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Stochastic Calculus and the Continuity of Local Times of Lévy Processes

Richard Bass* and Davar Khoshnevisan

1. Introduction. Let Z_t be a one dimensional Lévy process with characteristic function

$$E \exp(iuZ_t) = \exp(-t\psi(u)),$$

where

$$(1.1) \quad \psi(u) = -iau + \frac{1}{2}\sigma^2 u^2 - \int_{-\infty}^{\infty} (e^{iuz} - 1 - iuz1_{(|z|\leq 1)})\nu(dz).$$

Here ν satisfies $\int(1 \wedge z^2)\nu(dz) < \infty$.

We are interested in those Lévy processes for which 0 is regular for $\{0\}$ and either $\sigma^2 > 0$ or $\nu(\mathbb{R} - \{0\}) = \infty$. In this case (see [K]) there exists a bounded continuous function g that is a density for the 1-resolvent:

$$(1.2) \quad \int f(x)g(x-y)dx = E^y \int_0^\infty e^{-t} f(Z_t)dt, \quad f \geq 0, \quad y \in \mathbb{R}.$$

(If $G(x, y)$ is the Green function for Z_t killed at an independent exponential time with parameter 1, the relationship between g and G is given by $g(x) = G(0, x) = G(a, a+x)$ for any $a \in \mathbb{R}$ and $G(x, y) = g(y-x)$.)

For each x ,

$$(1.3) \quad g(x) = \frac{1}{2\pi} \int e^{-iux} \frac{1}{1 + \psi(u)} du.$$

For each x , $g(x - \cdot)$ is the 1-potential of an additive functional L_t^x that is continuous in t . Moreover, a version of $L_t^x(\omega)$ may be chosen that is jointly measurable in (x, t, ω) . See [GK] for details. L_t^x is called the local time of Z_t at x . L_t^x is also a density of occupation time measure: if $f \geq 0$,

$$(1.4) \quad \int_0^t f(Z_s)ds = \int f(x)L_t^x dx, \text{ a.s..}$$

A number of people have studied the question of the continuity of L_t^x in the space variable (see [Bo], [Me], [GK] and [MT]), culminating in the works [B1], [BH], and [B2], where a necessary and sufficient condition for the joint continuity of L_t^x in t and x is given.

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The purpose of this paper is to give a stochastic calculus proof of the following sufficient condition for joint continuity. Let $\varphi : \mathbb{R} \rightarrow [0, \infty)$ be defined by

$$(1.5) \quad \varphi^2(x) = \frac{1}{\pi} \int (1 - \cos ux) \operatorname{Re} \frac{1}{1 + \psi(u)} du.$$

Let $d(a, b) = \varphi(b - a)$ and let $H(u)$ be the logarithm of the smallest number of d -balls of radius less than u that are needed to cover $[-1, 1]$. Define

$$(1.6) \quad F(\delta) = \int_0^\delta (H(u))^{\frac{1}{2}} du.$$

Theorem 1.1. (a) *If $F(0+) < \infty$, then L_t^x has a jointly continuous version.*

(b) *For each t ,*

$$\limsup_{\delta \downarrow 0} \sup_{\{a, b: \varphi(a-b) < \delta\}} \sup_{s \leq t} \frac{|L_s^a - L_s^b|}{F(\varphi(a-b))} \leq 2(\sup_x L_t^x)^{1/2}, \quad \text{a.s.}$$

Theorem 1.1(a) was first proved in [BH], where it was also remarked that the entropy condition was equivalent to one involving the monotone rearrangement of φ . Part (b) was also proved in [BH], with, however, the constant 2 replaced by a larger constant (namely 416). In [B2] it was shown that part (b) holds with the constant 2 under the additional assumption that φ is regularly varying (but not slowly varying) and that the constant 2 is sharp. (The principle result of [B2] was that the condition $F(0+) < \infty$ is necessary as well as sufficient for joint continuity.) Marcus and Rosen [MR] have recently obtained necessary and sufficient conditions for the joint continuity of local times of certain Markov processes. Theorem 1.1 for symmetric Lévy processes is a special case of their results.

In Section 2 we prove Theorem 1.1 assuming that $\operatorname{ess\,sup}_x L_t^x < \infty$, a.s. We establish this latter fact in Section 3.

2. Modulus of continuity. Our proof is modeled after that of [McK]. Let us begin by assuming for this section that $\operatorname{ess\,sup}_x L_t^x < \infty$, a.s. Let R be an exponential variable with parameter 1, independent of Z_t . Since $g(x - \cdot)$ is the 1-potential of L_t^x , we have

$$(2.1) \quad E^a L_R^b = g(b - a).$$

Proposition 2.1. $|g(a) - g(b)| \leq \varphi^2(a - b)$.

Proof. Let $T_x = \inf\{t : Z_t = x\}$, $S = T_a \wedge T_b$. Since L_t^x increases only when Z_t is at x , the strong Markov property at time S yields

$$\begin{aligned} |g(a) - g(b)| &= |E^0 L_R^a - E^0 L_R^b| = |E^0[E^{Z_S} L_R^a - E^{Z_S} L_R^b; S \leq R]| \\ &\leq E^0 |E^{Z_S} L_R^a - E^{Z_S} L_R^b| \\ &= E^0 [|E^a L_R^a - E^a L_R^b; S = T_a] + E^0 [|E^b L_R^a - E^b L_R^b; S = T_b] \\ &= |g(0) - g(b - a)|P^0(S = T_a) + |g(a - b) - g(0)|P^0(S = T_b) \end{aligned}$$

Since $g(x) = E^0 L_R^x \leq E^x L_R^x = g(0)$, then

$$|g(a) - g(b)| \leq 2g(0) - g(b - a) - g(a - b).$$

By (1.3) and (1.5), the right hand side equals $\varphi^2(a - b)$. $\square$

Using (2.1) and the Markov property,

$$(2.2) \quad M_t^a = g(a - Z_{t \wedge R}) - g(a - Z_0) - L_{t \wedge R}^a$$

is a martingale with $M_0 = 0$. Fix a and b and let $N_t = M_t^a - M_t^b$. Let $L_t^* = \text{ess sup}_x L_t^x$.

Proposition 2.2. $\langle N, N \rangle_t \leq 2\varphi^2(a - b)L_t^*$

Proof. Let N^c, N^d be the continuous and purely discontinuous parts of N_t , respectively. We first estimate $\langle N^d, N^d \rangle_t$.

Let

$$(2.3) \quad W(x, z) = [\{g(a - (x + z)) - g(a - x)\} - \{g(b - (x + z)) - g(b - x)\}].$$

Since L_t^a and L_t^b are both continuous in t , the jumps of N_t are the jumps of $g(a - Z_t) - g(b - Z_t)$. Hence

$$[N^d, N^d]_t = \sum_{s \leq t} \Delta N_s^2 = \sum_{s \leq t \wedge R} (W(Z_{s-}, \Delta Z_s))^2.$$

By the definition of Lévy measure, $E \sum_{s \leq t} 1_A(\Delta Z_s) = \nu(A)t$ if A is a subset of $\mathbb{R}$ that is a positive distance from 0. By the Markov property and the translation invariance of the increments of Z_t , $\sum_{s \leq t} 1_A(\Delta Z_s) - \nu(A)t$ is a martingale. Taking the stochastic integral of $1_B(Z_{s-})$ with respect to this martingale, we see that $\sum_{s \leq t} h(Z_{s-}, \Delta Z_s) -$

$\int_0^t \int h(Z_{s-}, z) \nu(dz) ds$ is again a martingale, where $h(x, z) = 1_B(x)1_A(z)$. Taking linear combinations and limits, we deduce that

$$[N^d, N^d]_{t \wedge R} - \int_0^{t \wedge R} \int W(Z_{s-}, z)^2 \nu(dz) ds$$

is a local martingale. Hence it follows that

$$\langle N^d, N^d \rangle_t = \int_0^{t \wedge R} \int W(Z_{s-}, z)^2 \nu(dz) ds.$$

Since Z_t has only countably many jumps, we get

$$\begin{aligned} (2.4) \quad \langle N^d, N^d \rangle_t &= \int_0^{t \wedge R} \int (W(Z_s, z))^2 \nu(dz) ds \\ &\leq \int \int (W(x, z))^2 L_t^x dx \nu(dz) \\ &\leq L_t^* \int \int (W(x, z))^2 dx \nu(dz) \\ &= \frac{L_t^*}{2\pi} \int \int |\widehat{W}(u, z)|^2 du \nu(dz) \quad (\text{Plancherel's theorem}) \end{aligned}$$

where $\widehat{W}(u, z)$ is the Fourier transform of $W(\cdot, z)$, z fixed.

By (2.3),

$$\begin{aligned} \widehat{W}(u, z) &= \widehat{g}(-u) (\{e^{iu(a-z)} - e^{iua}\} - \{e^{iu(b-z)} - e^{iub}\}) \\ &= \widehat{g}(-u) e^{iua} (e^{-iuz} - 1)(1 - e^{iu(b-a)}). \end{aligned}$$

Since $|e^{iu\theta} - 1|^2 = 2(1 - \cos \theta)$,

$$\begin{aligned} (2.5) \quad \int \int |\widehat{W}(u, z)|^2 \nu(dz) du &= 2 \int |\widehat{g}(-u)|^2 |1 - e^{iu(b-a)}|^2 \int (1 - \cos uz) \nu(dz) du \\ &= 4 \int (1 - \cos(u(b-a))) |\widehat{g}(u)|^2 \text{Re} \psi^d(u) du, \end{aligned}$$

where $\psi^d(u) = \psi(u) - \frac{1}{2}\sigma^2 u^2$. Substituting (2.5) in (2.4), we obtain

$$(2.6) \quad \langle N^d, N^d \rangle_t \leq \frac{2L_t^*}{\pi} \int (1 - \cos(u(b-a))) |\widehat{g}(u)|^2 \text{Re} \psi^d(u) du.$$

Next we estimate $\langle N^c, N^c \rangle_t$. If f is a smooth function and we write K_t for the martingale part of $f(Z_{t \wedge R})$, then by Itô's formula,

$$K_t^c = \int_0^{t \wedge R} f'(Z_{s-}) \sigma dB_s,$$

where B_t is a standard Brownian motion. Then

$$\begin{aligned}
 (2.7) \quad \langle K^c, K^c \rangle_t &= \sigma^2 \int_0^{t \wedge R} (f'(Z_{s-}))^2 ds = \sigma^2 \int_0^{t \wedge R} (f'(Z_s))^2 ds \\
 &\leq \sigma^2 \int (f'(x))^2 L_t^x dx \\
 &\leq \sigma^2 L_t^* \int (f'(x))^2 dx \\
 &= \sigma^2 L_t^* \frac{1}{2\pi} \int |\hat{f}'(u)|^2 du \quad (\text{Plancherel}) \\
 &= \sigma^2 L_t^* \frac{1}{2\pi} \int |u|^2 |\hat{f}(u)|^2 du.
 \end{aligned}$$

Approximating $g_{ab}(\cdot) = g(a - \cdot) - g(b - \cdot)$ by smooth functions in a suitable way, taking limits, and noting that $\hat{g}_{ab}(u) = \hat{g}(-u)(e^{iua} - e^{iub})$, we get

$$\begin{aligned}
 \langle N^c, N^c \rangle_t &\leq \sigma^2 \frac{L_t^*}{2\pi} \int u^2 |\hat{g}(-u)|^2 |e^{iua} - e^{iub}|^2 du \\
 &= \frac{2L_t^*}{\pi} \int \frac{\sigma^2 u^2}{2} |\hat{g}(u)|^2 (1 - \cos(u(b-a))) du.
 \end{aligned}$$

Adding to (2.6) yields

$$\begin{aligned}
 (2.8) \quad \langle N, N \rangle_t &= \langle N^c, N^c \rangle_t + \langle N^d, N^d \rangle_t \\
 &\leq \frac{2L_t^*}{\pi} \int |\hat{g}(u)|^2 \text{Re } \psi(u) (1 - \cos(u(b-a))) du.
 \end{aligned}$$

Finally, from (1.1), $\text{Re } \psi(u) \geq 0$. So

$$\begin{aligned}
 (2.9) \quad \varphi^2(x) &= \frac{1}{\pi} \int (1 - \cos ux) \text{Re} \frac{1}{1 + \psi(u)} du \\
 &= \frac{1}{\pi} \int (1 - \cos ux) \frac{\text{Re}(1 + \overline{\psi(u)})}{|1 + \psi(u)|^2} du \\
 &= \frac{1}{\pi} \int (1 - \cos ux) |\hat{g}(u)|^2 (1 + \text{Re } \psi(u)) du,
 \end{aligned}$$

since $\hat{g}(u) = (1 + \psi(u))^{-1}$. Comparing (2.9) to (2.8) proves the proposition. $\square$

Proposition 2.3. *Let $\epsilon > 0$. There exists $J_0 > 0$ depending on ϵ such that if X_t is any square integrable martingale with jumps bounded in absolute value by J_0 and with $\langle X, X \rangle_t$ continuous, then $\exp(X_t - (1 + \epsilon)\langle X, X \rangle_t/2)$ is a positive supermartingale.*

Proof. Take J_0 small enough so that $|e^x - 1 - x| \leq (1 + \epsilon)x^2/2$ if $|x| \leq J_0$. Let

$$Y_t = X_t - (1 + \epsilon)\langle X, X \rangle_t/2.$$

By Itô's formula,

$$\begin{aligned}
e^{Y_t} &= 1 + \int_0^t e^{Y_{s-}} dY_s + \frac{1}{2} \int_0^t e^{Y_{s-}} d\langle Y^c, Y^c \rangle_s + \sum_{s \leq t} (e^{Y_s} - e^{Y_{s-}} - e^{Y_{s-}} \Delta Y_s) \\
&= 1 + \int_0^t e^{Y_{s-}} dX_s - \frac{(1+\epsilon)}{2} \int_0^t e^{Y_{s-}} d\langle X, X \rangle_s + \frac{1}{2} \int_0^t e^{Y_{s-}} d\langle X^c, X^c \rangle_s \\
&\quad + \sum_{s \leq t} e^{Y_{s-}} (e^{\Delta Y_s} - 1 - \Delta Y_s) \\
&= 1 + \text{local martingale} - \frac{\epsilon}{2} \int_0^t e^{Y_{s-}} d\langle X^c, X^c \rangle_s - \frac{(1+\epsilon)}{2} \int_0^t e^{Y_{s-}} d\langle X^d, X^d \rangle_s \\
&\quad + \sum_{s \leq t} e^{Y_{s-}} (e^{\Delta X_s} - 1 - \Delta X_s)
\end{aligned}$$

Since $\langle X^d, X^d \rangle_t - \sum_{s \leq t} (\Delta X_s)^2$ is a local martingale,

$$\begin{aligned}
(2.10) \quad e^{Y_t} &= 1 + \text{local martingale} - \frac{\epsilon}{2} \int_0^t e^{Y_{s-}} d\langle X^c, X^c \rangle_s + \text{local martingale} \\
&\quad - \frac{1+\epsilon}{2} \sum_{s \leq t} e^{Y_{s-}} (\Delta X_s)^2 + \sum_{s \leq t} e^{Y_{s-}} (e^{\Delta X_s} - 1 - \Delta X_s).
\end{aligned}$$

But $e^{\Delta X_s} - 1 - \Delta X_s - (1+\epsilon)(\Delta X_s)^2/2 \leq 0$ by our selection of J_0 . Hence (2.10) exhibits $\exp(Y_t)$ as a local martingale minus an increasing process. $\square$

Write P for P^0 .

Corollary 2.4. $P(\sup_{s \leq t} |X_s| > \lambda + (1+\epsilon)\langle X, X \rangle_t/2) \leq 2e^{-\lambda}$.

Proof. Reducing the continuous part of X_t by stopping times, we may assume X_t bounded, as long as our probability bound does not depend on the L^∞ norm of X_t . We can then write $e^{Y_t} = K_t - V_t$, where K_t is a martingale with $K_0 \equiv 1$ and V_t an increasing process with $V_0 \equiv 0$. Then by Doob's inequality,

$$\begin{aligned}
P(\sup_{s \leq t} e^{Y_s} > e^\lambda) &\leq P(\sup_{s \leq t} K_s > e^\lambda) \\
&\leq e^{-\lambda} EK_t = e^{-\lambda} EK_0 = e^{-\lambda}.
\end{aligned}$$

This proves $P(\sup_{s \leq t} X_s > \lambda + (1+\epsilon)\langle X, X \rangle_t/2) \leq \exp(-\lambda)$. Applying the same argument to $-X$ proves the corollary. $\square$

Under the assumption $L_t^* < \infty$, a.s., we can now prove Theorem 1.1.

Proof of Theorem 1.1. Let $N_t = M_t^a - M_t^b$ as above, $F(\delta)$ defined by (1.5). Since the potentials of $L_{t \wedge R}^a$ and $L_{t \wedge R}^b$ are bounded, N_t is square integrable ([DM], p.193).

Clearly $F(\delta) \rightarrow 0$ as $\delta \rightarrow 0$. Also, $\varphi(\delta) \rightarrow 0$ as $\delta \rightarrow 0$ by the continuity of g , hence $H(u) \rightarrow \infty$ as $u \rightarrow 0$, hence $\delta/F(\delta) \rightarrow 0$ as $\delta \rightarrow 0$.

Let α, β be > 0 such that $\alpha\beta > 1$, let $\epsilon > 0$, set $\delta = |b - a|$, and set $\eta = \varphi(\delta)$. Let

$$X_t = \beta F(\eta) \eta^{-2} N_t.$$

Since the jumps of N_t are bounded by $2 \sup_x |g(x - a) - g(x - b)| \leq 2\varphi^2(b - a)$, the jumps of X_t are bounded by $2\beta F(\eta)$, which will be less than the J_0 of Proposition 2.3 if δ is small.

Now apply Corollary 2.4: if δ is sufficiently small,

$$\begin{aligned} (2.11) \quad & P(\sup_{s \leq t} |M_s^a - M_s^b| > \alpha F(\eta) + (1 + \epsilon)\beta F(\eta) L_t^*) \\ & \leq P(\sup_{s \leq t} |N_s| > \alpha F(\eta) + \frac{(1 + \epsilon)}{2} \beta \frac{F(\eta)}{\eta^2} \langle N, N \rangle_t) \quad (\text{Proposition 2.2}) \\ & = P(\sup_{s \leq t} |X_s| > \alpha \beta F^2(\eta)/\eta^2 + \frac{1 + \epsilon}{2} \langle X, X \rangle_t) \\ & \leq \exp(-\alpha \beta F^2(\eta)/\eta^2). \end{aligned}$$

A standard metric entropy argument (see, e.g., [D]) and (2.11) shows that we can find a version of M_t^x that is jointly continuous in $t \in [0, R)$ and $x \in \mathbb{Q}$ and such that for each $K > 0$,

$$(2.12) \quad P(\limsup_{\eta \downarrow 0} \sup_{\{a, b \in \mathbb{Q} \cap [-K, K] : \varphi(a - b) < \eta\}} \sup_{s \leq t} \frac{|M_s^a - M_s^b|}{F(\varphi(a - b))} > c(\alpha + (1 + \epsilon)\beta L_t^*) = 0$$

for each $\alpha, \beta > 0$ such that $\alpha\beta > 1$. Here $\mathbb{Q}$ denotes the rationals. By being a bit more careful with the constants in the metric entropy argument, one can show that one can in fact take $c = 1$.

Fix an ω not in the null set for any α, β, ϵ rational, K a positive integer, take $K \geq \sup_{s \leq t} (|Z_s| + 1)$, $\alpha \in [(L_t^*(\omega))^{1/2}, (1 + \epsilon)(L_t^*(\omega))^{1/2}]$, and $\beta = (1 + \epsilon)/\alpha$, and then let $\epsilon \rightarrow 0$. We thus get

$$(2.13) \quad \limsup_{\eta \downarrow 0} \sup_{\{a, b \in \mathbb{Q} : \varphi(a - b) < \eta\}} \sup_{s \leq t} \frac{|M_s^a - M_s^b|}{F(\varphi(a - b))} \leq 2(L_t^*)^{1/2}, \quad \text{a.s.}$$

By Proposition 2.1, $|g(x - a) - g(x - b)| \leq \varphi^2(\delta)$. Since $\eta = o(F(\eta))$ as $\eta \rightarrow 0$, (2.2) yields (2.13) with $M_s^a - M_s^b$ replaced by $L_{s \wedge R}^a - L_{s \wedge R}^b$. Arguing as in [GK], one can

find a version of L_t^x that is jointly continuous in $t \in [0, R)$ and $x \in \mathbb{R}$, that is still an occupation time density, and that satisfies

$$\limsup_{\eta \downarrow 0} \sup_{\{a, b: \varphi(a-b) < \eta\}} \sup_{s \leq t \wedge R} \frac{|L_s^a - L_s^b|}{F(\varphi(a-b))} \leq 2(L_t^*)^{1/2}, \quad \text{a.s.}$$

Finally, using the strong Markov property at R and performing a renewal argument yields Theorem 1.1. $\square$

3. Essential boundedness. It remains to show that $L_t^* < \infty$, a.s., under the hypotheses of Theorem 1.1. Let

$$(3.1) \quad J_t^x(r) = \int_0^t \frac{1}{2r} 1_{[x-r, x+r]}(Z_s) ds.$$

Clearly $J_t^x(r)$ is bounded by $t/2r$. Let $\rho(x) = 1_{[-1, 1]}(x)/2$, $\rho_r(x) = r^{-1}\rho(x/r)$. Note $J_t^x(r) = L_t^* * \rho_r(x)$, where $*$ denotes convolution.

Proposition 3.1. *If $\epsilon > 0$, there exists $K > 0$ such that*

$$\sup_{r \leq 1} P(\sup_x J_t^x(r) > K) < \epsilon.$$

Proof. Since $J_t^x(r) = L_t^* * \rho_r(x)$, the 1-potential of $J_t^x(r)$ is $g * \rho_r(x - \cdot)$. By Proposition 2.1,

$$(3.2) \quad |g * \rho_r(a) - g * \rho_r(b)| \leq \varphi^2(a - b).$$

Let $W_r(x, z) = \int W(y, z) \rho_r(x - y) dy$. If we let $N_t(r)$ be the martingale part of $g * \rho_r(a - Z_{t \wedge R}) - g * \rho_r(b - Z_{t \wedge R})$, then as in the proof of Proposition 2.2,

$$\langle N^d(r), N^d(r) \rangle_t \leq \int \int (W_r(x, z))^2 L_t^x dx \nu(dz).$$

But by Jensen's inequality,

$$(W_r(x, z))^2 \leq \int W(y, z)^2 \rho_r(x - y) dy,$$

hence

$$\begin{aligned} \langle N^d(r), N^d(r) \rangle_t &\leq \int \int \int W(y, z)^2 \rho_r(x - y) L_t^x dx \nu(dz) dy \\ &= \int \int W(y, z)^2 J_t^y(r) \nu(dz) dy \\ &\leq (\sup_x J_t^x(r)) \int \int W(y, z)^2 dy \nu(dz) \end{aligned}$$

With a similar change to estimate $\langle N^c(r), N^c(r) \rangle_t$, we get

$$(3.3) \quad \langle N(r), N(r) \rangle_t \leq 2\varphi^2(a-b) \sup_x J_t^x(r).$$

Proceeding as in Section 2, we get the joint continuity of $J_t^x(r)$ in x , with probability estimates independent of r . Take K_0 large so that $P(\sup_{s \leq t} |Z_s| > K_0 - 1) < \epsilon/2$. Using the probability estimates for the continuity of $J_t^x(r)$ in x , take $K > K_0$ large enough so that

$$P(\sup_{|x| \leq K_0} J_t^x(r) > K) < \epsilon/2.$$

This proves the proposition. $\square$

Theorem 3.1. *Under the assumptions of Theorem 1.1, $\text{ess sup}_x L_t^x < \infty$, a.s.*

Proof. Let $r_n = 2^{-n}$. Let $\epsilon > 0$ and choose K as in Proposition 3.1. Let

$$A_n = \{\sup_x J_t^x(r_n) > K\}$$

If $J_t^x(r_n) > K$ for some x , then since

$$J_t^x(r_n) = \frac{1}{2}[J_t^{x+r_{n+1}}(r_{n+1}) + J_t^{x-r_{n+1}}(r_{n+1})],$$

we get $\sup_y J_t^y(r_{n+1}) > K$. Therefore $A_n \subseteq A_{n+1}$.

It follows that

$$P(\sup_{x,n} J_t^x(r_n) = \infty) \leq P(\bigcup_{n=1}^{\infty} A_n) = \lim_n P(A_n) \leq \epsilon.$$

Since ϵ is arbitrary, $\sup_{x,n} J_t^x(r_n) < \infty$, a.s. Our result then follows by Lebesgue's differentiation theorem. $\square$

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