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Infinitesimal Behaviour of a Continuous Local Martingale

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a. Summary

We investigate the behaviour of a continuous local martingale $(M_t)_{t\geq 0}$ in the neighbourhood of $t = 0$. We prove that under suitable conditions $\frac{M_t}{\sqrt{t}}$ tends in law to a normal variable as $t \to 0$. A convergence theorem to Brownian motion as well as an application to continuous Markov processes are also given.

b. The main result

In this paragraph we suppose that $(M_t)_{t\geq 0}$ is a continuous d-dimensional local martingale with $M_0 = 0$. The filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions and $\mathcal{F}_0$ is degenerate. The process $(W_t)_{t\geq 0}$ is a standard d-dimensional Wiener process defined on some probability space. The $\cdot$ denotes the obvious Euclidean inproduct between vectors in $\mathbf{R}^d$. If u is a vector in $\mathbf{R}^d$ then $|u|$ denotes the Euclidean norm of u. We use the results and the notation of [2] and [3].

Theorem:

If the continuous local martingale $(M_t)_{t \geq 0}$, $M_0 = 0$ satisfies

$\frac{1}{t} <M^i,M^j>_t \to a_{ij}$ a.e. as $t \to 0$.

then for all $0 = s_0 < s_1 ... < s_n = 1$ we have

$$\left(\frac{M_{s_1 t}}{\sqrt{t}}, \ldots\ldots\ldots\ldots, \frac{M_{s_n t}}{\sqrt{t}}\right) \to \left(A^{1/2}W_{s_1}, \ldots, A^{1/2}W_{s_n}\right) \quad \text{in law as } t \to 0.$$

Here $A^{1/2}$ is the symmetric positive definite square root of the positive definite symmetric matrix $A=(a_{ij})$ and W is a standard d-dimensional Wiener process.

Proof :

We will work with complex valued martingales and we will show that for

$(u_1, \dots, u_n) \in (\mathbb{R}^d)^n$

$$E\left[\exp \frac{i}{\sqrt{t}}\left\{u_1\cdot(M_{s_1t}) + u_2\left(M_{s_2t} - M_{s_1t}\right) + \dots\dots\dots + u_n\left(M_{s_nt} - M_{s_{n-1}t}\right)\right\}\right]$$

$$\to \exp\left(-\frac{1}{2}\left(s_1 u_1\cdot Au_1 + (s_2 - s_1)\, u_2\cdot Au_2 + \dots + (s_n - s_{n-1})\, u_n\cdot Au_n\right)\right)$$

Let $\sigma = \inf\{t \mid \operatorname{trace}(\langle M,M\rangle_t) \ge t\,(\operatorname{trace}(A)+1)\}$

Since $\frac{1}{t}\langle M,M\rangle_t \to A$ as $t \to o$ we certainly have $\sigma > o$ a.e.

Stopping M at time σ, the difference between

$$E\left[\exp \frac{i}{\sqrt{t}}\left\{u_1\left(M_{s_1t\wedge\sigma}\right)u_1 + u_2\left(M_{s_2t\wedge\sigma} - M_{s_1t\wedge\sigma}\right)u_2 + \dots + u_n\left(M_{s_nt\wedge\sigma} - M_{s_{n-1}t\wedge\sigma}\right)u_n\right\}\right]$$

and

$$E\left[\exp \frac{i}{\sqrt{t}}\left\{u_1\left(M_{s_1t}\right)u_1 + u_2\left(M_{s_2t} - M_{s_1t}\right)u_2 + \dots + u_n\left(M_{s_nt} - M_{s_{n-1}t}\right)u_n\right\}\right]$$

is clearly bounded by $2\,P[\sigma < t]$,

and hence we only have to prove the theorem for M^σ instead of M. From now on we therefore suppose w.l.o.g. that $\langle M,M\rangle_t \le t\,(\operatorname{trace}(A)+1)$.

Let $h = u_1 1_{[0,s_1t]} + u_2 1_{]s_1t,s_2t]} + \dots + u_n 1_{]s_{n-1}t,s_nt]}$.

h is a deterministic process and we have to calculate

$$E\left[\exp\left(i\int_0^t h(s)\cdot dM_s\right)\right]$$

Since the martingale M is continuous we know from Itô's lemma that the process (indexed by v):

$$\exp\left(i\left(\int_0^v h(s)\cdot dM_s\right) + \frac{1}{2}\int_0^v h(s)\cdot d\langle M,M\rangle_s\, h(s)\right)$$

is a martingale.

Therefore for all $(u_1, \dots, u_n) \in (\mathbb{R}^d)^n$

$$E\left[\exp\left\{i\left(u_1\cdot M_{s_1t} + u_2\cdot\left(M_{s_2t} - M_{s_1t}\right) + \dots + u_n\cdot\left(M_{s_nt} - M_{s_{n-1}t}\right)\right)\right.\right.$$

$$+ \frac{1}{2} u_1 \cdot \langle M,M\rangle_{s_1 t} u_1 + \ldots + \frac{1}{2} u_n \cdot \left(\langle M,M\rangle_{s_n t} - \langle M,M\rangle_{s_{n-1} t}\right) u_n \Big\}\Big] = 1 .$$

Replacing u_i by $\frac{u_i}{\sqrt{t}}$ gives

$$E\left[\exp i \left\{ \left(u_1 \cdot \frac{M_{s_1 t}}{\sqrt{t}} + u_2 \cdot \left(\frac{M_{s_2 t}}{\sqrt{t}} - \frac{M_{s_1 t}}{\sqrt{t}} \right) + \ldots + u_n \cdot \left(\frac{M_{s_n t}}{\sqrt{t}} - \frac{M_{s_{n-1} t}}{\sqrt{t}} \right) \right) \right.\right.$$

$$\left.\left. + \frac{1}{2t}\left(u_1 \cdot \langle M,M\rangle_{s_1 t} u_1 + \ldots + u_n \cdot \left(\langle M,M\rangle_{s_n t} - \langle M,M\rangle_{s_{n-1} t} \right) u_n \right) \right\}\right] = 1.$$

We will use this equality to prove the theorem. Let K denote the quantity

$$K = \exp\left(\frac{1}{2}\left(s_1 u_1 \cdot A u_1 + (s_2 - s_1)\, u_2 \cdot A u_2 + \ldots + (s_n - s_{n-1})\, u_n \cdot A u_n\right)\right) .$$

$$E\left[\exp i \sum_{m=1}^{n} \frac{u_m}{\sqrt{t}} \cdot \left(M_{s_m t} - M_{s_{m-1} t} \right)\right]$$

$$= E\left[\exp\left(i \sum_{m=1}^{n} \frac{u_m}{\sqrt{t}} \cdot \left(M_{s_m t} - M_{s_{m-1} t} \right) + \frac{1}{2t} \sum_{m=1}^{n} u_m \cdot \left(\langle M,M\rangle_{s_m t} - \langle M,M\rangle_{s_{m-1} t} \right) u_m \right) \cdot \right.$$

$$\left. \exp\left\{ \frac{1}{2t} \sum_{m=1}^{n} (s_m - s_{m-1})\, u_m \cdot A u_m - \frac{1}{2t} \sum_{m=1}^{n} u_m \cdot \left(\langle M,M\rangle_{s_m t} - \langle M,M\rangle_{s_{m-1} t} \right) u_m \right\}\right]$$

The integrand is bounded since $\frac{1}{v} \langle M,M\rangle_v \leq (\text{trace}\,(A)+1)$ for all v. We therefore can apply Lebesgue's theorem

$|K - 1| \leq$

$$\left| E\left[\exp\left\{ i \sum_{m=1}^{n} \frac{u_m}{\sqrt{t}} \cdot \left(M_{s_m t} - M_{s_{m-1} t} \right) + \frac{1}{2t} \sum_{m=1}^{n} u_m \cdot \left(\langle M,M\rangle_{s_m t} - \langle M,M\rangle_{s_{m-1} t} \right) u_m \right\} \cdot \right.\right.$$

$$\exp\left\{\frac{1}{2t}\sum_{m=1}^{n}(s_m - s_{m-1})\, u_m\cdot A u_m - \frac{1}{2t}\sum_{m=1}^{n} u_m\left(<M,M>_{s_m t} - <M,M>_{s_{m-1}t}\right)u_m\right\} - 1\Big]\Big|$$

$$\leq E\left[\exp\left\{\sum_{m=1}^{n}\frac{1}{2t}\sum_{m=1}^{n} u_m\left(<M,M>_{s_m t} - <M,M>_{s_{m-1}t}\right)u_m\right\}\cdot\right.$$

$$\left.\left|\exp\left\{\frac{1}{2t}\sum_{m=1}^{n}\left((s_m - s_{m-1})\, u_m\cdot A u_m - u_m\left(<M,M>_{s_m t} - <M,M>_{s_{m-1}t}\right)u_m\right)\right\} - 1\right|\right].$$

Since $(s_m - s_{m-1})\, u_m\cdot A u_m - \frac{1}{t}\, u_m\left(<M,M>_{s_m t} - <M,M>_{s_{m-1}t}\right)u_m \to 0$ as $t \to 0$

we obtain $K \to 1$ as $t \to 0$. qed.

Remarks

1. If the 1-dimensional martingale $(M_t)_{t\geq 0}$ is such that $\left(\frac{M_t}{\sqrt{t}}\right)_{t\geq 0}$ is bounded in L^p and $\frac{1}{t}<M,M>_t \to c$ then we find for all $r < p$

$$E\left[\left|\frac{M_t}{\sqrt{t}}\right|^r\right] \to \gamma(r)\, c^{r/2}.$$

Indeed $\frac{M_t}{\sqrt{t}}$ tends to a normal variable with mean zero and variance c. The theorem now follows with $\gamma(r) = \sqrt{\frac{2}{\pi}}\int_0^\infty x^r e^{-x^2/2}\, dx$.

Theorem.

Let $(M_t)_{t\geq 0}$ be a d-dimensional continuous local martingale as in the previous theorem.
Let A be as in the previous theorem.

Put $B^t_s = \frac{M_{st}}{\sqrt{t}}$ for $s \in [0,1]$.

Then $B^t \xrightarrow{law} A^{1/2}\, W$ as $t \to 0$,

where W is a standard d-dimensional Wiener process.

Proof.

Since the finite dimensional distributions converge we only need to prove that the image laws on $\mathcal{C}[0,1]$, the space of d-dimensional continuous functions on $[0,1]$, form a tight family as $t \to 0$. We will use the Aldous' criterion [1].

As in the previous theorem we may suppose that the martingale is bounded and satisfies

$$\left| \frac{1}{t} <M^i,M^j>_t \right| \leq c$$

for a fixed constant c.

(a) We first verify the uniform boundedness

$$P\left[\sup_{s\in[0,1]} \left|B_s^t\right| > k\right] = P\left[\sup_{s\in[0,1]} \left|\frac{M_{st}}{\sqrt{t}}\right| > k\right]$$

$$= P\left[\sup_{s\in[0,1]} \left|M_{st}\right| > k\sqrt{t}\right] \leq (t k^2)^{-1} E\left[\text{trace} <M,M>_t\right] \leq \frac{d c}{k^2}$$

this quantity tends to zero uniformly in t.

(b) For fixed stopping times $S \leq T \leq S+\theta$, (with respect to the filtration $(\mathcal{F}_{st})_{0\leq s\leq 1}$) we have

$$P\left[\left|B_S^t - B_T^t\right| \geq \varepsilon\right]$$

$$= P\left[\left|M_{St} - M_{Tt}\right| \geq \varepsilon\sqrt{t}\right]$$

$$\leq (t\varepsilon^2)^{-1} E\left[\left|M_{St} - M_{Tt}\right|^2\right]$$

$$\leq (t\varepsilon^2)^{-1} E\left[\text{trace}\left(<M,M>_{Tt} - <M,M>_{St}\right)\right]$$

$$\leq (t\varepsilon^2)^{-1} E\left[\text{trace}(<M,M>_{(S+\theta)t} - <M,M>_{St})\right]$$

$$\to \frac{\theta}{\varepsilon^2} \text{trace}(A)$$

as $t \to 0$.

Aldous' criterion is therefore satisfied and the theorem is proved. qed.

c. Application to continuous Markov processes

Let E be a locally compact space on which a strongly continuous Feller semi group $(P_t)_{t\geq o}$ is given. We suppose that the domain $\mathcal{D}_A$ of the infinitesimal generator A is an algebra and we denote by $\Gamma(f.g) = A(f g) - f A g - g A f$ the carré du champ operator. The semigroup $(P_t)_{t\geq o}$ is supposed to generate a continuous Markov process with values in E. For $x \in E$ we denote by E_x the corresponding expectation operator. Clearly $X_o = x$ P_x a.e.

Theorem

Let $f_1 \ldots f_n \in \mathcal{D}_A$ and let α is the $n \times n$ matrix consisting of the elements $\alpha_{ij}=\Gamma(f_i, f_j)(x)$.

Let $(B_s^{t1} \dots B_s^{tn})_{0 \le s \le 1}$ denote the n-dimensional process

$$B_s^{ti} = \frac{f_i(X_{ts}) - f_i(x) - \int_0^{st} Af_i(X_u)\,du}{\sqrt{t}} \qquad \text{viewed under } P_x .$$

Then $B^t \to \alpha^{1/2}\, W$ in distribution, where W is a standard n-dimensional Wiener process.

Proof

Let $M_t^i = f_i(X_t) - f_i(x) - \int_0^t Af_i(X_u)\,du$

This is a martingale and $\langle M^i, M^j\rangle_t = \int_0^t \Gamma(f_i, f_j)(X_u)\,du$.

Clearly $\frac{1}{t}\langle M^i, M^j\rangle_t = \frac{1}{t}\int_0^t \Gamma(f_i, f_j)(X_u)\,du \to \Gamma(f_i, f_j)(x)$.

We now can apply the main theorem.

Corollary.

If $f \in \mathcal{D}_A$ then

$$E_x\left[\left|\frac{f(X_t) - f(x)}{\sqrt{t}}\right|^p\right] \to \gamma(p)\,\Gamma(f,f)(x)^{p/2} \qquad \text{as } t \to 0$$

where $\gamma(p) = \sqrt{\frac{2}{\pi}}\int_0^\infty x^p e^{-x^2/2}\,dx$.

Proof.

Putting $M_t = f(X_t) - f(x) - \int_0^t Af(X_u)\,du$, we have $$\langle M,M\rangle_t = \int_0^t \Gamma(f,f)(X_u)\,du$$

Clearly $$\left\| \int_0^t Af(X_u)\,du\,\frac{1}{\sqrt{t}} \right\|_p \to 0 \qquad \text{as } t \to 0,$$

so that $E_x\left[\left|\frac{f(X_t) - f(x)}{\sqrt{t}}\right|^p\right]$ and $E_x\left[\left|\frac{M_t}{\sqrt{t}}\right|^p\right]$ have the same limit

Since $\left\|\frac{M_t}{\sqrt{t}}\right\|_p \le c \left\| \left(\frac{1}{t}\int_0^t \Gamma(f,f)(X_u)\,du \right)^{1/2} \right\|_p$ by the Burkholder inequality, we obtain $\left\|\frac{M_t}{\sqrt{t}}\right\|_p \le c \left(\max_{y \in E} \Gamma(f,f)(y)^{1/2} \right) t^{1/2}$.

We can therefore apply a previous remark for all p between o and ∞. qed.

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