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EVGENII I. TROFIMOV

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SOBOLEV TOPOLOGIES
IN SEMIMARTINGALE THEORY

E.I.Trofimov

Introduction

This article is a short presentation of the author's studies on construction of semimartingale distributions via topologies of the fractional Sobolev- Slobodetskii hilbertian spaces $\mathbb{H}^s_2$ and some others of the kind (see [T1], for instance).

The $\mathbb{H}^s$-topologies are considerably finer than the pseudo-paths topology Λ (that of convergence in measure) used by P.A.Meyer and W.A.Zheng [MeZ] and Ch.Stricker [St]. We give here some new conditions, being weaker than those from [MeZ] and [St], to insure tightness of semimartingale sequences in $\mathbb{H}^s$. We prove equivalence of the tightness conditions from [MeZ], [St], and [T1], under which every limit point of a tight sequence of semimartingales is a semimartingale law.

Sobolev spaces

Throughout all functions and processes are defined on the time interval $[0,1]$. We denote by $\mathbb{D}$ the usual space of functions, which are right continuous with left limits in this interval. For $0<s<1$ let $\mathcal{X}^s$ be the completion of the space of $\mathbb{C}^\infty$ functions in $[0,1]$ with respect to the standard Sobolev-Slobodetskii norm $\|\cdot|\mathbb{H}^s\|$ (see the well known books of Lions-Magenes (1967), Besov-Il'in-Nokolski (1975), Triebel (1986), for instance) defined by

$$(1)\qquad \|x|\mathbb{H}^s\|^2 = \|x\|^2_{L^2} + \int_0^1\!\!\int_0^1 \frac{|x(u) - x(v)|^2}{|u - v|^{1+2s}}\lambda(du)\lambda(dv)$$

where λ is the Lebesgue measure on $(o,1)$. Then $\mathbb{H}^s$ is the space of equivalence classes of functions w.r.to λ. Let τ^s be the norm topology in $\mathbb{H}^s$.

It is a general fact about these spaces that for $s < t$, $\mathbb{H}^t$ is contained in $\mathbb{H}^s$, and the embedding operator is compact. This property is convenient in the considerations below about probability distributions in the scale $(\mathbb{H}^s)$.

Preliminaries

Let $\mathbb{D}\mathbb{H}^s \subset \mathbb{D}$ be the subset of $\mathbb{D}$ consisting of functions admitting a finite $\mathbb{H}^s$-norm. Since two elements of $\mathbb{D}$ equal a.e. are equal everywhere, one may consider $\mathbb{D}\mathbb{H}^s$ as a subspace of $\mathbb{H}^s$. The limiting case $\mathbb{D}\mathbb{H}^o$ of $\mathbb{D}\mathbb{H}^s$ is simply the set $\mathbb{D}$, with the topology induced from $L^2(\lambda)$. We use notations of topologies also for their traces into function sub-classes. The reader is reffered to the author's works for details on construction of modified Sobolev-Slobodetski norms, which seem somewhat more convenient in theory of stochastic processes.

On the set $\mathbb{D}\mathbb{H}^s$, with $s > 0$, the topology of $\mathbb{H}^s$ isn't comparable with either the uniform topology, or that of Skorokhod. It is easy to see that there exist functions in $\mathbb{H}^s$ with infinite and oscillating discontinuites, and there exist continuous functions which do not belong to $\mathbb{H}^s$. On the other hand, these norms somehow measure the "non-Hölderness" of a function $x(\cdot)$.

We restrict our study to the interval $0 < s < 1/2$ for the following reasons. First, using the definition of the $\mathbb{H}^s$ norm and the Fubini-Tonelli theorem, we may calculate the mean square norm $\mathbb{E}(\|W.|\mathbb{H}^s\|^2)$ of the standard Wiener process (W_t) on $[o,1]$ and obtain quite easy that it is finite for $s < 1/2$ and infinite for $s \geq 1/2$. The same result can be deduced from Lévy's modulus of continuity for the Wiener process. Similarly, the $\mathbb{H}^s$ norm of the indicator function of an open interval from $[0,1]$ is finite for $s < 1/2$ and infinite for $s \geq 1/2$, and this applies also to the paths of the standard Poisson process.

Sobolev norms and p-variations

To estimate the $\mathbb{H}^s$ norm of semimartingale paths, we need to recall some inequalities. For $x \in \mathbb{D}$, $p \in [1,\infty[$, and finite subdivisions $\pi = \{ 0 = t_0 < \ldots < t_n = 1 \}$ of the interval $[0,1]$, put

$$\mathrm{Var}_p^{\pi}(x) = \Big[\sum_i |x(t_{i+1}) - x(t_i)|^p\Big]^{1/p},$$

and

$$\mathrm{Var}_p(x) = \sup_\pi \mathrm{Var}_p^{\pi}(x) \; ;$$

and let $\mathbb{V}_p$ be the subspace of $\mathbb{D}$ consisting of functions of finite p-variation. It is easy to see that $\mathrm{Var}_p(x) \geq \mathrm{Var}_q(x)$ for $p < q$, and the embedding $\mathbb{V}_p \subset \mathbb{V}_q$ is strict.

The following inequality of Hirschmann, known in function theory, is of importance below (see [T1] for a generalization): for $x \in \mathbb{D}$, $p \geq 1$ and $\theta \in \mathbb{R}$

$$\int_{(0,1)} dt \; |x(t+\theta) - x(t)|^p \;\; \leq \;\; 4\,|\theta|\,(\mathrm{Var}_p(x))^p$$

Using this inequality and the estimate (for $p > 2$)

$$\int_{(0,1)} dt \; |x(t+\theta) - x(t)|^2 \;\; \leq \;\; \Big[\int_{(0,1)} dt \; |x(t+\theta) - x(t)|^p\Big]^{2/p},$$

for $ps < 1$ we obtain that,

$$\|x|\mathbb{H}^s\| \leq c_{p,s}\,\mathrm{Var}_p(x). \tag{2}$$

Thus space $\mathbb{V}_p$ endowed with the p-variation norm is continuously embedded in $\mathbb{H}^s$ for $ps < 1$, and moreover it is a compact embedding, by (2) and the compactness of those in the scale $(\mathbb{H}^s)$.

While the $\mathbb{H}^s$ norm of the paths of the drift process of a semimartingale can be estimated by (2), its local martingale part requires an additional inequality. A first estimate for the norms of martingales can be deduced simply from the Fubini-Tonelli theorem, as above for the Wiener process.

D. Lepingle has proved [Lep] that the paths of a martingale M a.s. have finite p-variation for $p > 2$, and he gave the estimate:

$$(3) \qquad \mathbb{E}[(\mathrm{Var}_p(M))^q] \le C_{pq}\, \mathbb{E}[(M_t^*)^q],$$

for every $q \ge 1$, where $M_t^* \equiv \sup_{s \le t} |M_s^n|$.

(2,3) together imply an estimate of the mean $\mathbb{H}^s$ norm of a martingale, and an estimate in probability of the norm of a local martingale. Furthermore, by [Lep] and (2) we have that:

almost all semimartingale paths belong to the function classes $\mathbb{DH}^s$, *for all* $s < 1/2$, *and to the three following ones*

$$\mathbb{H}^{1/2-} \equiv \bigcap_{s<1/2} \mathbb{H}^s, \quad \mathbb{DH}^{1/2-} \equiv \bigcap_{s<1/2} \mathbb{DH}^s, \quad \mathbb{V}_{2+} \equiv \bigcap_{p>2} \mathbb{V}_p .$$

In the framework of the Sobolev distribution model, it is natural to consider semimartingale distributions in the latter three, smaller support-spaces, with a Sobolev countably normed topology, which we denote $\tau^{1/2-}$. Similarly we define a topology τ^{u-}, for any $u > 0$.

Measurability.

It was shown in [T1] that the Borel σ-algebra in $(\mathbb{DH}^s, \tau^s)$ is equal to the usual cylindrical (*i.e.* coordinate) σ-algebra in $\mathbb{DH}^s$, and thus to the Borel σ-algebra of the Skorokhod topology, and to that of Λ, the topology of convergence in measure (see [MeZ]). Let us notice here, that the cylindrical measurability properties in the countably normed support-spaces defined above are simply verifiable.

Indeed, any set from $\mathbb{V}_p$, closed and bounded in $\mathrm{Var}_p(\cdot)$, is closed in the topology Λ, since the p-variation functional is lower-semicontinuous in this topology. The latter property is a simple extension of the result from [MeZ] for the case $p=1$. Hence by (2) and the compactness of embeddings in the Sobolev scale this set is compact in τ^s, for any $s<1/p$.

Then the measurability properties follow from the remarks:

1) the set $\mathbb{V}_{2+}$ is itself borelian in the spaces $\mathbb{H}^{1/2-}$ and $\mathbb{DH}^{1/2-}$ w.r. to the topology $\tau^{1/2-}$.

2) Since $\Lambda=\tau^s$, when restricted to a bounded and closed set from $\mathbb{V}_p$ w.r. to the norm $\mathrm{Var}_p(\cdot)$ for $ps<1$, we have that $\sigma(\mathbb{V}_{2+},\tau^{1/2-})$ is the restriction of the cylindrical σ-algebra to $\mathbb{V}_{2+}$. Here comes our first theorem.

Theorem 1. *For any $s\in[0,1/2[$, every semimartingale (X_t) defines random elements with values in the spaces $(\mathbb{DH}^s,\tau^s)$, $(\mathbb{H}^{1/2-},\tau^{1/2-})$, $(\mathbb{DH}^{1/2-},\tau^{1/2-})$, and $(\mathbb{V}_{2+},\tau^{1/2-})$.*

Tightness.

Meyer, Zheng and Stricker had given [MeZ],[St] simple conditions under which a sequence of semimartingales is Λ-tight in $\mathbb{D}$, and each of its limit laws P also defines a semimartingale into the canonical space $\mathbb{D}$. First we show that weaker conditions imply tightness w.r.to the $\tau^{1/2-}$ topologies. Throughout we do not give details whenever standard martingale machinery is used.

Let (P_n) be a sequence of probability laws on $\mathbb{D}$ such that the coordinate process $(X_t)_{[0,1]}$ is a P_n-semimartingale, for all $n\geq 1$. As usual we consider the canonical space $\mathbb{D}$, with its natural, right continuous filtration $(F_t)_{[0,1]}$ of σ-algebras; X^n denotes the process X on the probability base $\beta_n=(\mathbb{D},F_1,(F_t),P_n)$; and we assume it to be completed as usual. We consider decompositions $X = V^n + M^n$ on this base into a process V^n of finite 1-variation and a local martingale M^n, with $M^n_0 \equiv 0$.

Let us introduce a condition implying the tightness of the sequence P_n (here $p \geq 1$).

(A_p^{loc}) *For any $\varepsilon > 0$ there exist decompositions $X^n = V^n + M^n$ and β_n-stopping times S_n such that* $\sup_n P_n\{ S_n < 1 \} < \varepsilon$, and

$$E = \sup_n \mathbb{E}\left[(M^n)^*_{S_n} \right] < \infty ,$$

$$\sup_n P_n\{ Var_p(V^n)_{S_n} > a \} < \varepsilon ,$$

for sufficiently large a, where $Var_p(\cdot)_t$ denotes the p-variation taken upon the time interval $[0,1]$.

This is our first tightness result. Note that it does not imply that the limit laws are semimartingale laws. This will be the object of the stronger condition (A^r) below and theorem 3.

Theorem 2. *(i) The sequence P_n is tight in the space $(\mathbb{V}_p, \tau^{1/p-})$ for $p > 2$, if it fulfills the condition* (A_q^{loc}) *for* some $q \leq p$.

(ii) The sequence P_n is tight in $(\mathbb{V}_{2+}, \tau^{1/2-})$, if it fulfills the condition (A_q^{loc}) *for* all $q > 2$.

Note. As $\mathrm{Var}_q(x) \geq \mathrm{Var}_p(x)$ for $q < p$, we have: $(A_q^{loc}) \Rightarrow (A_p^{loc})$; the assumption $p > 2$ is necessary to handle the semimartingale laws of general kind, and the theorem allows further precisiions in particular situations.

Proof. Fix some $s < 1/p$. Since the embedding $\mathbb{V}_p \subset \mathbb{H}^s$ is compact, condition (A_q^{loc}) entails tightness of the family V^n in $(\mathbb{V}_p, \tau^s)$. For the martingale parts M^n the tightness property follows from condition (A_p^{loc}) and the estimates (2.3) :

$$P_n\{ \| M^n | \mathbb{H}^u \| > b \} < \varepsilon + cE/b < 2\varepsilon$$

for $u \in (s, 1/p)$ and sufficiently large b. The tightness of the sums $V^n + M^n$ follows by linearity of the topologies τ^s.

To extend the result onto the spaces $(\mathbb{V}_p, \tau^{1/p-})$ and $(\mathbb{V}_{2+}, \tau^{1/2-})$, let us note that the above considerations entail: for any $\delta > 0$ and sequences $s(i) \nearrow 1/p$, $q(i) \searrow p$ (resp., $q(i) \searrow 2$), with $s(i)q(i) < 1$, there exists a sequence $\alpha(i) \nearrow \infty$ sufficiently fast, such that

$$\sup_n P_n \left\{ Var_{q(i)}(X^n) > \alpha(i), \text{ for some } i \right\} < \delta .$$

Then the $\tau^{1/p-}$ (resp., $\tau^{1/2-}$) tightness follows, because δ is arbitrarily small, the set $C = \{ x : Var_{q(i)}(x) \le \alpha(i), \ for \ i \ge 1 \}$ is compact in $\mathbb{V}_p$ (resp., $\mathbb{V}_{2+}$) w.r.to the topology $\tau^{1/p-}$ (resp., $\tau^{1/2-}$), and

$$\sup_n P_n\{ X^n \in (\mathbb{V}_p - C) \} < \delta \qquad \square$$

Remark. The condition (A_p^{loc}), with $p > 1$, may be of use when sequences of semimartingales with asymptotically oscillating drifts, or processes of finite quadratic energy are under consideration.

To consider tightness conditions implying the semimartingale property of all limit laws we modify condition (A_1^{loc}) to make it stronger as follows (A^r). From now on, we suppose $X_0^n \equiv 0$, $n \ge 1$, for the sake of simplicity.

Given $r \in [1, \infty]$, and a sequencce of decompositions $X^n = V^n + M^n$, we put

$$(4) \qquad J_r = \sup_n \sup_{S \in \mathcal{T}} \| \Delta M_S^n \|_r < \infty ,$$

where $\mathcal{T}$ is the set of all (F_t)-stopping times.

(A^r) *There exist decompositions* $X^n = V^n + M^n$, *for which* $J_r < \infty$, *such that*

$$\sup_n P_n\{ Var_1(V^n) + [M^n, M^n]_1 > a\} \to 0 \quad as\ a \to \infty$$

(*i.e.*, the family of the r.v.s $Var_1(V^n) + [M^n, M^n]_1$, $n \geq 1$, is *stochasticaly bounded*).

R e m a r k. Obviously, ($A^{r'}$) $\Rightarrow$ (A^1) $\Rightarrow$ (A_1^{loc}) for $r' > r$. Condition (A^r) remains essentially the same, if the brackets $[M^n, M^n]_1$ are replaced respectively by the r.v.s $(M^n)_1^*$, or the predictable brackets $\langle M^n \rangle_1$ when $r \geq 2$. Indeed, by the Davis' inequality, for some constant c, each from the processes $(M^n)^*$, $c\,[M^n, M^n]^{1/2}$ (put Y^n for it) dominates the other (denote it Z^n) in the Lenglart sense. Since the quantity (4) $J_1 < \infty$, one has by the Lenglart-Rebolledo (L-R) domination inequality:

$$P_n\{ Y_1^n > a\} < (J_1 + b)/a + P_n\{ Z_1^n > b\}, \tag{5}$$

i.e., the family of the *r.v.s* Y_1^n is stochasticaly bounded, if the family Z_1^n is.

We now recall the two tightness conditions from [MeZ] and [St], denoted here (B) and (C), respectively.

(B) *For any* $\varepsilon > 0$ *there exist* P_n*-quasimartingales* Y^n *such that*

$$\sup_n P_n\{(X^n - Y^n)_1^* > 0\} < \varepsilon, \quad \text{and} \quad \sup_n P_n \mathcal{V}(Y^n) < \infty,$$

where $\mathcal{V}(Y^n)$ denotes the quasimartingale variation of Y^n ([DM]).

$$\text{(C)} \qquad \sup_n \sup_U P_n\{ |U \cdot X_1| > a\} \to 0 \quad \text{as } a \to \infty,$$

where U varies in the set Π of all (F_t)-predictable simple processes, bounded by 1 in absolute value, and left continuous, and $U \cdot X_1$ is the elementary stochastic integral upon the interval $[0,1]$ (*i.e.*, the family of the r.v.s $\{ U \cdot X_1^n,\ U \in \Pi,\ n \in \mathbb{N}\}$ is stochasticaly bounded).

Notice, that by the Lepingle's theorem on p-variations of semimartingale paths, one can consider the sequence P_n restricted to the new canonical space $\mathbb{V}_{2+} \subset \mathbb{D}$ endowed with the filtration, which is the trace of (F_t).

P r o p o s i t i o n. *For all* $r \leq \infty$, $(\mathrm{A}^r) \Longleftrightarrow (\mathrm{B}) \Longleftrightarrow (\mathrm{C})$.

T h e o r e m 3. *Under the condition* (C), *the sequence* P_n *is tight in* $(\mathbb{V}_{2+}, \tau^{1/2-})$, *and the canonical process* X *is a semimartingale w.r.to every law* P, *which is a limit point of the sequence* P_n.

Proofs. The theorem is an obvious consequence of the Λ-tightness theorems [MeZ], [St], and the two previous ones, and the proposition. To prove the proposition, it is sufficient to show implications: $(\mathrm{A}^1) \Rightarrow (\mathrm{C}) \Rightarrow (\mathrm{A}^\infty)$; and $(\mathrm{C}) \Rightarrow (\mathrm{B})$, or $(\mathrm{A}^r) \Rightarrow (\mathrm{B})$ either. The converse implication $(\mathrm{B}) \Rightarrow (\mathrm{C})$ was shown in [St]. The first one follows from the estimate

$$\sup_U P_n\{ |U\cdot X_1| > 2a \} \leq \leq P_n\{ \mathrm{Var}_1(V^n) > a \} +$$

$$+ (J_1 + b)/a + P_n\{ c\,([M^n, M^n]_1)^{1/2} > b \}.$$

In the martingale part, it is again a consequence of the Davis' and L-R inequalities (*cf.* (5)).

Let now the sequence P_n, or equivalently the sequence X^n, satisfy the condition (C). For short, we denote this property by the following symbolic inclusion (it will be used below also to denote such a property of other semimartingale sequences):

(*a*) $$X^n \in (\mathrm{C})$$

Stricker had shown [St], in particular, that (*a*) entails:

(*b*) *The sequence* $[X^n, X^n]_1$, $n \geq 1$, *is stochasticaly bounded.*

Indeed, notice that $[X^n,X^n]_1=(X_1^n)^2-2(X_-^n\cdot X^n)_1$, and the property **(C)** imply the stochastic boundedness of the family $(X^n)_1^*$, $n\geq 1$. Then for every *a priori* given $\varepsilon>0$, the integrands X_-^n, $n\geq 1$, are uniformly bounded, out of some sets of P_n-probabilities less than ε. Hence by **(C)**, the family of the integrals is *stochasticaly* bounded, out of these sets, in virtue of the known local property.

The implication $(\mathbf{C})\Rightarrow(\mathbf{A}^\infty)$ is a result of the following chain of simple consequences of the statements (a,b). Put

$$U^n = \sum_{s\leq\cdot} \Delta X_s^n\, 1_{(|X_s^n|>1)}$$

and consider the decompositions $X^n=U^n+A^n+M^{nd}+M^{nc}$, where M^{nc} is the continuous local martingale part of X^n, and A^n+M^{nd} is the canonical decomposition of the special semimartingale $(X^n-U^n-M^{nc})$ into a (predictable) process A^n of finite variation and a purely discontinuous local martingale N^{nd} Notice, that the decompositions $X^n=V^n+M^n$, $n\geq 1$, where $V^n=U^n+A^n$, $M^n=M^{nc}+M^{nd}$, satisfy the condition $J_\infty\leq 2$.

Since $Var_1(U^n)+[M^{nd},M^{nd}]_1\leq [X^n,X^n]_1$, the property (b) entails:

(c) *The sequence* $(Var_1(U^n)+[M^{nd},M^{nd}]_1)$, $n\geq 1$, *is stochasticaly bounded.*

As above, by the property $J_\infty\leq 2$, and the L-R inequality, one has from (a,c):

(d) $$(Z^n=U^n+M^{nd})\in(\mathbf{C}),$$

(*i.e.*, the P_n-semimartingales Z^n satisfy the Stricker's condition); and (a,d) entail:

(e) $$(Y^n=A^n+M^{nc})\in(\mathbf{C})$$

Again by Stricker, (e) entails:

(f) *The sequence* $\langle M^{nc}\rangle_1$, $n \geq 1$, *is stochasticaly bounded.*

Applying the Lenglart domination inequality, one has from (f):

$$(g) \qquad M^{nc} \in (\mathbb{C}),$$

and (a,d,e,g) entail: $A^n \in (\mathbb{C})$. Since the processes A^n are predictable, the last inclusion is equivalent to the property:

(h) *The sequence* $Var_1(A^n)$, $n \geq 1$, *is stochasticaly bounded.*

Finally, (c,f,h,) result in the property (A^{∞}).

It remains to show the implication $(\mathrm{A}^2) \Longrightarrow (\mathrm{B})$, for instance. Define the pre-localizations

$$Y^n = (X^n)^{R(n,l)-},$$

where $R(n,l) = \inf\{ t : (Var_1(V^n))_t + [M^n, M^n]_t^{1/2} \geq l \}$. The processes Y^n are special $\mathcal{S}^2$-semimartingales and hence quasi-martingales. By (A^2) they satisfy (B), because

$$\sup_n P_n\{ (X^n - Y^n) > 0 \} \leq \sup_n P_n\{ R(n,l) < \infty \} < \varepsilon ,$$

for sufficiently large l; and $\sup_n \mathcal{V}(Y^n) \leq c\, l$ □

R e m a r k. The reader acquainted with the Emery's topological semimartingale space can see that the weak relative compactness conditions (A^r) and $(\mathbb{C})$ resemble boundedness ones in that space. An examination of such relations, and the systematic development of semimartingale limit theorems w.r. to various topologies, which are of use for applicatiions, make up the subject of the book [T2].

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Kazan University, Russia