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Séminaire de probabilités (Strasbourg), tome 26 (1992), p. 533-539
<http://www.numdam.org/item?id=SPS_1992__26__533_0>

A NOTE ON THE ENERGY INEQUALITIES FOR INCREASING PROCESSES

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1. Introduction.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$ which satisfies the usual conditions. It is well-known that if $A=(A_t)_{t\geq 0}$ is an adapted right continuous increasing process whose left potential is bounded by a constant $c>0$, then

$$E[A_\infty^n] \leqq n!c^{n-1}E[A_\infty] \leqq n!c^n$$

holds for every integer $n \geqq 0$. This is called the "energy inequality".

In this short note, we establish the more general inequality

$$E[\Phi(A_\infty)] \leqq \frac{1}{c}\left(\int_0^\infty \Phi(cs)\, e^{-s}\, ds\right) E[A_\infty] \tag{1}$$

where Φ is a convex function on $\mathbb{R}_+$ such that $\lim_{t\downarrow 0}\Phi(t)=\Phi(0)=0$ (which is not necessarily positive). (1) is used for an alternative proof of the theorem on the equivalence of BMO_Φ - norms of martingales given in [1].

Dellacherie and Meyer in [3, p.189] investigated the case where $\Phi(t)=t^p$, $p>0$: their proof of the inequality

$$E[A_\infty^p] \leqq \Gamma(p+1)\, c^p \tag{2}$$

contains an unavoidable error. In fact, (2) is valid only for $p \geqq 1$, though they asserted otherwise : for $p<1$, the constant process $A_t=c$ does not satisfy (2). Their proof is based on the following assertion : if g is a convex function on the interval $[1, +\infty[$ such that

$$g(n) = \log\Gamma(n) \quad \text{for every integer } n \geqq 1 \;; \tag{3}$$

$$g(x+1) - g(x) \leqq \log\Gamma(x) \quad \text{for every } x \in [1, +\infty[, \tag{4}$$

then $g(x) \leqq \log\Gamma(x)$ holds for every $x \geqq 1$. But the following example shows that this is false. For $x \geqq 1$, we set

$$g(x) = (\log[x])(x - [x]) + \log \Gamma([x]) ,$$

where $[x]$ denotes the integer part of x. It is clear that g is convex and satisfies (3) and (4), but unless x is an integer, we have $g(x) > \log \Gamma(x)$. Actually, if f is a convex function satisfying (3) and (4), then $\log \Gamma \leqq f \leqq g$ (cf. [2, Chap.7, §1, no.1, Prop.1]).

2. Analytic results.

In this section, we shall give an upper bound of the integral of $\Phi \circ f$, where f is an increasing BMO - function on the interval $[0, 1[$ and Φ is as in (1).

For each right continuous integrable function f on $[0, 1[$ (which is not necessarily increasing), we define the function $f^{\#}$ by

$$f^{\#}(t) = \frac{1}{1-t} \int_t^1 (f(s) - f(t))ds , \qquad t \in [0, 1[.$$

Note that f is uniquely determined by $f^{\#}$ and $\int_0^1 f(t)dt$. In fact, if $f_1^{\#} = f_2^{\#}$, then the function

$$F(t) = \int_t^1 (f_1(s) - f_2(s))ds, \qquad t \in [0, 1[$$

is the unique solution of the equation $F(t) = F(0) - \int_0^t (1-s)^{-1} F(s)ds$ and hence we have $F(t) = F(0)(1-t)$. It follows that $f_1 - f_2 \equiv F(0)$ and therefore $f_1 = f_2$ if $\int f_1 dt = \int f_2 dt$.

Furthermore f can be expressed by $f^{\#}$ as follows :

<u>Lemma 1</u>. Let f be a right continuous integrable function on $[0, 1[$. If $f^{\#}$ is integrable over $[0, 1[$, then

$$f(t) = \int_0^t (1-s)^{-1} f^{\#}(s)ds + \alpha - f^{\#}(t), \qquad t \in [0, 1[, \tag{5}$$

where $\alpha = \int_0^1 f(t)dt$. In particular, if $\log(\frac{1}{1-t}) f(t)$ is integrable or f^p is integrable for some $p > 1$, then $f^{\#}$ is integrable and (5) holds.

Proof. Let g(t) be the function defined by the right-hand side of (5). Using Fubini's theorem, we easily verify that $g^{\#} = f^{\#}$ and $\int_0^1 g(t)dt = \int_0^1 f(t)dt$. Hence we obtain $f = g$, as noted above.

If the function $\log(\frac{1}{1-t})\, f(t)$ is integrable, then

$$\int_0^1 |f^{\#}(t)|dt \leqq \int_0^1 \log(\frac{1}{1-t}) \circ |f(t)|dt + \int_0^1 |f(t)|dt < +\infty \quad .$$

If f^p is integrable for some $p > 1$, then $\log(\frac{1}{1-t})\, f(t)$ is also integrable by Hölder's inequality. Thus the proof is complete. []

Now let Φ be a convex function on $[0, +\infty[$ such that $\lim_{t\downarrow 0} \Phi(t) = \Phi(0) = 0$ and let φ be its right-hand derivative. Note that if Φ is a positive convex function such that $\Phi(0) = 0$, it is necessarily right continuous at 0, that is, the required condition is satisfied. In general, we cannot affirm that $\varphi(0) > -\infty$, but in the following lemma, we assume this.

Lemma 2. Let Φ and φ be as above and let f and g be positive Borel functions on [0, 1[. If $\varphi(0) (= \lim_{t\downarrow 0} \varphi(t)) > -\infty$ and f, g satisfy the conditions

$$\int_0^1 f(t)\, \varphi\circ f(t)dt \leqq \int_0^1 g(t)\, \varphi\circ f(t)dt, \qquad \int_0^1 f(t)\, \varphi\circ f(t)dt < +\infty$$

then

$$\int_0^1 \Phi\circ f(t)dt \leqq \int_0^1 \Phi\circ g(t)dt \quad . \tag{6}$$

This is a well-known lemma for positive convex functions Φ ([3, p.180]). The proof is almost the same as that of the case where Φ is positive. By the formula for integration by parts, we have for $u, v \geqq 0$,

$$v\varphi(u) \leqq \int_{]0,\, u]} t\, d\varphi(t) + \Phi(v) \quad ;$$

the equality holds if $u = v$. From this it follows that

$$\int_0^1 f(t)\, \varphi\circ f(t)dt = \int_0^1 dt \left(\int_0^\infty s\, I_{\{ f(t) \geqq s \}}\, d\varphi(s) \right) + \int_0^1 \Phi\circ f(t)dt \quad ;$$

$$\int_0^1 g(t)\, \varphi\circ f(t)dt \leqq \int_0^1 dt \left(\int_0^\infty s\, I_{\{ f(t) \geqq s \}}\, d\varphi(s) \right) + \int_0^1 \Phi\circ g(t)dt \quad .$$

To prove (6), we may assume that $\int_0^1 \Phi\circ f(t)dt > -\infty$ and $\int_0^1 \Phi\circ g(t)dt < +\infty$. Then all the integrals in the preceding inequalities are finite, and (6) follows.

Proposition 3. Let Φ be as in (1) and f be a positive right continuous increasing function on [0, 1[. If $f^{\#} \leqq 1$, then

$$\int_0^1 \Phi\circ f(t)\,dt \leqq \left(\int_0^\infty \Phi(t)\,e^{-t}\,dt\right)\left(\int_0^1 f(t)\,dt\right). \tag{7}$$

Note. As Φ is negative or bounded to the below on [0, 1[, integrals in (7) make sense.

Proof. First let ψ be a bounded increasing function on [0, 1[. From (5) and Fubini's theorem, it follows that

$$\int_0^1 f(t)\psi(t)dt = \int_0^1 f^{\#}(t)\psi^{\#}(t)dt + \left(\int_0^1 f(t)dt\right)\left(\int_0^1 \psi(t)dt\right). \tag{8}$$

Now let g be an integrable function on [0, 1[such that $f^{\#} \leqq g^{\#}$, $g^{\#}$ is integrable, and $\int_0^1 g(t)dt = \int_0^1 f(t)dt$. Since (8) is also valid for g, we have

$$\int_0^1 f(t)\psi(t)dt \leqq \int_0^1 g(t)\psi(t)dt .$$

If the right-hand derivative φ of Φ is bounded, we can set $\psi = \varphi\circ f$. Then by Lemma 2 we have

$$\int_0^1 \Phi\circ f(t)dt \leqq \int_0^1 \Phi\circ g(t)dt . \tag{9}$$

By the monotone convergence theorem, we obtain (9) for any Φ as in (1).

To prove (7), it is sufficient to show that (9) applies to the function $g(t) = \log^+\{\alpha(1-t)^{-1}\}$, where $\alpha = \int_0^1 f(t)dt$. So we must verify that $\int_0^1 g(t)dt = \alpha$ and $f^{\#}(t) \leqq g^{\#}(t) = \alpha(1-t)^{-1} \wedge 1$. But the first condition is obvious and the other follows from the definition of $f^{\#}$ and the inequality $f^{\#} \leqq 1$. Thus the proof is complete. []

3. Application.

The following theorem is an easy consequence of Proposition 3, and as remarked later, they are equivalent.

Theorem 4. Let $A=(A_t)_{t\geqq 0}$ be an adapted, right continuous, incrasing process (resp. predictable, right continuous, increasing process which is zero at $t=0$), and let Φ be a convex function on $[0,+\infty[$ such that $\lim_{t\downarrow 0}\Phi(t)=\Phi(0)=0$. If

$$(10)\qquad E[A_\infty - A_{T-}\,|\,F_T] \leqq c \quad \text{a.s.} \qquad (\text{resp. } E[A_\infty - A_T\,|\,F_T] \leqq c \quad \text{a.s.})$$

holds for every stopping time (resp. predictable stopping time) T, then

$$(1)\qquad E[\Phi(A_\infty)] \leqq \frac{1}{c}\left(\int_0^\infty \Phi(ct)\, e^{-t}\, dt\right) E[A_\infty] \quad .$$

Corollary. Let $A=(A_t)$ be as in Theorem 4. Then

$$E[A_\infty^p] \leqq c^{p-1}\,\Gamma(p+1)\,E[A_\infty] \leqq c^p\,\Gamma(p+1) \qquad (1 \leqq p < +\infty);$$

$$E[A_\infty^p] \geqq c^{p-1}\,\Gamma(p+1)\,E[A_\infty] \qquad (0 < p < 1);$$

$$E[\exp(\alpha A_\infty)] \leqq \frac{\alpha}{1-c\alpha}\,E[A_\infty] + 1 \leqq \frac{1}{1-c\alpha} \qquad (0 \leqq \alpha < 1/c).$$

Previous to proving Theorem 4, we note that for every $A=(A_t)$ as in the statement, we have

$$(11)\qquad \int_{\{A_\infty > \lambda\}} (A_\infty - \lambda)\, dP \leqq c\, P\{A_\infty > \lambda\} \quad \text{for all } \lambda > 0 \;,$$

(for the proof see [4, p.346]). We use only this inequality to prove (1), so (1) is true for every random variable A_∞ satisfying (11).

Proof of Theorem 4. Without loss of generality, we can assume that $c=1$. Let f be the (unique) right continuous increasing function on $[0,1[$ with the same distribution as A_∞, with respect to the Lebesgue measure. We set, for each $t \in [0,1[$,

$$\tau(t) = \inf\{s \in [0,1[\,;\ f(s) > f(t)\} \wedge 1 \;.$$

It is obvious that $]\tau(t),1[\, \subset \{s : f(s) > f(t)\} \subset [\tau(t),1[$ and hence that

$$P\{A_\infty > f(t)\} = 1 - \tau(t) \qquad \text{and} \qquad \int_{\{A_\infty > f(t)\}} A_\infty\, dP = \int_{\tau(t)}^1 f(s)\,ds \;.$$

Since the function $t \longrightarrow \frac{1}{1-t}\int_t^1 f(s)ds$ is increasing and $\tau(t) \geqq t$, we have

$$f^{\#}(t) \leqq \frac{1}{1-\tau(t)}\int_{\tau(t)}^{1} (f(s)-f(t))\, ds$$

$$= \frac{1}{P\{A_\infty > f(t)\}} \int_{\{A_\infty > f(t)\}} (A_\infty - f(t))\, dP \leqq 1 ,$$

where the last inequality follows from (11) with $c = 1$. In order to obtain (1), it only remains to apply Proposition 3 to this f. []

Remark. Although Theorem 4 is probabilistic, it is equivalent to Proposition 3, which is purely analytic. To see this, let $\Omega = [0, 1[$, dP be the Lebesgue measure on Ω, and F_t be the augmentation of the σ-field generated by the set $]t\wedge 1, 1[$ and the Borel subsets of $[0, t\wedge 1]$. If f is a function as in Proposition 3, the increasing process $A_t(\omega) = f(t\wedge\omega)$ satisfies (10), and hence (7) follows from (1). Furthermore, if we set $f(t) = c\log^+(\alpha(1-t)^{-1})$ for $\alpha \in]0, 1]$ and $c > 0$ and if we define A_t as above, then (A_t) satisfies (10) and the equality holds in (1). Therefore (1) cannot be improved any more. There is a more interesting such example in [3].

We now give an application of Theorem 4. Using general Young functions, Bassily and Mogyoródi in [1] introduced the BMO_Φ-norm corresponding to Φ, and proved that it is equivalent to the usual BMO_1-norm, if Φ has a finite power. Their proof is elementary, but somewhat complicated. We give a more straightforward proof of it.

Let Φ be an increasing convex function on $[0, +\infty[$ such that $\Phi(0) = 0$, and let $M = (M_t)_{t\geqq 0}$ be a right continuous, uniformly integrable martingale. We set

$$\|M\|_{BMO_\Phi} = \inf\{\lambda > 0 : \sup_T \|E[\Phi(\frac{1}{\lambda}|M_\infty - M_{T-}|)|F_T]\|_{L^\infty} \leqq 1\} ,$$

where the supremum is taken over all stopping times. The original definition of the BMO_Φ-norm is seemingly different from this definition, but they are identical ; see the proof of Theorem 6 in [1].

Theorem 5 (Bassily and Mogyoródi). Let Φ be an increasing convex function on $[0, +\infty[$ such that $\Phi(0) = 0$. If $\int_0^\infty \Phi(ct)\, e^{-t}\, dt < +\infty$ for some constant

$c > 0$, then, for every right continous uniformly integrable martingale $M = (M_t)$ we have

$$c_\Phi \|M\|_{BMO_1} \leq \|M\|_{BMO_\Phi} \leq C_\Phi \|M\|_{BMO_1},$$

where $c_\Phi > 0$ and $C_\Phi > 0$ depend only on Φ, and $\|\cdot\|_{BMO_1}$ denotes the norm corresponding to the function $\Psi(t) = t$.

Proof. We prove the right-hand inequality only : the left-hand inequality is an easy consequence of Jensen's inequality. We set $M_t^* = \sup_{s \leq t} |M_s|$, $t \geq 0$. It is well-known (e.g. [3, p.193]) that

$$E[M_\infty^* - M_{T-}^* | \mathcal{F}_T] \leq 4 \|M\|_{BMO_1} .$$

Let $C_\Phi^{-1} = \inf\{c > 0 : \int_0^\infty \Phi(ct)\, e^{-t}\, dt > 1\}$. It then follows from the hypotheses that $0 < C_\Phi < +\infty$. By Theorem 4, setting $\beta = 4\|M\|_{BMO_1}$, we have

$$E[\Phi(\frac{1}{\beta C_\Phi} M_\infty^*)] \leq \frac{1}{\beta} (\int_0^\infty \Phi(\frac{t}{C_\Phi})\, e^{-t}\, dt)\, E[M_\infty^*] \leq 1 .$$

We put this inequality in conditional form in the usual manner (cf. [3, p.190]). Then we have

$$E[\Phi(\frac{1}{\beta C_\Phi} \sup_t |M_{T+t} - M_{T-}|) | \mathcal{F}_T] \leq 1 \quad \text{a.s.}$$

and hence $\|M\|_{BMO_\Phi} \leq \beta C_\Phi = 4 C_\Phi \|M\|_{BMO_1}$. This completes the proof. []

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