac{1}{2}\int_0^\infty K_s^2\,ds\Big) = \exp\Big(-\tfrac{1}{2}\int_0^\infty H_{A_{s-}}^2\,ds\Big)$$
$$= \exp\Big(-\tfrac{1}{2}\int_0^\infty H_s^2\,d\langle M\rangle_s\Big)\,.$$

This proves (iv) when $H$ is bounded; the general case follows by taking limits.

(iii) $\Rightarrow$ (i). Let $s < t$. Since $M' = \int \big(\mathbb{1}_{[0,s]} - \mathbb{1}_{(s,\infty)}\big)\,dM$ has the same law as $M$, the triple $(M^s, \langle M\rangle, M_t - M_s)$ has the same law as $(M'^s, \langle M'\rangle, M'_t - M'_s)$. But the stopped processes $M^s$ and $M'^s$ are the same, the quadratic variations $\langle M\rangle$ and $\langle M'\rangle$ are equal and $M'_t - M'_s = -(M_t - M_s)$, yielding

$$(M^s, \langle M\rangle, M_t - M_s) \overset{\text{(law)}}{=} (M^s, \langle M\rangle, -(M_t - M_s))\,.$$

Denoting by $\mathcal{G}_s$ the $\sigma$-field generated by the processes $M^s$ and $\langle M\rangle$ and the null events, this implies that $M$ is a martingale for the filtration $\mathcal{G}$, whence also for its right-continuous enlargement $\mathcal{H}$ (Lemma 1).

The random variables $A_t = \inf\{s : \langle M\rangle_s > t\}$ are stopping times for the filtration $\mathcal{H}$ (they are $\mathcal{H}_0$-measurable!); the stopped martingales $M^{A_t}$ are square-integrable and one has $\langle M\rangle_{A_t} = t$. Introducing the filtration $\mathcal{K}_t = \mathcal{H}_{A_t}$ and the Brownian motion $\beta = \beta^M$, one can write

$$\mathbb{E}[\beta_t - \beta_s | \mathcal{K}_s] = \mathbb{E}[\beta_{\langle M\rangle_{A_t}} - \beta_{\langle M\rangle_{A_s}} | \mathcal{H}_{A_s}] = \mathbb{E}[M_{A_t} - M_{A_s} | \mathcal{H}_{A_s}] = 0\,.$$

Consequently $\beta$ is a $\mathcal{K}$-martingale, hence (Lévy's characterization) a $\mathcal{K}$-Brownian motion; so it is independent of $\mathcal{K}_0$, and a fortiori of $\mathcal{G}_0 = \sigma(\langle M\rangle)$.

(v) $\Rightarrow$ (i). Let $B$ be a Brownian motion independent of $\langle M\rangle$. Applying (i) $\Rightarrow$ (v) to the martingale $N = B_{\langle M\rangle}$ and remarking that, since $\langle M\rangle = \langle N\rangle$, the right-hand side of (v) is the same for $M$ and $N$, we see that $M$ and $N$ have the same law. Consequently $\big(\beta^M, \langle M\rangle\big)$ and $\big(\beta^N, \langle N\rangle\big) = \big(B, \langle M\rangle\big)$ also have the same law and $\beta^M$ is independent of $\langle M\rangle$. ∎

REMARKS. — a) The hypotheses that $\mathcal{B}$ is the natural filtration of $\beta$ in Lemma 2 and $\mathcal{F}$ that of $M$ in (ii) cannot be dropped.

If one supposes only that $\mathcal{B}$ is a filtration such that $\beta$ is a $\mathcal{B}$-Brownian motion, Lemma 2 becomes false: Take a Brownian motion $B$ with natural filtration $\mathcal{B}$, call $\beta = \int \operatorname{sgn}(B)\,dB$ the Lévy transform of $B$ and $\mathcal{G}$ the $\sigma$-field generated by $\operatorname{sgn}(B_1)$. Now $H = \operatorname{sgn}(B)$ is $\mathcal{B}$-predictable and a fortiori $\mathcal{E}$-predictable, where $\mathcal{E}_t = \bigcap_{\varepsilon>0}(\mathcal{B}_{t+\varepsilon} \vee \mathcal{G})$. But $\int H\,d\beta = B$ is certainly not independent of $\operatorname{sgn}(B_1)$. (What makes this example work is that for $t < 1$ both random variables $\operatorname{sgn}(B_t)$ and $\operatorname{sgn}(B_1)$ are independent of $\beta$, but the pair $(\operatorname{sgn}(B_t), \operatorname{sgn}(B_1))$ is not.)

Similarly, the theorem becomes false if $\mathcal{F}$ is no longer the natural filtration of $M$, but only some filtration for which $M$ is a martingale. In that case, (i), (iii), (iv) and (v) are still equivalent, but (ii) may become stronger, as shown by the following example. Take as above $B$ with natural filtration $\mathcal{B}$ and Lévy transform $\beta$; define an increasing process $A$ independent of $\beta$ by $A_t = t$ for $t \leqslant 1$ and $A_t = 1 + [u\mathbb{1}_{\{B_1>0\}} + v\mathbb{1}_{\{B_1\leqslant 0\}}](t-1)$ for $t > 1$, where $u$ and $v$ are strictly positive real numbers. Our martingale verifying (i) will be the Lévy transform $M_t = \beta_{A_t}$ of $B_{A_t}$; as the latter is a martingale for the filtration $\mathcal{F}_t = \mathcal{B}_{A_t}$, so is also $M$. The process

$$H_t = \begin{cases} \operatorname{sgn}(B_t) & \text{if } t \leqslant 1 \\ \operatorname{sgn}(B_1)\operatorname{sgn}(B_{A_t}) & \text{if } t > 1 \end{cases}$$

is $\mathcal{F}$-predictable, but the random variables

$$M_2 = \begin{cases} \beta_{1+u} & \text{if } B_1 > 0 \\ \beta_{1+v} & \text{if } B_1 \leqslant 0 \end{cases} \quad \text{and} \quad \int_0^2 H_s\,dM_s = \begin{cases} B_{1+u} & \text{if } B_1 > 0 \\ 2B_1 - B_{1+v} & \text{if } B_1 \leqslant 0 \end{cases}$$

do not have the same law in general. Indeed, on the one hand the law of $M_2$ is symmetric ($\beta$ is independent of $\operatorname{sgn}(B_1)$) and on the other hand, if $u$ is chosen large and $v$ small, $\mathbb{P}[B_{1+u} > 0 \text{ and } B_1 > 0]$ is close to 1/4 and $\mathbb{P}[2B_1 - B_{1+v} > 0 \text{ and } B_1 \leqslant 0]$ to 0, yielding by addition

$$\mathbb{P}\big[\textstyle\int_0^2 H_s\,dM_s > 0\big] \approx \frac{1}{4} \neq \frac{1}{2} = \mathbb{P}[M_2 > 0]\ .$$

This shows that the filtration $\mathcal{F}$ is too large for (ii) to hold.

b) *If $M$ is a martingale on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ for a filtration $\mathcal{F} = (\mathcal{F}_t)_{t\geqslant 0}$ and if $\mathcal{C}$ is a sub-$\sigma$-field of $\mathcal{A}$, then $M$ is still a martingale for the enlarged filtration $(\mathcal{F}_t \vee \mathcal{C})$ if and only if $\mathbb{E}[\int_0^t H_s\,dM_s|\mathcal{C}] = 0$ for each $t$ and each simple, $\mathcal{F}$-predictable process $H$ verifying $|H| = 1$.* Indeed, if $M$ is a martingale for $(\mathcal{F}_t \vee \mathcal{C})$, so is also $\int H\,dM$, yielding $\mathbb{E}[\int_0^t H_s\,dM_s|\mathcal{F}_0 \vee \mathcal{C}] = 0$. Conversely, if the condition holds, $\mathbb{E}\big[(M_t - M_s)U\mathbb{1}_C\big] = 0$ for each $C \in \mathcal{C}$ and each $\mathcal{F}_s$-measurable $U$ with values in $\{-1, 1\}$; but for $A \in \mathcal{F}_s$, $2\mathbb{1}_A = (\mathbb{1}_A - \mathbb{1}_{A^c}) + 1$ is the sum of two such $U$'s, whence $\mathbb{E}\big[(M_t - M_s)\mathbb{1}_A\mathbb{1}_C\big] = 0$, and $M$ is a martingale for the large filtration.

c) *If $\mathcal{F} = (\mathcal{F}_t)_{t\geqslant 0}$ is the natural filtration of some Brownian motion $B$ and if $\mathcal{C}$ is a non-trivial sub-$\sigma$-field of $\bigvee_t \mathcal{F}_t$, no $\mathcal{F}$-Brownian motion can be a martingale for $(\mathcal{F}_t \vee \mathcal{C})$.* For such a Brownian motion $\beta$ would have the form $\int H\,dB$ for an $\mathcal{F}$-predictable $H$ with $|H| = 1$; so $B = \int H\,d\beta$ would also be a $(\mathcal{F}_t \vee \mathcal{C})$-martingale, hence a $(\mathcal{F}_t \vee \mathcal{C})$-Brownian motion and would be independent of $\mathcal{F}_0 \vee \mathcal{C} = \mathcal{C}$.

d) Yet, keeping the notations of c), *there exist a non-trivial sub-$\sigma$-field $\mathcal{C}$ of $\bigvee_t \mathcal{F}_t$ and a process that is both a $(\mathcal{F}_t)$- and a $(\mathcal{F}_t \vee \mathcal{C})$-martingale,* for instance the $\sigma$-field $\mathcal{C} = \mathcal{F}_1$ and the process $\int h\,dB$, with $h = \mathbb{1}_{[1,\infty)}$. This example generalizes as follows: Let $A$ be a $(\mathcal{F}_t)$-predictable set and assume, for simplicity, that (almost) all sections $A^c(\omega)$ of its complementary have infinite Lebesgue measure. Let $M = \int \mathbb{1}_A \, dB$, $N = \int \mathbb{1}_{A^c} \, dB$ and denote by $\mathcal{C}$ the $\sigma$-field generated by the Brownian motion $\beta^N$. Time-changing by $\langle N \rangle = \int \mathbb{1}_{A^c} \, dt$ the predictable representation property with respect to $\beta^N$ shows that every square-integrable, $\mathcal{C}$-measurable random variable assumes the form

$$U = \mathbb{E}[U] + \int_0^\infty K_s \mathbb{1}_{A^c}(s) \, dB_s \, ,$$

where $K$ is predictable and such that $\mathbb{E}\big[\int_0^\infty K_s^2 \mathbb{1}_{A^c}(s) \, ds\big] < \infty$. This easily implies that $M = \int \mathbb{1}_A \, dB$ satisfies the condition in b) above, showing that $M$ is a $(\mathcal{F}_t \vee \mathcal{C})$-martingale.

It seems worthwhile to present such examples here as they play an important rôle in some martingale proofs of the Ray-Knight theorems for Brownian local times (see, for instance, Exercises (2.8) and (2.9) pages 426–427 of Revuz & Yor [6] and Jeulin [4]).

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