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Generalised transforms, quasi-diffusions, and Désiré André's equation.

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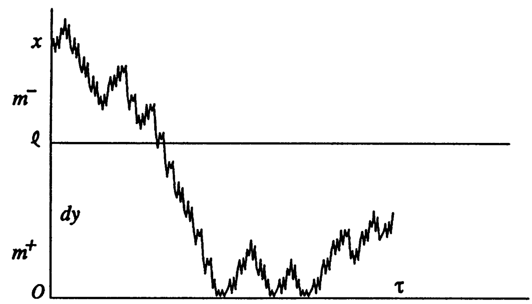
Suppose B_t is a real Brownian motion started at $x > \ell > 0$. Writing $L(a,t)$ for the B_t local time, we define the fluctuating additive functional $V_t = \int L(a,t)m(da)$ and let τ denote its equalisation time $\inf\{t > 0 : V_t > 0\}$. Hypotheses on m are: it is a signed measure on $[0,\infty)$ with Hahn decomposition $m^+ - m^-$ and m^+ is supported on $[0,\ell]$. The problem of computing $\mathbf{P}_x[B_\tau \in dy; \tau < \infty] = \Pi(x,dy)$ was posed in [5].



There are two suggestions; the first comes from [5]. Suppose $f(x) = f_\theta(x)$ is a bounded function satisfying $df_x/dm = -\theta^2 f$ with θ real. Then $f_\theta(B_t)e^{\theta^2 V_t/2}$ is a bounded martingale for $t \le \tau$, and by [12] we know that on the set where τ is infinite V_t explodes negatively. Applying the Doob stopping theorem gives us

$$\int_0^\ell f_\theta(y)\Pi(x,dy) = \mathbf{E}_x\left[f_\theta(B_\tau); \tau < \infty\right] = f_\theta(x) \tag{0.1}$$

Since there are examples [8] where one can solve 0.1 by inspection, we would like to know whether this 'eigenvalue relation' uniquely determines $\Pi(x,dy)$. See [13] for an idea of how 0.1 fits into a more general setting.

The other idea is to derive an equation for $\Pi(x,dy)$. We will use the integral equation named for Désiré André [9], perhaps better known as the first passage relation for Lévy processes. Désiré André's equation is formally of convolution type but it has a singular kernel. In [7] we worked with its Fourier transform. Solving amounts to calculating $\int_0^\ell \mathbf{E}_y\left[e^{-z^2 V_T/2}\right]\Pi(x,dy)$ (T is the first passage time to the boundary point ℓ) and then hoping that this integral transform will uniquely determine $\Pi(x,dy)$.

So both methods for computing $\Pi(x,dy)$ depend on proving a uniqueness result for a certain transform. The first method, embodied in 0.1, necessitates a proof of spectral uniqueness; in section three we will see the connection with the spectrum of the Désiré André equation. The second method has to do with first passage uniqueness. It can be posed for any quasi-diffusion without reference to the problem stated at the beginning, though in general the answer is still unknown. See [11] for an idea of the difficulties involved.

In this note we examine both uniqueness questions under the restriction that m^+ comes from a short string in the sense of Krein [2] — precise conditions are stated at the beginning of the next section. This assumption permits the use of analytic function theory and yields simpler proofs than in [5] (without, however, subsuming the work done there).

As in [5], our method uses Krein's spectral representation for strings but we take zero as base point instead of ℓ. Working with this 'reversed string' has the drawback that Krein's theory applies directly only if the mass at zero is finite. If there is an infinite mass at zero then we need the notion of a 'killed string' as discussed in [4].

The advantage of reversing the string is that it allows us to reduce the two problems to similar conditions, listed at 2.1-2 below. This is important because we prove spectral uniqueness by interpolating from the eigenvalues at 0.1 to the eigenvalues of a string for m^+, which is the spectrum at 2.1. Remark that we do not allow m^+ to charge the boundary point ℓ.

The paper begins with a simplification. Using excursion theory we show that $\Pi(x,.) \ll m^+$. This is a crucial part of the argument: our uniqueness proof uses Krein's spectral theory in $L_1([0,\ell),m^+)$. Section two contains the main results. The first step is to rewrite the first passage problem, formulating it as an eigenvalue relation on the eigenvalues of the positive string (this step is not necessary but it does emphasise the difficulty with 0.1 — it is not a Krein transform). So, modulo the discussion in section four, first passage uniqueness follows from our simplification. From there on the argument is a straightforward extension of results in [2]. The basic idea is that an analytic function of finite type cannot have too many zeros. We show, in effect, that there are (at least) as many eigenvalues at 0.1 as there are points in the spectrum of the positive string.

Section three shows the connection with a singular integral equation. This equation (3.2 below) has Wiener-Hopf structure but we study it using properties of the symbol; one can identify the imaginary zeros of the symbol with the eigenvalues of 0.1.

1. Krein's strings and absolute continuity.

We now formulate our problem more precisely, by using the language of Krein's theory of strings. Notation is adapted from [2], to which we refer the reader for more detail.

The positive mass m^+ is the mass of a Krein string (m^+, k, ℓ). This means that $0, \ell$ are in the closed support of m^+, which is itself a subset of $[0, \ell]$. The case $m^+\{\ell\} = \infty$ is trivial here and will be omitted without further comment. We shall suppose throughout that $\ell + m^+(0, \ell] < \infty$, so if there is a finite mass at zero then the string is *short* in the sense of [2]. The first three sections deal exclusively with this case.

As in [5], the measure m^+ is not allowed to charge the boundary point ℓ. The value of k can be any positive constant or infinity; we can even take $k = 0$ since m^+ does not charge ℓ, but we will not do so here.

The requirements on m^- are less stringent. We allow any positive measure on (ℓ, ∞) and we define

$$D^-(z, x) = \mathbf{E}_x \left[e^{z^2 V_T / 2} \right] \tag{1.1}$$

where $x \geq \ell$ and T is the hitting time of ℓ. This function is decreasing and satisfies $D^-(z, \ell) = 1$.

The first step is to simplify 0.1. We show that $\Pi(x, .) \ll m^+$, thereby reducing uniqueness to a problem in L_1. The proof uses results from [3] on the transition density of quasi-diffusions, applied to the quasi-diffusion X_t with speed measure m^+; probabilistically X_t is the time-change of B_t by the increasing additive functional $V_t^+ = \int L(a, t) m^+(da)$.

For the proof of $\Pi(x, .) \ll m^+$ we look at excursions of B_t *downwards* from the point ℓ. These run in the inverse local time scale at ℓ, denoted here by σ_t. Write $\mathcal{E}_t$ for the excursion process so that a generic excursion path Γ is defined by $\Gamma \circ \mathcal{E}_t = \{B_{s+\sigma_{t-}} : 0 < s < \Delta\sigma_t\}$; an excursion functional is any function defined on the excursions.

At the heart of excursion theory lies the

Master Formula. There is a measure $\mathcal{Q}$ on the space of excursion paths with the property that if $\mathcal{Q}[\mathcal{A}]$ is finite then

$$\sum_{0 < s \leq t} \mathcal{A} \circ \mathcal{E}_s - \mathcal{Q}[\mathcal{A}] t$$

is a $\mathcal{B}_{\sigma_t}$ martingale.

Terminology is from [10]. As written here $\mathcal{A}$ is constant but the result extends. For example, if $\mathcal{A}$ depends (measurably) on time then the right side becomes $\int_0^t \mathcal{Q}[\mathcal{A}_s] ds$. More generally, one can make $\mathcal{A}$ vary predictably in the sense that on the excursion starting at time σ_{t-} the functional may depend on the entire process $\{B_s : 0 < s < \sigma_{t-}\}$. This extension is to be found in [6] where one also finds a more detailed description of the excursion measure $\mathcal{Q}$.

Suppose ξ is the minimum functional on excursion space and take T_ε as the B_t hitting time for $\ell - \varepsilon$. Recalling that τ is the equalisation time for the additive functional V_t,

$$\tau'(\varepsilon, t) = 1_{(\tau > \sigma_{t-})} \left[1_{(\tau > T_\varepsilon)}\, \tau\right] \circ \mathcal{E}_t$$

defines a predictably randomised excursion stopping time.

Before beginning the proof we note that $\mathbf{P}_x[B_\tau = \ell] = 0$; the law of V_T, alias the X_t first passage time to ℓ, has no atoms.

Theorem 1.2 The kernel $\Pi(x, dy)$ is absolutely continuous with respect to m^+.

Proof: Choose $\varepsilon > 0$ arbitrarily, and, using the above definition of $\tau'(\varepsilon, t)$, we write the indicator function

$$\mathrm{Ind}\left[B_\tau \in dy; \exists\, t \text{ with } \sigma_{t-} + T_\varepsilon \circ \theta_{\sigma_{t-}} < \tau < \sigma_t\right] = \sum_{t>0} \mathrm{Ind}[\Gamma_{\tau'(\varepsilon,t)} \in dy; \xi < \ell - \varepsilon] \circ \mathcal{E}_t$$

The right side is now of a form where we can apply Maisonneuve's extension of the Master formula to see that

$$\sum_{0<s\leq t} \mathrm{Ind}[\Gamma_{\tau'(\varepsilon,s)} \in dy; \xi < \ell - \varepsilon] \circ \mathcal{E}_s - \int_0^t \mathcal{Q}[\Gamma_{\tau'(\varepsilon,s)} \in dy; \xi < \ell - \varepsilon] ds$$

is a uniformly integrable B_{σ_t} martingale. Taking the expectation, we use the standard description of the excursion measure in terms of the killed process $\bar{B}_t$ to find

$$\mathbf{P}\left[B_\tau \in dy; \exists\, t \text{ with } \sigma_{t-} + T_\varepsilon \circ \theta_{\sigma_{t-}} < \tau < \sigma_t\right] = \mathbf{E}\left[\int_0^{L(\ell,\tau)} \mathcal{Q}\left[\mathbf{P}_{\varepsilon,V}\left[\bar{B}_\tau \in dy\right]; \xi < \ell - \varepsilon\right] dt\right]$$

where (abuse of notation) we take $V = V(\sigma_{t-} + T_\varepsilon \circ \theta_{\sigma_{t-}})$. Recalling that X_t is the time-change of B_t by V_t^+, we note that if $S = \inf\{t > 0 : V_t^+ = c\}$ then $B_S = X_c$; this allows us to write $\mathbf{P}_\varepsilon[\bar{B}_\tau \in dy]$ in terms of the transition semigroup for X_t. Letting $\varepsilon \downarrow 0$, it follows from [3] that $\Pi(x, dy)$ is absolutely continuous wrt m^+ on $[0, \ell)$.

Remark that 1.2 is valid even for long strings, though for them we need to work with ℓ as the initial (and not the final) point of the string.

Next, we look at some spectral theory. Associated to any string (m^+, k, ℓ) are two basic functions, usually denoted by A and D. We recall the definitions but remark that our conventions differ slightly from [2]. For us $A(z, x)$ is the unique solution of the boundary value problem

$$\frac{d}{dm^+} A_x = z^2 A \quad ; \quad A(z, 0) = 1, \quad A_x(z, 0-) = 0$$

One can show [2] p.163 that for $z > 0$ the function $A(z, .)$ is convex increasing. The complementary solution $D(z, x)$ solves

$$\frac{d}{dm^+} D_x = z^2 D \quad ; \quad D_x(z, 0-) = -1, \quad D(z, \ell) + k D_x(z, \ell) = 0 \qquad (z > 0)$$

$D(z,.)$ is convex, decreasing, and never negative on $[0,\ell)$.

The Green's function for the string (m^+,ℓ,k) is defined, as usual, by $G_z(x,y)=A(z,x)D(z,y)$ for $x<y$ and it has spectral representation

$$G_z(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}A(i\gamma,x)A(i\gamma,y)\frac{\Delta(d\gamma)}{z^2+\gamma^2} \tag{1.3}$$

From now on we simplify notation by writing m in place of m^+, unless there is some possibility of confusion.

For real z, the above kernel defines an operator $G_zu(x)=\int G_z(x,y)u(y)m(dy)$ on $L_2([0,\ell),m)$ which is the inverse of $z^2-\mathcal{G}$ (the latter is defined on the closed subspace $\mathcal{D}_2(\mathcal{G})$ of $\{u: ||u||_2+||\mathcal{G}u||_2<+\infty\}$ determined by the boundary conditions for the string; there we have $(z^2-\mathcal{G})f=z^2f-df_x/dm$). However we want to work on $L_1([0,\ell),m)$. The argument below is adapted from [2].

Theorem 1.4 If $(x,y)\to G_z(x,y)$ is uniformly bounded then G_z is an isomorphism on $L_1([0,\ell),m)$.

Proof: Any function $u\in L_1$ is the limit of $L_2\cap L_1$ functions u_n and, since $\mathcal{G}G_zu_n=z^2G_zu_n-u_n$ converges in L_1, we can define its limit, independently of u_n, as $\mathcal{G}G_zu$. Since G_z satisfies the boundary conditions we see, with obvious notation, that G_z maps L_1 into $\mathcal{D}_1(\mathcal{G})$ and by definition $(z^2-\mathcal{G})G_z=I$. We claim G_z is onto. For the proof, taking $u\in\mathcal{D}_1(\mathcal{G})$ we write $v=G_z(z^2-\mathcal{G})u-u\in\mathcal{D}_1(\mathcal{G})$, so $(z^2-\mathcal{G})v=0$. But if $v(0,z)>0$, then this *function* is monotone increasing and hence $v\notin\mathcal{D}_1(\mathcal{G})$ since it fails to satisfy the boundary condition at ℓ. Replacing v by $-v$ if required, the contradiction shows us that $v=0$. So $G_z: L_1\to\mathcal{D}_1(\mathcal{G})$ is an isomorphism with inverse $z^2-\mathcal{G}$.

There is clearly a problem here if m^+ puts an infinite mass at the origin. We will look into this in section four. However, in the case of short strings the kernel $G_z(x,y)$ is uniformly bounded and 1.4 applies.

2. Uniqueness Results.

In this section we consider the two uniqueness problems posed in the introduction, retaining the assumption that the positive string is short. We begin by showing that they are related; each can be formulated as a spectral uniqueness question, but on different spectra.

Recall that T denotes the first passage time to the boundary point ℓ. Then $\mathbf{E}_y\left[e^{-z^2V_T/2}\right]=A(z,y)/A(z,\ell)$ and we want to study uniqueness of the transform $\mu^*(z)=\int_0^\ell\mathbf{E}_y\left[e^{-z^2V_T/2}\right]\mu(dy)$

where μ is a bounded signed measure. If we suppose that this vanishes then, using $G_z(y,\ell) = A(z,y)D(z,\ell)$ and the spectral representation at 1.3, our hypothesis becomes

$$\int_{[0,\ell)} \mu(dy)\frac{1}{\pi}\int A(i\gamma,y)A(i\gamma,\ell)\frac{\Delta(d\gamma)}{\gamma^2+z^2} = 0$$

For z real the measure $(\gamma^2+z^2)^{-1}\Delta(d\gamma)$ is finite (just put $x = y = 0$ in 1.3), and since

$$\int_{[0,\ell)} |\mu|(dy)\frac{1}{\pi}\int |A(i\gamma,y)A(i\gamma,\ell)|\frac{\Delta(d\gamma)}{\gamma^2+z^2}$$

is bounded by $\int|\mu|(dy)||A(.,\ell)||_2||A(.,y)||_2 = \int|\mu|(dy)G_z(\ell,\ell)G_z(y,y)$, we have the estimate needed to commute the order of integration. Taking $\widehat{\mu}(\gamma) = \int A(\gamma,y)\mu(dy)$ our hypothesis reads

$$\frac{1}{\pi}\int \widehat{\mu}(\gamma)A(i\gamma,\ell)\frac{\Delta(d\gamma)}{\gamma^2+z^2} = 0$$

But recall from [2] p.177 (see also p.171) that the spectrum consists of those (imaginary) points $\pm i\psi_n$ satisfying

$$A(i\psi_n,\ell) + kA_x(i\psi_n,\ell+) = 0 \tag{2.1}$$

By uniqueness of the Stieltjes transform, using the fact that A and A_x cannot vanish simultaneously — recall how $k > 0$ — this means that $\widehat{\mu}(\gamma) = 0$ on the spectrum of $\mathcal{G}$. So uniqueness for the transform $\mu \to \mu^*(z)$ can be formulated as the spectral uniqueness question: do the conditions $\int A(\pm i\psi_n,y)\mu(dy) = 0$ necessarily imply $\mu \equiv 0$?

To make the resemblance between 2.1 and 0.1 more explicit, notice that the functions f_θ defined at 0.1 can be written

$$f_\theta(x) = \begin{cases} A(i\theta,x) & x < \ell; \\ A(i\theta,\ell)D^-(\theta,x) & x > \ell; \end{cases}$$

where D^- is defined at 1.1. These functions are continuous but we need their derivatives to match at ℓ, which requires choosing (real) $\theta = \pm\theta_n$ so that

$$A_x(i\theta_n,\ell+) - D_x^-(\theta_n,\ell)A(i\theta_n,\ell) = 0 \tag{2.2}$$

Since 0.1 says that $\int A(\pm i\theta_n,x)\mu(dx) = 0$, we find the two problems differ only in their spectra. Note that k plays no role in 2.2.

Remark 2.3. We are glossing over an important point here, namely the role of zero in our eigenvalue conditions. For 2.1 the answer is very simple: zero is an eigenvalue if and only if k is infinite. Condition 2.2 is more subtle. Clearly zero is an eigenvalue if and only if τ is finite almost surely. A criterion for this can be formulated in terms of the Lévy process Y_t of the next section. It says that τ is finite if and only if

$$\frac{\partial}{\partial\theta}\left[\frac{A_x(i\theta^{1/2},\ell)}{A(i\theta^{1/2},\ell)} - D_x^-(\theta^{1/2},\ell)\right]_{\theta=0} \leq 0$$

This follows from the strong law of large numbers except in the case of equality, when we can use [1].

To state our main results we introduce $\mathcal{M}_m[0,\ell]$, the space of bounded signed measures on $[0,\ell]$ which are absolutely continuous with respect to m.

Theorem 2.4 Suppose the string (m^+,ℓ,k) is short. Then the transform

$$\mu \to \int_0^\ell A(i\psi_n,y)\mu(dy) \quad ; \quad A(i\psi_n,\ell)+kA_x(i\psi_n,\ell+)=0$$

is a 1-1 mapping on $\mathcal{M}_m[0,\ell)$.

Proof: If $\int_0^\ell A(i\psi,y)\mu(dy)=0$ for $\psi=\pm i\psi_n$ then

$$G_z\mu(x)=\int \mu(dy)\frac{1}{\pi}\int A(i\gamma,x)A(i\gamma,y)\frac{\Delta(d\gamma)}{z^2+\gamma^2}$$

vanishes if we can commute the two integrals — justified by the estimate

$$||A(.,x)A(.,y)||_1 \le ||A(.,x)||_2||A(.,y)||_2,$$

noting from 1.1 that $||A(.,x)||_2^2=G_z(x,x)\in L_\infty([0,\ell),m)$ since the string is short. Thus $G_zu=0$ and uniqueness follows from 1.4.

The corresponding result for 2.2 is not so easy.

We introduce the notation $0<\chi_1<\chi_2\ldots<\chi_n<\ldots$ for the positive zeros of $\theta\to A(i\theta,\ell)$ — zero is never a root. The non-negative eigenvalues of 2.2 we list as $0\le\theta_1<\theta_2\ldots$.

Theorem 2.5 If the string (m^+,ℓ,k) is short then the transform

$$\mu \to \int_0^\ell \mu(dy)A(i\theta_n,y) \quad ; \quad A_x(i\theta_n,\ell)-D_x^-(\theta_n,\ell)A(i\theta_n,\ell)=0$$

uniquely determines $\mu\in\mathcal{M}_m[0,\ell]$.

Our argument can be outlined as follows. Because the string is short, $A(z,.)$ is bounded on $[0,\ell)$ and we have a ready made *analytic* function $z\to\widehat{\mu}(z)=\int A(z,x)\mu(dx)$. We will prove that if $\widehat{\mu}$ vanishes on the spectrum $\{\pm i\theta_n : n\ge 1\}$ then it vanishes everywhere. The argument involves comparing the eigenvalues with the zeros of $A(.,\ell)$. Using a growth estimate and the Phragmen-Lindelöf theorem we find $\hat{\mu}$ is a constant multiple of $w(z)=\Pi_n\left(1+z^2\theta_n^{-2}\right)$, and examining the ratio for z large gives us $\hat{\mu}(z)=0$.

Now for the details.

Definition We say that an integral function f has type T if $|f(z)|<Ce^{(T+\delta)|z|}$ for all $\delta>0$ but for no $\delta<0$. If we can take $T=0$ then f has minimal type; f has finite type if $T<+\infty$.

We have the following basic type estimate for string functions. The proof is given in [2] p. 238.

Lemma 2.6 The string functions $z \to A(z,x)$ and $z \to A_x(z,x)$ have exact exponential type

$$T(x) = \int_0^x [m'(y)]^{1/2} dy$$

where m' is the derivative of m.

The Hölder inequality

$$T^2(x) \le x \int_0^x m'(y)dy \le xm[0,x]$$

shows that the type is always finite; the type is minimal whenever m is singular.

We will carry out the proof of 2.5 in five steps, starting from the hypothesis that $\hat{\mu}$ vanishes on the eigenvalues $\pm i\theta_n$. It helps to keep in mind the classical case: m^+ is Lebesgue measure on $[0,\ell)$ and $A(z,x) = \cosh zx$.

1) We may suppose that the eigenvalues θ_n at 2.2 are interlaced with the zeros χ_n of $A(i\theta,\ell)$, and that $0 \le \theta_1 < \chi_1 < \theta_2 < \chi_2 < \ldots$.

By [2] p.232 the roots of $A(z,\ell)$ and $A_x(z,\ell)$ are purely imaginary and interlaced. This means that $A_x(i\theta,\ell)/A(i\theta,\ell)$, which is zero at the origin, takes all real values on each (χ_{n-1},χ_n) for $n \ge 2$. The only problem is with choosing θ_1. But if θ_1 is not zero (i.e. if τ is not finite) the criterion at 2.3 shows that the function

$$\theta \to \frac{A_x(i\theta,\ell)}{A(i\theta,\ell)} - D_x^-(\theta,\ell),$$

which may start at zero, becomes positive for a time before bending back down and tending to $-\infty$ as θ approaches the point χ_1. So it must have a root in between.

2) $w(z) = \Pi_n \left(1 + z^2\theta_n^{-2}\right)$ is an analytic function of type T — the product is over all eigenvalues, with the convention that if $\theta_1 = 0$ the corresponding factor just z^2.

$A(z,\ell)$ has type $T = T(\ell)$ by 2.6. By [2] p.232 its zeros $\pm i\chi_n$ satisfy $\chi_n \sim \pi n T^{-1}$ for n large, so by 1) the infinite product converges absolutely and defines an analytic function. To see that $w(z)$ is of finite type notice that its maximum values are along the real axis, something which is true also for $A(z,\ell)$ and $A_x(z,\ell)$. But $A(z,\ell)$ is an even function, and by the Hadamard product representation it is a constant multiple of $\Pi_n \left(1 + z^2\chi_n^{-2}\right)$, so by 1) $w(z)/A(z,\ell)$ grows at most quadratically on the real axis. It follows that $w(z)$ also has type T.

3) $x \to |A(z,x)|$ is strictly increasing on the support of m provided $z^2 = a + ib$ with $a > 0$.

For this we write $A = u + iv$ and we obtain the system of equations

$$\frac{d}{dm}u_x = au - bv \quad ; \quad \frac{d}{dm}v_x = av + bu$$

We claim $u^2 + v^2$ is increasing. Starting from

$$\frac{1}{2}d\left(u^2+v^2\right)_x = a(u^2+v^2)dm + (u_x^2+v_x^2)dx$$

we see that for $y > x$

$$\frac{1}{2}(u^2+v^2)_x(y) - \frac{1}{2}(u^2+v^2)_x(x) = a\int_x^y (u^2+v^2)dm + \int_x^y (v_x^2+u_x^2)dx > 0$$

But $(u^2+v^2)_x$ vanishes at the origin (actually at $0-$), so u^2+v^2 is increasing as required.

4) $\widehat{\mu}(z) = \int A(z,x)\mu(dx)$ is constant multiple of $w(z)$.

By 3) and 2.6 we see that $\hat{\mu}$ has finite type. Hence by 2), our hypothesis, and [2] p.20, $f = \widehat{\mu}/w$, which is analytic, is of finite type also. We can apply the Phragmen-Lindelöf theorem: *If f is analytic and of finite type in a sector of opening less than π, and is bounded by L on the boundary, then f is bounded by L everywhere in the sector.* For this we check, using the Hadamard product representation, that $|A(z,\ell)/w(z)|$ is bounded on rays close to the real axis for $|z| > 1$ say. By 3) the same holds for f, so by the Phragmen-Lindelöf theorem the even function f is bounded on the upper (lower) half plane, thereby proving by the Liouville theorem that $\widehat{\mu}/w$ is constant.

5) $\hat{\mu} = 0$.

We examine the behaviour of $\hat{\mu}/w$ as z tends to infinity along the real axis, bearing in mind that m^+ does not charge ℓ. Then, since ℓ is a point of increase of m^+, we find that $A(z,x)/A(z,\ell)$ converges to zero for $x < \ell$ and by the dominated convergence theorem $\hat{\mu}(z)/A(z,\ell)$ converges to zero also. But, as we saw for 4), the ratio $A(z,\ell)/w(z)$ is bounded when $z > 1$. Hence $\hat{\mu}(z)/w(z) \to 0$, and $\hat{\mu} = 0$ by 4).

This completes the main part of the proof of 2.5, and in particular it leaves us in the hypotheses of 2.4. So we are finished.

3. Désiré André's equation.

In this section we examine the connection between 0.1 and the first passage relation as it applies to diffusion factorisation ([7] but see also [9]).

Recall that σ_t is the right continuous inverse of the B_t local time at ℓ. Then $Y_t = V_{\sigma_t}$ is a Lévy process and we define its Laplace exponent $\kappa(z)$ by $\mathbf{E}\left[e^{-z(Y_t-Y_0)}\right] = e^{-\kappa(z)t}$. The time of return to ℓ after equalisation, namely $\tau + T\circ\theta_\tau$, corresponds in the σ_t time scale to $U = \inf\{t : Y_t > 0\}$. The random variable Y_U is called *the overshoot of level zero* for the process Y_t; note that $V_T = Y_0 < 0$ here.

One can derive an equation for the distribution of Y_U. If f is a bounded continously differentiable function then by Ito's formula the process

$$e^{-\lambda t}f(Y_t)-\int_0^t e^{-\lambda s}ds\int[f(Y_s+y)-f(Y_s)]\nu(dy)+\lambda\int_0^t e^{-\lambda s}f(Y_s)ds \qquad (\lambda>0)$$

is a uniformly bounded (purely discontinuous) martingale. Taking $f(x)=e^{-zx}$ and stopping at the time $t=U$, we compute the expectation, first in Y_t and then in V_T, to obtain

$$\mathbf{E}_x\left[\mathbf{E}_{V_T}\left[e^{-\lambda U}e^{-zY_U}\right]\right]=\mathbf{E}_x\left[e^{-zV_T}\right]-[\lambda+\kappa(z)]\mathbf{E}_x\left[\mathbf{E}_{V_T}\left[\int_0^U e^{-\lambda t}e^{-zY_t}dt\right]\right] \tag{3.1}$$

This is the equation of Désiré André [9]; it is valid (at least) when z is purely imaginary.

To connect with the problem at 0.1, we remark that, by definition, $Y_U=V_T\circ\theta_\tau$. So using the strong Markov property of B_t at time τ we can rewrite the first term of 3.1

$$\int_0^\ell \Pi(x,dy)\mathbf{E}_x\left[e^{-\lambda L(0,\tau)}|B_\tau=y\right]\mathbf{E}_y\left[e^{-zV_T}\right]=$$

$$\mathbf{E}_x\left[e^{-zV_T}\right]-[\lambda+\kappa(z)]\mathbf{E}_x\left[\mathbf{E}_{V_T}\left[\int_0^U e^{-\lambda t}e^{-zY_t}dt\right]\right]$$

We want to solve this equation for $\Pi(x,dy)$.

The above is the Laplace transform of a convolution equation of Wiener-Hopf type. The left side is bounded analytic on the right half plane, while the first and last terms on the right are bounded analytic on the left half plane. The given data are $\mathbf{E}_x\left[e^{-zV_T}\right]$, the initial condition, and $\kappa(z)$ which we call the *symbol* of the equation.

There are several difficulties with this interpretation. Our equation is defined only in a limited region of the complex plane. Also, the 'convolution kernel' is not well-behaved. In general it has a singularity at the origin — see [7].

Let us try another approach. We begin by writing the above equation as

$$\int_0^\ell \Pi(x,dy)\mathbf{E}_x\left[e^{-\lambda L(0,\tau)}|B_\tau=y\right]\mathbf{E}_y\left[e^{-z^2V_T/2}\right]$$

$$=\mathbf{E}_x\left[e^{-z^2V_T/2}\right]-[\lambda+\kappa(z^2/2)]\mathbf{E}_x\left[\mathbf{E}_{V_T}\left[\int_0^U e^{-\lambda t}e^{-z^2Y_t/2}dt\right]\right] \tag{3.2}$$

which we consider as a function of z. If $0<y<\ell$ then $\mathbf{E}_y\left[e^{-z^2V_T/2}\right]=A(z,y)/A(z,\ell)$ is a meromorphic function in z. Moreover, $\kappa(z^2/2)=\kappa_+(z^2/2)+\kappa_-(z^2/2)$ where, in the notation of 1.1,

$$\kappa_+(z^2/2)=\frac{A_x(z,\ell)}{2A(z,\ell)}\quad;\quad\kappa_-(-z^2/2)=-\frac{1}{2}D_x^-(z,\ell) \tag{3.3}$$

Here κ_+ (resp. κ_-) is the exponent of the positive (resp. negative) jumps of Y_t.

Lemma 3.4 Equation 3.2 holds in the sectors $\pi/4 < |\mathrm{Arg}(z)| < 3\pi/4$.

Proof: We know 3.2 holds when $z^2/2$ is purely imaginary We will extend by analytic continuation, using the following.

1) For $0 < y < \ell$ the function $z \to \mathbf{E}_y\left[e^{-z^2 V_T/2}\right]$ is meromorphic, so the left side of 3.2 extends as a meromorphic function to the entire complex plane.

2) Since for $\ell < x$ the function $z \to \mathbf{E}_x\left[e^{-z^2 V_T/2}\right]$ is a bounded analytic function in the sectors $\pi/4 < |\mathrm{Arg}(z)| < 3\pi/4$, the same is true of $z \to \kappa_-(z^2/2)$ and $z \to \mathbf{E}_x\left[\mathbf{E}_{V_T}\left[\int_0^U e^{-\lambda t} e^{-z^2 Y_t/2} dt\right]\right]$.

3) From 1), 3.3, and 2) we find the symbol $\kappa(z^2/2) = \kappa_+(z^2/2) + \kappa_-(z^2/2)$ is meromorphic in the sectors $\pi/4 < |\mathrm{Arg}\ (z)| < 3\pi/4$.

The required extension property of 3.2 follows.

Remark that by 3.3 the purely imaginary zeros of the symbol $\kappa(z^2/2)$ coincide with the spectrum defined at 2.2 — the origin is again a special case which we treat according to 2.3. This allows us to prove the following.

Theorem 3.5 If the positive string is short then 3.2 has a unique solution $\Pi(x, dy)$.

Proof: We show that 3.2 leads to 0.1. The result will then follow by 2.5. From 3.4, taking $z = i\theta_n \neq 0$ as a spectral point, equation 3.2 gives us

$$\int_0^\ell \Pi(x,dy)\mathbf{E}_x\left[e^{-\lambda L(0,\tau)}|B_\tau = y\right]\mathbf{E}_y\left[e^{\theta^2 V_T/2}\right] = \mathbf{E}_x[e^{\theta^2 V_T/2}] - \lambda \mathbf{E}_x\left[\mathbf{E}_{V_T}\left[\int_0^U e^{-\lambda t} e^{\theta^2 Y_t/2} dt\right]\right]$$

The claim is that the last term tends to zero as $\lambda \downarrow 0$. To see why, we split it as

$$\lambda \mathbf{E}_x\left[\int_0^U e^{-\lambda s} e^{\theta_n^2 Y_s/2} ds; U < \infty\right] + \lambda \mathbf{E}_x\left[\int_0^U e^{-\lambda s} e^{\theta_n^2 Y_s/2} ds; U = \infty\right]$$

Then the first term is bounded by $\mathbf{E}\left[1 - e^{-\lambda U}; U < +\infty\right]$ and tends to zero by the dominated convergence theorem. The second term we can bound in terms of the lowest eigenvalue $\theta_1 \neq 0$ by

$$\lambda \mathbf{E}_x\left[\int_0^U e^{-\lambda s} e^{\theta_1^2 Y_s/4} ds; U = \infty\right] \le \lambda \mathbf{E}_x\left[\int_0^\infty e^{-\lambda s} e^{\theta_1^2 Y_s/4} ds\right] = \frac{\lambda}{\lambda + \kappa(-\theta_1^2/4)} \mathbf{E}_x\left[e^{\theta_1^2 V_T/2}\right]$$

(there are no zeros of the symbol between θ_1 and the origin). Since this tends to zero with λ, we have verified our claim.

Thus we can justify letting $\lambda \downarrow 0$ when we substitute $z = i\theta_n$ in the above equation, and this yields the eigenvalue conditions

$$\int_0^\ell \Pi(x,dy)\mathbf{E}_y\left[e^{\theta_n^2 V_T/2}\right] = \mathbf{E}_x\left[e^{\theta_n^2 V_T/2}\right]$$

which we can rewrite as

$$\int_0^\ell \Pi(x,dy)A(i\theta_n,y) = A(i\theta_n,\ell)D^-(\theta_n,x)$$

Comparing with the formula for $f_\theta(x)$ given above 2.2, we find this is the same as 0.1. So uniqueness follows by 2.5.

All this calls for some explanation. In the introduction we claimed two methods for computing $\Pi(x,dy)$ — the choice is between solving 0.1 or 3.2. But we have just proved that 0.1 is a special case of 3.2 (for short strings only, though the result seems to hold in general).

The point is that each method has its own advantages. Equation 3.2 yields some extra information not easily extracted from the rather cryptic relation at 0.1. Note how the symbol $\kappa(z)$, which uniquely defines m, is the crucial component of any solution. Yet κ does not appear explicitly in 0.1. The other side of the coin is that it can be difficult to solve 3.2 directly. Edwin Beggs (private communication) pointed out connections with the Riemann-Hilbert problem. The idea is that one can solve quite readily for $\Phi_0(z^2/2,x)$; but we are then faced with the problem of inverting the relevant Krein transform.

The easiest method for explicitly computing $\Pi(x,dy)$ is to guess the answer using 3.2, and then verify it from 0.1. So the uniqueness result is indispensable.

4. Killed strings.

The above results do not cover all the eigenvalue problems encountered in [8]. In this section we treat the remaining case: when there is an infinite mass at the origin so that the triple (m^+,k,ℓ) represents a killed string. See [4] for applications of killed strings to excursion theory; remark, however, that their convention on the mass at zero is different from ours.

For a killed string we use the function $C(z,x)$, defined as the unique solution of

$$\frac{d}{dm}C_x = z^2 C \quad ; \quad C(z,0)=0,\, C_x(z,0)=1$$

The boundary conditions show us that this is an odd function of z (see also [2] p.172). It is shown in [4] that we have the spectral representation

$$G_z(x,y) = \frac{1}{\pi}\int C(i\gamma,x)C(i\gamma,y)\frac{\Delta(d\gamma)}{z^2+\gamma^2}$$

for the Green function. The analogue of 2.5 is the transform

$$\mu \rightarrow \int_0^\ell \mu(dy)C(i\theta_n,y) \quad ; \quad C_x(i\theta_n,\ell) - D_x^-(\theta_n,\ell)C(i\theta_n,\ell) = 0 \tag{4.1}$$

and it is clear that this cannot uniquely determine $\mu \in \mathcal{M}_m[0,\ell]$ since the mass at the origin does not register. Note also that zero is never an eigenvalue of this problem.

We keep the notation of section two by writing $w(z) = \prod_n \left(1 + z^2\theta_n^{-2}\right)$, which we will compare with the even function $z^{-1}C(z,\ell)$. The roots of $z^{-1}C(z,\ell)$ are $\pm i\chi_n$, ordered by $0 < \theta_1 < \chi_1 < \theta_2 < \chi_2 \ldots$ as before. Then, following the arguments of [2], it is not hard to see that $C(z,\ell)$ is of finite type, has no zeros off the imaginary axis, and that the roots of $C_x(z,\ell)$ and $C(z,\ell)$ are interlaced.

This is essentially all we need to deduce that $G_z\mu = 0$, thereby uniquely determining μ except for the mass at the origin.

To determine $\mu\{0\}$ we need some extra information. For this we go back to the problem posed in the introduction; there are two cases.

Suppose m^- is a Radon measure. Then, since m^+ has infinite mass at zero, we have $\tau < +\infty$ and $\Pi(x, dy)$ has total mass one. This is our extra equation.

On the other hand, if m^- puts an infinite mass on a finite interval, then τ is infinite with positive probability. Working from 3.1, we interpret $\zeta = \inf\{t > 0 : Y_t = -\infty\}$ as a killing time for Y_t. If U is the overshoot time with this killing removed, then 3.1 with $z = 0$ yields

$$\mathbf{E}\left[e^{-\lambda U}; U < \zeta\right] = \mathbf{P}_x[T < +\infty] - [\lambda + \kappa(0)]\lambda^{-1}\mathbf{E}\left[1 - e^{-\lambda U \wedge \zeta}\right]$$

But, since ζ is independent exponential of rate $\gamma = \kappa_-(0)$, we can rewrite this as

$$\mathbf{E}\left[e^{-(\lambda+\gamma)U}\right] = \mathbf{P}_x[T < +\infty] - [\lambda + \kappa(0)](\lambda+\gamma)^{-1}\mathbf{E}\left[1 - e^{-(\lambda+\gamma)U}\right],$$

whence our extra equation by taking $\lambda = 0$ and solving for $\mathbf{P}[U < \zeta] = \mathbf{E}\left[e^{-\gamma U}\right]$.

Notice that, in both cases, zero behaves like a phantom eigenvalue of 4.1.

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