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Extending Probability Spaces and Adapted Distribution

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Abstract. We develop adapted distribution, a notion of equivalence for processes with filtration, in terms of extensions of probability spaces with filtration.

Introduction

In probability theory, one often argues so:

> Enlarging the space, if necessary, there exists a random variable Y such that ...

Constructing the extension is close to trivial, so no more is generally said. In the general theory of processes, in which a filtration is present, some care must be exercised, because recklessly enlarging a filtration may destroy essential properties of the original processes relative to the filtration, such as the martingale property or the Markov property.

It turns out that the safe way to extend a space with filtration is as follows. This method of extension and related conditions have been discussed by [Je], [AB] and [JM].

0.1 Definition. *An* extension *of a probability space with filtration* $(\Omega, \mathcal{F}, P, \boldsymbol{\mathcal{F}})$ *is a space* $(\dot{\Omega}, \mathcal{G}, Q, \boldsymbol{\mathcal{G}})$ *satisfying*

(1) $\dot{\Omega} = \Lambda^1 \times \Omega \times \Lambda^2$ *for some sets* Λ^1, Λ^2, *and* $\mathcal{G} = \mathcal{H}^1 \times \mathcal{F} \times \mathcal{H}^2$, $\boldsymbol{\mathcal{G}}$ *is the smallest filtration such that for each* $t \in [0, \infty]$, $\mathcal{G}_t \supseteq \mathcal{H}^1_t \times \mathcal{F}_t \times \mathcal{H}^2_t$, *for some* $\mathcal{H}^i$ σ*-algebras and* $\boldsymbol{\mathcal{H}}^i$ *filtrations on* Λ^i, $i = 1, 2$.
(2) *for each* $F \in \mathcal{F}$, $\dot{F} = \Lambda^1 \times F \times \Lambda^2 \in \mathcal{G}$ *and* $Q(\dot{F}) = P(F)$.
(3) *For all* $s \in [0, \infty]$, $\dot{\mathcal{F}}_\infty$ *and* $\mathcal{G}_s$ *are conditionally independent given* $\dot{\mathcal{F}}_s$. *Equivalently, for all* $F \in \mathcal{F}_\infty$,
$$Q(\dot{F} \mid \mathcal{G}_s) = \dot{}(P(F \mid \mathcal{F}_s))$$

The conditional probability need not be regular; it denotes only a set version of conditional expectation. Equality is always almost sure or up to indistinguishability, whichever is appropriate.

This notion of extension could be given in a more abstract and general form, but the concrete form we have given suffices for our purposes.

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The purpose of defining this notion of extension is to make sure that an induced process $\dot{X}$ on $\dot{\Omega}$, given by

$$\dot{X}(\lambda_1, \omega, \lambda_2) = X(\omega),$$

where X is a process on Ω, has the same properties relative to the filtration $\boldsymbol{\mathcal{G}}$ as X has relative to $\boldsymbol{\mathcal{F}}$. A very general definition of "having the same properties with respect to filtrations" is that of having the same *adapted distribution*, defined in [HK]. Processes with filtration that have the same adapted distribution agree on the usual properties of processes, such as the martingale (or semimartingale, etc.) property, the Markov property, and also on properties that may involve extensions of their space, such as having weak solutions of a given stochastic differential equation (see [H1], [HK]).

The first purpose of this paper is to observe that adapted distribution is exactly the notion of equivalence of processes that goes with extensions of probability spaces. That is, $\dot{\Omega}$ satisfying (1) and (2) of the definition is an extension of Ω (that is, satisfies (3)) iff for every process X on Ω, $(X,\boldsymbol{\mathcal{F}})$ and $(\dot{X},\boldsymbol{\mathcal{G}})$ have the same adapted distribution. The second purpose is to show how using the method of extensions of spaces, instead of the method of saturated spaces, leads to much simpler and slightly more general versions of theorems proved in [HK] and [H2].

Because we feel that our present approach improves considerably on [HK], and because some changes were required in order to discuss convergence in adapted distribution (in [H4]), we have given all the definitions concerned with adapted distribution. In cases when we generalize results in [HK] or [H2], however, we will refer to the original papers for details, since it would be idle to repeat them.

Weaker invariants of processes with filtration have been given by Jacod [J] (local, or predictable, characteristic), and Aldous [A] (synonymity). Neither adapted distribution nor either of the others is easy to compute. Our feeling is that in most cases when any of them can be computed, they are equivalent (as in the case of independent increment processes). The advantage of adapted distribution is that it is designed to have a good structural theory.

Preliminary versions of some of the results in this paper were announced in [H3].

1. Preliminaries on Topologies and Filtrations

A probability space with filtration (or *stochastic base* or *adapted probability space*) is a structure $\boldsymbol{\Omega} = (\Omega, \mathcal{F}, P, \boldsymbol{\mathcal{F}})$, where the filtration $\boldsymbol{\mathcal{F}} = (\mathcal{F}_t)_{t\in[0,\infty]}$ is an increasing family of sub-σ-algebras of $\mathcal{F}$. We assume here that $\boldsymbol{\mathcal{F}}$ satisfies the usual conditions, that is, it is right continuous, $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$, and complete in the sense that $\mathcal{F}_0$ contains the nullsets of P. $\mathcal{F}_\infty$ need not be the same as either $\mathcal{F}$ or $\mathcal{F}_{\infty-} = \bigvee_t \mathcal{F}_t$, but except in measure theoretic proofs, it will usually be $\mathcal{F}$. Having a distinct $\mathcal{F}_\infty$ makes it easy for us to handle processes which are not adapted. We could allow arbitrary subsets of $[0,\infty]$ as filtration parameter sets, but a filtration $(\mathcal{G}_t)_{t\in T}$ on $(\Omega, \mathcal{F}, P)$ satisfying the usual conditions can be extended to a filtration $(\mathcal{F}_t)_{t\in[0,\infty]}$ by letting

$$\begin{aligned}\mathcal{F}_t &= \mathcal{G}_t, \quad t \in T,\\ &= \bigcap_{s\in T\cap[t,\infty]} \mathcal{G}_s, \quad t = \inf(T\cap[t,\infty]),\\ &= \bigvee_{s\in T\cap[0,t]} \mathcal{G}_s, \text{ otherwise.}\end{aligned}$$

Here, to assure completeness, we take the empty join to consist of the null and co-null sets of P, and we take the empty intersection to be $\mathcal{F}$.

A stochastic process on $(\Omega, \mathcal{F}, P)$ is a pair $\mathbf{X} = (X, \boldsymbol{\mathcal{F}})$ where X is a random variable, $\boldsymbol{\mathcal{F}}$ is a filtration on $(\Omega, \mathcal{F}, P)$, and X is $\mathcal{F}_\infty$-measurable. Equality of random variables is always taken as almost sure. We assume that X takes values in a metric space M, though we do not generally require M to be separable or complete.

An object that is a stochastic process in the usual sense, that is, a function $X : [0,\infty] \times \boldsymbol{\Omega} \to M$, will be treated as a random variable whose values are functions from $[0,\infty] \to M$. Different topologies may be applied to the trajectories, depending on their properties.

(1) For trajectories that are r.c.l.l. (right continuous with left limits, but not necessarily left continuous at ∞), we may use a modified Skorokhod topology, so that $x_n \to x$ iff the restrictions of the functions to $[0,\infty)$ converge in the Skorokhod topology and $x_n(\infty) \to x(\infty)$. Functions without values at ∞ may be given such values in an arbitrary but uniform manner.

(2) If trajectories are measurable functions we may use the topology of convergence in measure ([MZh]). For processes which we construct, we will normally use this topology, even when trajectories are r.c.l.l. . This topology is given as follows. Let μ be a finite measure which is equivalent to Lebesgue measure on $[0,\infty)$ and has $\mu(\infty) > 0$—the topology does not depend on just which μ is used. Convergence is convergence in measure with respect to μ.

(3) Uniform convergence or pointwise convergence. These topologies are, of course, nonseparable.

The topology of convergence in measure does not separate trajectories which differ only on a Lebesgue null subset of $[0,\infty)$. It is, however, Hausdorff on r.c.l.l. trajectories, hence the same as the Skorokhod topology on Skorokhod-compact sets of trajectories. It follows that a Skorokhod-tight sequence of processes X_n, $n \in \mathbf{N}$, converges in distribution to X iff convergence in distribution holds for the topology of convergence in measure.

By the martingale convergence theorem, if x has a bounded, measurable trajectory in the reals, then

$$x = \lim_{n\to\infty} x_n \text{ a.e. } \mu,$$

where

$$\begin{aligned}x_n(t) &= \frac{1}{\mu[\frac{k}{n}, \frac{k+1}{n})} \int_{\frac{k}{n}}^{\frac{k+1}{n}} x(s)\mu(ds), \quad t \in [\frac{k}{n}, \frac{k+1}{n}),\\ x_n(\infty) &= x(\infty).\end{aligned}$$

This gives a Borel measurable (for convergence in measure) way of extracting $x(t)$ for μ-a.e. t from x which is Borel measurable for the topology of convergence in measure. Those who wish to look into the matter will see that this can be used to show that the relation of "having a.e. the same adapted distribution", defined in [HK] is the same (modulo presence of ∞) to "having the same AD" as defined herein for processes with measurable trajectories and the topology of convergence on those trajectories.

For each separable metric space M, we fix a countable set C_M of functions $M \to [0,1]$ which are dense among functions $K \to [0,1]$ for each compact subset K of the completion $\hat{M}$ of M. A canonical choice would be the linear lattice generated over the rationals by the functions $d(a,x) \wedge 1$, where d is the metric, and a belongs to a given countable dense subset of M. The *separable topology* (relative to C_M) on the space of measures on M is the weakest topology such that the map $\mu \to \mu(\phi)$ is a continuous function of μ for each $\phi \in C_M$. This topology is Hausdorff, because all measures on M are tight considered as measures on $\hat{M}$. Hence it is the same as the topology of weak convergence on precompact, hence $\hat{M}$-tight, families of measures on M.

1.1 DEFINITION. *$\boldsymbol{\mathcal{F}}$ is a subfiltration of $\boldsymbol{\mathcal{G}}$, $\boldsymbol{\mathcal{F}} \subseteq \boldsymbol{\mathcal{G}}$, if $\mathcal{F}_t \subseteq \mathcal{G}_t$ for every $t \in [0,\infty]$. If $\boldsymbol{\mathcal{F}} \subseteq \boldsymbol{\mathcal{G}}$, we say that $\boldsymbol{\mathcal{F}}$ is self-contained in $\boldsymbol{\mathcal{G}}$, or $\boldsymbol{\mathcal{G}}$ is an extension of $\boldsymbol{\mathcal{F}}$ (Aldous and Barlow [AB], condition H in [Je]), if for each $t \in [0,\infty]$ $\mathcal{F}_\infty$ and $\mathcal{G}_t$ are conditionally independent (c.i.) given $\mathcal{F}_t$).*

We write $\boldsymbol{\mathcal{F}} \prec \boldsymbol{\mathcal{G}}$.

A family of σ-algebras $\mathcal{H}^i$, $i \in I$, is conditionally independent given $\mathcal{K}$ if for any distinct $i_1, \dots, i_n \in I$ and $H_j \in \mathcal{H}_{i_j}$, $j = 1, \dots, n$,

$$P(\bigcap_{j=1}^{n} H_{i_j} \mid \mathcal{K}) = \prod_{j=1}^{n} P(H_{i_j} \mid \mathcal{K}).$$

In this paper we usually use it in the equivalent form given by [Lo] 25.3A. That is, since $\mathcal{F}_t \subseteq \mathcal{F}_\infty, \mathcal{G}_t$, $\mathcal{F}_\infty$ and $\mathcal{G}_t$ are c.i. given $\mathcal{F}_t$ iff for each, $F \in \mathcal{F}_\infty$,

$$P(F \mid \mathcal{G}_t) = P(F \mid \mathcal{F}_t),$$

iff for each $G \in \mathcal{G}_t$,

$$P(G \mid \mathcal{F}_\infty) = P(G \mid \mathcal{F}_t).$$

We see from the first equivalent form that $\boldsymbol{\mathcal{F}} \prec \boldsymbol{\mathcal{G}}$ iff every $\boldsymbol{\mathcal{F}}$-martingale is also a $\boldsymbol{\mathcal{G}}$-martingale. The reader will observe that the crucial clause (3) in the definition of extension of a probability space is that $\dot{\boldsymbol{\mathcal{F}}} \prec \boldsymbol{\mathcal{G}}$.

We observe that the definition of self-containment applies not only to filtrations, but in general to increasing families of σ-algebras (i.e. not necessarily complete or right continuous). If $(\mathcal{G}_t)_{t\in[0,\infty]}$ is an increasing family of σ-algebras, let $\langle(\mathcal{G}_t)\rangle$ denote the filtration generated by $(\mathcal{G}_t)$,

$$\langle(\mathcal{G}_t)\rangle_s = \bigcap_{s<t} \mathcal{G}_t \vee \mathcal{N},$$

where $\mathcal{N}$ is the ideal of nullsets of the relevant probability measure. Similarly, if $\mathcal{S}$ is a family of measurable sets or $S_1, \ldots, S_n$ are measurable sets, then $\langle \mathcal{S} \rangle$ or $\langle S_1, \ldots, S_n \rangle$ denotes the complete σ-algebra generated by the given family or sets.

The following results are clear.

1.2 LEMMA.

(1) *If* $\boldsymbol{\mathcal{F}} \prec \boldsymbol{\mathcal{G}}$ *and* $\boldsymbol{\mathcal{G}} \prec \boldsymbol{\mathcal{H}}$, *then* $\boldsymbol{\mathcal{F}} \prec \boldsymbol{\mathcal{H}}$.
(2) *The intersection of a family of self-contained subfiltrations of* $\boldsymbol{\mathcal{G}}$ *is self-contained.*
(3) *If* $(\mathcal{F}_t) \prec (\mathcal{G}_t)$, *then* $\langle (\mathcal{F}_t) \rangle \prec \langle (\mathcal{G}_t) \rangle$.

From (2), it follows that if $\mathbf{X} = (X, \boldsymbol{\mathcal{F}})$ is a random variable with filtration, then there exists a smallest self-contained subfiltration $\boldsymbol{\mathcal{I}}(\mathbf{X})$ of $\boldsymbol{\mathcal{F}}$ such that X is $\mathcal{I}(\mathbf{X})_\infty$-measurable. We call $\boldsymbol{\mathcal{I}}(\mathbf{X})$ the *intrinsic* filtration of $\mathbf{X}$.

If $(\Omega, \mathcal{F}, P)$ is a probability space, let $\mathcal{N}_P$ denote the nullsets of P. If $(\Lambda, \mathcal{G}, Q)$ is another probability space, a σ-algebra isomorphism of $(\mathcal{F}, P)$ and $(\mathcal{G}, Q)$ is a map $h : \mathcal{F}/\mathcal{N}_P \to \mathcal{G}/\mathcal{N}_Q$ such that whenever $F_1, F_2 \in \mathcal{F}$, and $h(F_i/\mathcal{N}_P) = G_i/\mathcal{N}_Q$, $G_i \in \mathcal{G}$,

(1) $P(F_i) = Q(G_i)$, $i = 1, 2$, and
(2) if $F_1 \subseteq F_2$ then $G_1 \subseteq G_2$ (a.s.).

It follows that all the σ-algebra operations are preserved a.s.

We abuse notation by writing $h(F_i) = G_i$, $i = 1, 2$, though this is only defined a.s. We extend the abuse to random variables by writing $h(X) = Y$ if $h(\{X \in B\}) = \{Y \in B\}$ for each open set B of their common value space M. Elementary measure theory gives us the following result.

1.3 PROPOSITION. *For any bounded Borel-measurable real function* ϕ,

(1) $h(\phi(X)) = \phi(h(X))$.
(2) $E^P[\phi(X)] = E^Q[\phi(h(X))]$.

Given $\boldsymbol{\mathcal{F}}$ a filtration on $(\Omega, \mathcal{F}, P)$, and $\boldsymbol{\mathcal{G}}$ a filtration on $(\Lambda, \mathcal{G}, Q)$, we say that a σ-algebra isomorphism h of $(\mathcal{F}_\infty, P)$ and $(\mathcal{G}_\infty, Q)$ is a *filtration isomorphism* of $(\boldsymbol{\mathcal{F}}, P)$ and $(\boldsymbol{\mathcal{G}}, Q)$ if for any $t \in [0, \infty]$, $F \in \mathcal{F}_t$ iff $h(F) \in \mathcal{G}_t$. The following result is clear from Proposition 1.3 and the definition of conditional expectation.

1.4 PROPOSITION. *If* $h : (\boldsymbol{\mathcal{F}}, P) \to (\boldsymbol{\mathcal{G}}, Q)$ *is a filtration isomorphism and* X *is an* $\mathcal{F}_\infty$-*measurable, integrable real random variable on* Ω, *then for each* $t \in [0, \infty]$,

$$h(E[X \mid \mathcal{F}_t]) = E[h(X) \mid \mathcal{G}_t].$$

2. DEFINITIONS OF ADAPTED DISTRIBUTION

We offer three definitions of adapted equivalence, each of which is the best for a particular purpose. For two of them, we need some preliminary definitions.

Let $\mathbf{X} = (X, \boldsymbol{\mathcal{F}})$ be a random variable with values in a metric space M.

2.1 DEFINITION. *Adapted functions for* M-*valued random variables are defined as follows (as in [H2], but with a slightly different basis clause, to make it easier to handle stochastic processes which do not have r.c.l.l. sample paths).*

(1) *A bounded continuous function* $\Phi : M \to \mathbf{R}$ *is an adapted function.*

(2) *If $f_1, \dots, f_n$ are adapted functions, and $\phi : \mathbf{R}^n \to \mathbf{R}$ is bounded and continuous, then $\phi(f_1, \dots, f_n)$ is an adapted function.*

(3) *If f is an adapted function, and $t \in [0, \infty)$, then $E[f \mid t]$ is an adapted function.*

The value of an adapted function f on $\mathbf{X} = (X, \mathcal{F})$ is defined in the obvious way by a similar recursion:

$$\begin{aligned}
&\Phi(\mathbf{X}) = \Phi(X), \\
&\phi(f_1, \dots, f_n)(\mathbf{X}) = \phi(f_1(\mathbf{X}), \dots, f_n(\mathbf{X})), \\
&E[f \mid t](\mathbf{X}) = E[f(\mathbf{X}) \mid \mathcal{F}_t].
\end{aligned}$$

The next definition applies when M is separable, though this condition may not be essential.

The *Knight-Aldous prediction process* $Z(\mathbf{X})$ ([Kn], [A]) of an M-valued random variable $\mathbf{X}$ is the measure-valued process given by

$$Z(\mathbf{X})_t(\Phi) = E[\Phi(X) \mid \mathcal{F}_t]$$

for $\Phi : M \to \mathbf{R}$ bounded and continuous. We may take each $Z(\mathbf{X})_t$ to be a regular conditional probability. Since each $Z(\mathbf{X})_t(\Phi)$ is a martingale, we may choose a version for which this is r.c.l.l. for each $\Phi \in C_M$. Giving the measures on M the separable topology, $Z(\mathbf{X})$ is r.c.l.l. . Give the trajectories the topology of convergence in measure. This makes the trajectory space separable, and we can repeat the construction.

2.2 DEFINITION. *$\mathrm{m}_n(\mathbf{X})$, $n \in \mathbf{N} = \{1, 2, 3, \dots\}$ and $\mathrm{m}(\mathbf{X})$ are given by*

$$\begin{aligned}
\mathrm{m}_1(\mathbf{X}) &= Z(\mathbf{X}), \\
\mathrm{m}_{n+1}(\mathbf{X}) &= Z(\mathrm{m}_n(\mathbf{X}), \mathcal{F}). \\
\mathrm{m}(\mathbf{X}) &= (\mathrm{m}_n(\mathbf{X}); n \in \mathbf{N})
\end{aligned}$$

This is a modification of the definition in [HK]. It fills the same rôle for adapted equivalence and is more suitable for convergence in adapted distribution, discussed in [H4].

2.3 THEOREM. *Let $\mathbf{X}$ and $\mathbf{Y}$ be random variables with values in the same metric space M. The following are equivalent.*

(1) *There is an filtration isomorphism $h : \mathcal{I}(\mathbf{X}) \to \mathcal{I}(\mathbf{Y})$ such that $h(X) = Y$.*

(2) *For any M-adapted function f, $E[f(\mathbf{X})] = E[f(\mathbf{Y})]$.*

(3) *$\mathrm{m}(\mathbf{X}) \sim \mathrm{m}(\mathbf{Y})$ (when M is separable).*

By $\sim$ we mean that the processes have the same distribution.

PROOF: That (1) implies each of (2) and (3) follows from iterated application of Props. 1.3 and 1.4. To show (2) implies (1), let $\mathcal{J}(\mathbf{X})$ be the filtration generated by the adapted functions of $\mathbf{X}$: $\mathcal{J}(\mathbf{X})_t$ is generated modulo nullsets by the random variables $f(\mathbf{X})$ such that f is an adapted function of the form $E[g \mid t]$, and $\mathcal{J}(\mathbf{X})_\infty$ is generated by all adapted functions of $\mathbf{X}$. If $F \in \mathcal{J}(\mathbf{X})_\infty$ and $\epsilon > 0$, then there is an adapted function f taking values in $[0, 1]$ such that

$$E|1(F) - f| < \epsilon.$$

Hence by Jensen's inequality,

$$E|P(F \mid \mathcal{F}_t) - E[f \mid t](\mathbf{X})| < \epsilon.$$

It follows that $\mathcal{J}(\mathbf{X})$ is self-contained in $\boldsymbol{\mathcal{F}}$. The argument that (1) implies (2) shows that $\mathcal{J}(\mathbf{X}) \subseteq \mathcal{I}(\mathbf{X})$, so the two must be equal. The same applies to $\mathbf{Y}$. The map h which sends $\{f(\mathbf{X}) \in B\} \to \{f(\mathbf{Y}) \in B\}$ for each Borel set $B \subseteq \mathbf{R}$ is measure preserving by (2), hence it is an isomorphism of $\mathcal{I}(\mathbf{X})_\infty$ and $\mathcal{I}(\mathbf{Y})_\infty$. Since it maps $\{E[f \mid t](\mathbf{X}) \in B\} \to \{E[f \mid t](\mathbf{Y}) \in B\}$, we see that it is adapted, so (1) holds. That (3) implies (1) is similar.

We say that $\mathbf{X}$ and $\mathbf{Y}$ are adaptedly (or stochastically) equivalent, or have the same adapted distribution, if any of the three conditions of the Theorem holds. Most people want to know what is the adapted distribution itself. We can give this name either to the operator which maps $f \to E[f(\mathbf{X})]$, or to the distribution of $\mathrm{m}(\mathbf{X})$. When $\mathbf{X}$ and $\mathbf{Y}$ have the same adapted distribution, we write

$$\mathbf{X} \equiv_{AD} \mathbf{Y}.$$

2.4 THEOREM. *If $(\dot{\Omega}, \mathcal{G}, Q, \boldsymbol{\mathcal{G}})$ is an extension of $(\Omega, \mathcal{F}, P, \boldsymbol{\mathcal{F}})$, and* $\mathbf{X}$ *is a process on $(\Omega, \mathcal{F}, P, \boldsymbol{\mathcal{F}})$, then $(X, \boldsymbol{\mathcal{F}}) \equiv_{AD} (\dot{X}, \boldsymbol{\mathcal{G}})$.*

PROOF: $h(F) = \Lambda_1 \times F \times \Lambda_2$ is a filtration isomorphism of $\boldsymbol{\mathcal{F}}$ and $\dot{\boldsymbol{\mathcal{F}}}$. By definition of extension, $\dot{\boldsymbol{\mathcal{F}}} \prec \boldsymbol{\mathcal{G}}$. Hence, by Lemma 1.2, $h(\mathcal{I}(\mathbf{X})) = \mathcal{I}(\dot{\mathbf{X}})$.

3. AMALGAMATION AND ADJUNCTION

The results of this section show how to extend an adapted probability space in order to add another process.

3.1 LEMMA. *Let $(\Omega^j, \mathcal{G}^j, P^j)$, $j \in J$, be probability spaces, and let $\mathcal{F}^j \subseteq \mathcal{G}^j$, $j \in J$, be sub-σ-fields, with measure algebra isomorphisms $h^j : (\mathcal{F}, P) \to (\mathcal{F}^j, P^j)$, where $(\mathcal{F}, P) = (\mathcal{F}^{j_0}, P^{j_0})$ for some $j_0 \in J$, and h^{j_0} is the identity. Then the map $Q : \prod_j \mathcal{G}^j \to \mathbf{R}$ given by*

$$Q(\prod_j G^j) = \int \prod_j (h^j)^{-1}(P^j(G^j) \mid \mathcal{F}^j))\, dP$$

is a probability measure on $(\prod_j \Omega^j, \prod_j \mathcal{G}^j)$. If π^j, $j \in J$ are the coordinate projections $\omega \to \omega_j$, then $(\pi^j)^{-1}(\mathcal{F}^j)$, $j \in J$ are identical modulo nullsets, and $(\pi^j)^{-1}(\mathcal{G}^j)$, $j \in J$ are c.i. given $(\pi^j)^{-1}(\mathcal{F}^j)$.

PROOF: Since any set in $\prod_j \mathcal{G}^j$ depends on only countably many coordinates, we may assume that J is countable. For each $j \in J$, the map $q^j : \Omega^{j_0} \times \mathcal{G}^j \to \mathbf{R}$ given by

$$q^j(\omega, G) = (h^j)^{-1}(P^j(G^j \mid \mathcal{F}^j))(\omega)$$

is nearly a kernel, in that any fixed version of it is a.s. countably additive on any countable subalgebra, and this is enough to make the usual product measurability theorem go through (as [Ash], Theorem 2.7.2). Hence Q defines a probability measure. The remaining statements follow trivially.

3.2 AMALGAMATION THEOREM. *For $j \in J$, let $(\Omega^j, \mathcal{F}^j, P^j, \boldsymbol{\mathcal{F}}^j)$ be adapted probability spaces, let $\boldsymbol{\mathcal{I}}^j \prec \boldsymbol{\mathcal{F}}^j$, and let h^j, be a filtration isomorphism $(\boldsymbol{\mathcal{I}}^{j_0}, P^{j_0}) \to (\boldsymbol{\mathcal{I}}^j, P^j)$, with h^{j_0} the identity. Then $(\Omega^j, \mathcal{F}^j, P^j, \boldsymbol{\mathcal{F}}^j)$, $j \in J$, have a common extension $(\dot{\Omega}, \mathcal{G}, Q, \boldsymbol{\mathcal{G}})$, with $\dot{\Omega} = \prod_{j \in J} \Omega^j$, such that*

(1) *for any $I \in \mathcal{I}^{j_0}_\infty$, and $j \in J$,*

$$(\pi^{j_0})^{-1}[I] = (\pi^j)^{-1}[h^j(I)] \text{ a.s.}[Q],$$

where for each j, π^j is the projection $\omega \to \omega_j$;

(2) *$\dot{\mathcal{F}}^j_\infty$, $j \in J$, are conditionally independent given $\dot{\mathcal{I}}_\infty$;*

(3) *$\dot{\boldsymbol{\mathcal{I}}} \prec \boldsymbol{\mathcal{G}}$.*

PROOF: Apply the Lemma to $\mathcal{F}^j, \mathcal{I}^j_\infty$, $j \in J$, to get Q on $\mathcal{G} = \prod_j \mathcal{F}^j$, and let $\boldsymbol{\mathcal{G}}$ be the filtration on $(\dot{\Omega}, \mathcal{G}, Q)$ generated by the increasing family of σ-algebras $(\prod_j \mathcal{F}^j_t)$. (1) and (2) hold trivially, so we need only show (3) and that $\dot{\boldsymbol{\Omega}}$ actually is an extension of each $\boldsymbol{\Omega}^j$, $j \in J$. In the following we will simplify notation by ignoring the distinction between $\boldsymbol{\mathcal{F}}^j$ and $\dot{\boldsymbol{\mathcal{F}}}^j$, $\boldsymbol{\mathcal{I}}^j$ and $\dot{\boldsymbol{\mathcal{I}}}$, and we will write the latter as $\boldsymbol{\mathcal{I}}$. Let us first show that $\dot{\boldsymbol{\Omega}}$ actually is an extension of each $\boldsymbol{\Omega}^j$. Fix j. By 1.2(3), we need only show that $\boldsymbol{\mathcal{F}}^j \prec (\prod_j \mathcal{F}^j_t)$. Let $F \in \mathcal{F}^j_\infty$, and $G \in \mathcal{F}^j_t$, $G^i \in \mathcal{F}^{j_i}_t$, $i = 1 \dots n$. Then

$$\begin{aligned} Q(FGG^1 \cdots G^n) &= E[Q(FG \mid \mathcal{I}_\infty)Q(G^1 \mid \mathcal{I}_\infty) \cdots Q(G^n \mid \mathcal{I}_\infty)] \\ &= E[Q(FG \mid \mathcal{I}_\infty)Q(G^1 \mid \mathcal{I}_t) \cdots Q(G^n \mid \mathcal{I}_t)] \\ &= E[Q(FG \mid \mathcal{F}^j_t)Q(G^1 \mid \mathcal{I}_t) \cdots Q(G^n \mid \mathcal{I}_t)] \\ &= E[Q(F \mid \mathcal{F}^j_t)GG^1 \cdots G^n]. \end{aligned}$$

The first equality follows by definition of Q, the second because $\boldsymbol{\mathcal{I}} \prec \boldsymbol{\mathcal{F}}^j$, $j \in J$, the third by definition of conditional probability, the last by moving G outside the conditional probability and reversing the chain of equalities. Since sets of the form $GG^1 \cdots G^n$ form a semiring generating $\prod_j \mathcal{F}^j_t$, this shows, by [Lo] 25.3A, that $\mathcal{F}^j_\infty$ and $\mathcal{G}_t$ are c.i. given $\mathcal{F}^j_t$, as was required. To prove (3), repeat this argument with $F \in \mathcal{I}^j_\infty = \mathcal{I}_\infty$, and in the last line of the equation replace $\mathcal{F}^j_t$ by $\mathcal{I}_t$ using $\boldsymbol{\mathcal{I}} \prec \boldsymbol{\mathcal{F}}^j$.

3.3 ADJUNCTION THEOREM. *Suppose $(X, \boldsymbol{\mathcal{F}})$ is a process on $(\Omega, \mathcal{F}, P)$, and $(X', Y', \boldsymbol{\mathcal{F}}')$ is a process on $(\Omega', \mathcal{F}', P')$ such that $(X, \boldsymbol{\mathcal{F}}) \equiv_{AD} (X', \boldsymbol{\mathcal{F}}')$. Then $(\Omega, \mathcal{F}, P, \boldsymbol{\mathcal{F}})$ has an extension $(\dot{\Omega}, \mathcal{G}, Q, \boldsymbol{\mathcal{G}})$ which carries a random variable Y such that*

$$(\dot{X}, Y, \boldsymbol{\mathcal{G}}) \equiv_{AD} (X', Y', \boldsymbol{\mathcal{F}}').$$

PROOF: Amalgamate $(\Omega, \mathcal{F}, P, \boldsymbol{\mathcal{F}})$ and $(\Omega', \mathcal{F}', P', \boldsymbol{\mathcal{F}}')$ over the isomorphism of $\boldsymbol{\mathcal{I}}(\mathrm{X})$ and $\boldsymbol{\mathcal{I}}(\mathrm{X}')$ given by $\mathrm{X} \equiv_{AD} \mathrm{X}'$ and let $Y = \dot{Y}'$.

The Adjunction Theorem is similar to the definition of a saturated space in [HK] (see §6, below). The differences are

(1) the Adjunction Theorem realizes a process on any space by enlarging it, whereas a saturated space is a particular kind of space that does not need to be enlarged;
(2) for saturated spaces, the process to be realized must have a separable target space, whereas the Adjunction Theorem has no such restriction.

Stronger forms of saturation could be given to get around (3), but our feeling is that the Adjunction Theorem is so much simpler that it makes saturation obsolete.

Here is an example showing that conditionally independent amalgamation of filtrations over a common self-contained subfiltration does not necessarily yield a right-continuous family of σ-algebras. We ignore the distinction between sets and their indicator functions.

3.4 EXAMPLE: Let U, V_n, $n \in \mathbf{N}$, be independent Bernoulli trials with probability 1/2. Then $W_n = U \oplus V_n$, $n \in \mathbf{N}$, where $\oplus$ is mod 2 addition, are also independent Bernoulli trials with probability 1/2, and are independent of U. Let

$$\mathcal{G}_t = \langle V_n; n \geq \frac{1}{t} \rangle, \quad t > 0,$$

$$\mathcal{G}_0 = \langle \emptyset \rangle,$$

$$\mathcal{H}_t = \langle W_n; n \geq \frac{1}{t} \rangle, \quad t > 0,$$

$$\mathcal{H}_0 = \langle \emptyset \rangle.$$

Both $\boldsymbol{\mathcal{G}}$ and $\boldsymbol{\mathcal{H}}$ are right continuous at zero by the Kolmogorov 0-1 law, and they are trivially right continuous elsewhere. Let

$$\mathcal{F}_t = \langle V_m \oplus V_n; m, n \geq \frac{1}{t} \rangle, \quad t > 0,$$

$$\mathcal{F}_0 = \langle \emptyset \rangle.$$

We claim that

(1) For each t, $\mathcal{F}_t = \mathcal{G}_t \cap \mathcal{H}_t$.
(2) $\mathcal{G} = \mathcal{G}_\infty$ and $\mathcal{H} = \mathcal{H}_\infty$ are conditionally independent given $\mathcal{F} = \mathcal{F}_\infty$.
(3) $\boldsymbol{\mathcal{F}} \prec \boldsymbol{\mathcal{G}}, \boldsymbol{\mathcal{H}}$.

That is, $\boldsymbol{\mathcal{G}}$ and $\boldsymbol{\mathcal{H}}$ are independent over their common self-contained subfiltration $\boldsymbol{\mathcal{F}}$. $(\mathcal{G}_t \vee \mathcal{H}_t)$ is not right continuous, because, $U \in \mathcal{G}_t \vee \mathcal{H}_t$ for each $t > 0$, but $\mathcal{G}_0 \vee \mathcal{H}_0$ is trivial.

Now to prove our claims. We will first prove (2), since everything follows from that. Clearly $\mathcal{F} \subseteq \mathcal{G} \cap \mathcal{H}$. By [Lo] 25.3A, it suffices to show that for any H of the form

$$W_{i_1} \cap \ldots \cap W_{i_n},$$

$P(H \mid \mathcal{G}) = P(H \mid \mathcal{F})$, since, by inclusion-exclusion, any set in the semiring generated by the W_n's can be obtained from these by linear operations. By definition of the W_n's, one sees that an element $\omega \in H$ iff ω is in U but in none of the V_{i_j}'s, or ω is not in U, but in all of the V_{i_j}'s. Thus

$$H = (U \cap \bigcap_{j \leq n} V_{i_j}^c) \cup (U^c \cap \bigcap_{j \leq n} V_{i_j}),$$

which is a disjoint union, hence

$$P(H \mid \mathcal{G}) = \frac{1}{2} \left(\prod_{j \leq n} V_{i_j}^c + \prod_{j \leq n} V_{i_j} \right),$$

Since U is independent of $\mathcal{G}$ and the others are $\mathcal{G}$-measurable. But the right hand side is equal to

$$\frac{1}{2}\left(\prod_{j<k\leq n} 1 \oplus V_{i_j} \oplus V_{i_k}\right),$$

which is $\mathcal{F}$-measurable. This proves the claim. If we replace $\mathcal{H}$ by $\mathcal{H}_t$, $t>0$, we get the same result, but with all the i_j's $\leq 1/t$, which shows that $\mathcal{G}$ and $\mathcal{H}_t$ are conditionally independent given $\mathcal{F}_t$. Since $\mathcal{F} \subseteq \mathcal{G}$, $\mathcal{F}$ and $\mathcal{H}_t$ are conditionally independent given $\mathcal{F}_t$, that is, $\boldsymbol{\mathcal{F}} \prec \boldsymbol{\mathcal{H}}$. By symmetry, $\boldsymbol{\mathcal{F}} \prec \boldsymbol{\mathcal{G}}$. $\mathcal{F}_t$ is certainly contained in $\mathcal{G}_t \cap \mathcal{H}_t$. Since it is complete, it must in fact equal the latter, because a set cannot be conditionally independent of itself given $\mathcal{F}_t$ unless it is in $\mathcal{F}_t$.

4. Characterization of AD by adapted adunction

In [H2], we gave a filtration embedding condition sufficient to imply that two processes have the same adapted distribution. In the present section, we generalize this condition using the idea of extension/adjunction in order to give a simple characterization of adapted distribution.

4.1 Definition. *Let $(X,\boldsymbol{\mathcal{F}})$ and $(Y,\boldsymbol{\mathcal{G}})$ be processes with filtration. We say that*

$$(X,\boldsymbol{\mathcal{F}}) \hookrightarrow (Y,\boldsymbol{\mathcal{G}})$$

iff

(4.1.1) *For every r.c.l.l. , $\boldsymbol{\mathcal{F}}$-adapted process X' taking values in a separable space, there exists an extension $\dot{\boldsymbol{\mathcal{G}}}$ of $\boldsymbol{\mathcal{G}}$ and a $\dot{\boldsymbol{\mathcal{G}}}$-adapted process Y' such that*

$$(X,X') \sim (\dot{Y},Y').$$

The relation $\hookrightarrow$ as here defined is stronger than $\hookrightarrow^+$ as defined in [H2], except that it allows the space of $(Y,\boldsymbol{\mathcal{G}})$ to be extended. Our first result is that if extensions are allowed in $\hookrightarrow$ as defined in [H2], the resulting notion is equivalent to $\hookrightarrow$ as we have defined it here.

4.2 Theorem. *The following condition is equivalent to $(X,\boldsymbol{\mathcal{F}}) \hookrightarrow (Y,\boldsymbol{\mathcal{G}})$.*

(4.2.1) *Same as (4.1.1), except that X' is restricted to be a step process, i.e. X' takes only finitely many values, and there is some finite n and fixed times $t_1,\ldots,t_n$ such that X' has jumps only at times $t_1,\ldots,t_n$.*

Proof: Assume the condition, and let an $\boldsymbol{\mathcal{F}}$-adapted process X' be given. Let X^n, $n \in \mathbb{N}$ be a sequence of finite-valued step processes which are $\boldsymbol{\mathcal{F}}$-adapted and such that $X^n \to X'$ a.s. Then $(X,X^n) \Rightarrow (X,X')$. Choose Y^n such that $(X,X^n) \sim (Y,Y^n)$ for each $n \in \mathbb{N}$. By [JM], Theorem 2.8, there is an enlargement of the original space with (Y,Y') such that some subsequence of the (Y,Y^n) converge weak-strongly to (Y,Y'). As in the proof of [JM] Theorem 7.6, the space of (Y,Y') can be taken to be an extension of the original space.

The reader will observe that the weak-strong convergence results of [JM] are a very useful tool in adapted distribution, though they are phrased in terms only of distribution.

A similar convergence notion, but in terms of convergence in adapted distribution, and so apparently less useful, was given in [HK], Definition 4.1. It will be observed in [H4], however, that any sequence which converges in distribution has a subsequence which converges in adapted distribution, so the two amount to the same thing.

4.3 THEOREM (CHARACTERIZATION OF AD). $\mathrm{X} \equiv_{AD} \mathrm{Y}$ *iff* $\mathrm{X} \hookrightarrow \mathrm{Y}$ *and* $\mathrm{Y} \hookrightarrow \mathrm{X}$.

PROOF: The ($\Rightarrow$) direction follows by the Adjunction Theorem. ($\Leftarrow$) is similar to the first part of the proof of Theorem 2.3 of [H2]. Since $\hookrightarrow$ as defined in this paper implies the $\hookrightarrow^+$ of [H2] modified to allow extensions, only the first part of that proof is needed.

Theorems 4.2 and 4.3 in [H2] can be made equivalences in the same way by changing the definition of $\hookrightarrow_{SDE}$ to allow an extension of the target space.

5. STOCHASTIC DIFFERENTIAL EQUATIONS

Adapted distribution was originally inspired by the following problem. To find a solution to a stochastic differential equation

$$X_t = K_t + \int_0^t f(s,X)dZ,$$

where Z is a semimartingale, it may be necessary to enlarge the space with filtration which carries the coefficients of the equation. Since this changes the coefficients, in what sense can we say that the original equation has been solved? In [HK] the approach was to show that on a saturated spaces, one need not enlarge the space to solve such an equation, and that furthermore every equation was "the same", in the sense of adapted distribution, as one on a saturated space, so that one might as well work only on saturated spaces in the first place. Here, we will review the results of [HK] about SDE's and adapted distribution, which were proved using saturated spaces, from the point of view of extensions of spaces.

The first observation to make, is that there is really no problem about enlarging spaces to solve equations, because, as shown in [JM], Theorem 7.6, the enlargement of the space can be done via an extension in the sense of this paper. Hence the equation on the enlarged space is equivalent, in the sense of adapted distribution, to the original equation.

The second observation, is that in [HK], processes must take values in separable spaces, so that the results given in Section 7 of [HK] do not cover path dependent integral equations such as those solved in [JM]. This separability restriction also makes it awkward for [HK] to state results about predictable processes without path properties (cf. [HK] Theorem 7.8). The extension approach does not have this restriction to separable value spaces, so there is no problem with path dependent integrals, and we can talk about processes up to indistinguishability by using the topology of uniform convergence, or uniform convergence on bounded sets, on their trajectories.

Let D be the space of r.c.l.l. functions $[0,\infty) \to \mathbf{R}^d$, $d \in \mathrm{N}$, with the uniform topology, and let $\boldsymbol{\mathcal{D}}$ be the natural filtration on D. $\mathcal{D}_t$ is generated by those F for which $x \in F$ depends only on x^t, which is x stopped at t,

$$x^t(s) = x(s \wedge t).$$

Following [JM], we say that $h : \Omega \times [0,\infty) \times D \to \mathbf{R}^{d\times m}$ (which we consider officially to be a random variable $\Omega \to (([0,\infty) \times D) \to \mathbf{R}^{d\times m})$, with the uniform topology on trajectories) is predictable if it is measurable with respect to the predictable σ-field, which is generated by sets of the form

$$F \times (s,t] \times G,$$

where $F \in \mathcal{F}_s$ and $G \in \mathcal{D}_s$. We get the following, cleaner version of [HK] Theorem 7.8.

5.1 THEOREM. *Let $(\Omega, \mathcal{F}, P, \boldsymbol{\mathcal{F}})$ be an adapted space, with h as above and Z an $\mathbf{R}^m$-valued semimartingale on Ω and X, K, and W processes on Ω such that*

$$X_t = K_t + \int_0^t h(s,X)dZ_s.$$

Let $(\Lambda, \mathcal{G}, Q, \boldsymbol{\mathcal{G}})$ be another space with processes h', Z', K', and W' such that

$$(h, Z, K, W, \boldsymbol{\mathcal{F}}) \equiv_{AD} (h', Z', K', W' \boldsymbol{\mathcal{G}}). \tag{5.1.1}$$

Then Λ has an extension on which h' is predictable and Z' is a semimartingale, and which carries a solution X' to

$$X'_t = K'_t + \int_0^t h'(s,X')dZ'_s,$$

such that

$$(X, h, Z, K, W, \boldsymbol{\mathcal{F}}) \equiv_{AD} (X', h', Z', K', W' \boldsymbol{\mathcal{G}}).$$

(We have written K' instead of $\dot{K}'$, etc., omitting the dots.)

In [H1] it was shown that it is not necessary to extend the space to make Z' a semimartingale, and the proof given there for the separable target space case can be extended to show that h' is predictable without extending the space.

PROOF: Amalgamate Ω and Λ over the isomorphism given by (5.1.1). Because the uniform topology was used for trajectories of h, h and h' are indistinguishable on the amalgamated space. Likewise K and K', and Z and Z' are indistinguishable on the amalgamation. So the desired X' is just X.

6. SATURATED SPACES FROM EXTENSIONS

The approach of [HK] was based on saturated spaces. We have shown in the preceding sections that suitable forms of the results in that paper can be advantageously rephrased in terms of extensions. In this section, we sketch a proof that saturated spaces themselves can be obtained using the Adjunction Theorem.

Let $\boldsymbol{\Omega} = (\Omega, \mathcal{F}, P, \boldsymbol{\mathcal{F}})$ be an adapted space. $\boldsymbol{\Omega}$ is *saturated* if it satisfies a restriction of the Adjunction Theorem without having to be extended. That is, if $(X, \boldsymbol{\mathcal{F}})$ is a process on $\boldsymbol{\Omega}$, and $(X', Y', \boldsymbol{\mathcal{F}}')$ is a process on another space such that

$$(X, \boldsymbol{\mathcal{F}}) \equiv_{AD} (X', \boldsymbol{\mathcal{F}}'),$$

X, X' and Y' all taking values in a separable space, then there is a process Y on $\mathbf{\Omega}$ such that

$$(X,Y,\mathcal{F}) \equiv_{AD} (X',Y',\mathcal{F}').$$

This is apparently weaker than the definition in [HK]. We will take up that point later.

A saturated space is a kind of universal object familiar in algebra, universal algebra, and model theory. A good exposition in model theory is [S] 16.4, though the usual proof that any field has an algebraic closure contains the main ideas. Existence of saturated spaces can be proved from the Adjunction Theorem by imitating [S] 16.4 and using the following result at limit ordinal steps.

6.1 Extension Chain Theorem. *Let $\mathbf{\Omega}_\beta$, $\beta < \alpha$, alpha an ordinal number, be a family of adapted spaces such that for each $\gamma < \beta < \alpha$, $\mathbf{\Omega}_\beta$ is an extension of $\mathbf{\Omega}_\gamma$ and in fact,*

$$\Omega_\beta = \Omega_0 \times \prod_{\gamma<\beta} \Lambda_\gamma.$$

Then $\mathbf{\Omega}_\beta$, $\beta < \alpha$ have a common extension.

Proof: Let $\Omega = \Omega_0 \times \prod_{\beta<\alpha} \Lambda_\beta$, and give it the probabilistic structure induced by that of the $\mathbf{\Omega}_\beta$'s. The resulting $\mathbf{\Omega}$ is the required extension.

This method does not require the value spaces to be separable, only that the value spaces permitted have dense sets of cardinality not more than some fixed bound. This guarantees that there is only a set, rather than a proper class, of target spaces (up to homeomorphism) to be considered.

The definition of saturated space given here is not the same as that given in [HK], which was as follows. Let $T \subseteq \mathbf{R}$. Given a filtration $\mathcal{F}$, let $\mathcal{F}^T$ be the smallest filtration such that $\mathcal{F}^T_t = \mathcal{F}_t$, $t \in T$, and $\mathcal{F}^T_\infty = \mathcal{F}_\infty$. Say that $\mathbf{\Omega}$ is T-saturated if it satisfies the definition of saturation with $\mathcal{F}^T$ and $\mathcal{F}'^T$ replacing $\mathcal{F}$ and $\mathcal{F}'$. [HK] defined saturation as T-saturation for each countable T. This definition of saturation is actually equivalent to the one we have given. In any case, Theorem 6.3 generalizes the Adjunction Theorem so that T-saturated (for all T) spaces could be constructed by the method outlined above.

6.2 Lemma. *If $\mathcal{F}$ is a right continuous filtration, and for each t, $\mathcal{H}$ and $\mathcal{F}_t$ are conditionally independent given $\mathcal{F}_0$, then $(\mathcal{F}_t \vee \mathcal{H})$ is right continuous.*

Proof: It suffices to show that for $H \in \mathcal{H}$ and $F \in \mathcal{F}_\infty$, the martingale $P(HF \mid \mathcal{F}_t \vee \mathcal{H})$ has a right continuous version. One verifies from the definition that

$$P(HF \mid \mathcal{F}_t \vee \mathcal{H}) = 1_H P(F \mid \mathcal{F}_t).$$

The right hand side has a right continuous version by right continuity of $\mathcal{F}$.

6.3 Lemma. *If $(X,\mathcal{F})$ and $(X',Y',\mathcal{F}')$ are processes with filtration such that $(X,\mathcal{F}) \equiv_{AD} (X',\mathcal{F}')$ on T, then there is an extension $\mathcal{G}$ of $\mathcal{F}$ and a process $\dot{Y}$ on $\mathcal{G}$ such that*

$$(\dot{X},\dot{Y},\mathcal{G}^T) \equiv_{AD} (X',Y',\mathcal{F}'^T) \tag{6.3.1}$$

PROOF: The proof of Theorem 3.2 goes through with T as the parameter set of the filtrations, to produce a filtration $\mathcal{H}$ such that $\mathcal{H}^T = \mathcal{H}$, $\mathcal{F}^T \prec \mathcal{H}$, and (6.3.1) holds with $\mathcal{H}$ replacing $\mathcal{G}^T$. Let $\mathcal{G} = (\mathcal{F}_t \vee \mathcal{H}_t)$. If s is a right limit point of T, then $\mathcal{G}_s = \mathcal{H}_s$ by right continuity of $\mathcal{H}$. Otherwise, $\mathcal{G}$ is right continuous at s by Lemma 6.3. Hence $\mathcal{G}$ is right continuous, and hence a filtration. By definition, $\mathcal{G}^T = \mathcal{H}$, so (6.3.1) holds.

6.4 THEOREM. *A filtration $\mathcal{F}$ is saturated iff it is T-saturated for every $T \subseteq \mathbf{R}$.*

PROOF: The right-to-left implication follows by considering T dense in $\mathbf{R}$ and applying right continuity of the filtration. To prove the other direction, assume that $\mathcal{F}$ is saturated, fix $T \subseteq \mathbf{R}$, and show that $\mathcal{F}$ is T-saturated. Suppose X is a process on $\mathcal{F}$, and X' and Y' are processes on $\mathcal{F}'$ such that $(X,\mathcal{F}) \equiv_{AD} (X',\mathcal{F}')$ on T. Carrying out the extension in Theorem 6.3 yields $(X'',Y'',\mathcal{F}'')$ such that

$$(X,\mathcal{F}) \equiv_{AD} (X'',\mathcal{F}'')$$

and

$$(X',Y',\mathcal{F}'^T) \equiv_{AD} (X'',Y'',\mathcal{F}''^T)$$

By saturation of $\mathcal{F}$, there is Y such that

$$(X,Y,\mathcal{F}) \equiv_{AD} (X'',Y'',\mathcal{F}'')$$

It follows that

$$(X,Y,\mathcal{F}^T) \equiv_{AD} (X',Y',\mathcal{F}'^T)$$

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