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Lester E. Dubins
Meir Smorodinsky

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The Modified, Discrete, Levy-Transformation Is Bernoulli

Lester E. Dubins and Meir Smorodinsky

Abstract. From the absolute value of a martingale, X, there is a unique increasing process that can be subtracted so as to obtain a martingale, Y. Paul Levy discovered that if X is Brownian motion, B, then Y, too, is a Brownian motion. Equivalently, Levy found that the transformation that maps B to Y is measure-preserving. Whether it is ergodic, a question raised by Marc Yor, is open. Here, the natural analogue of Levy's transformation for the symmetric random walk is modified and, thus modified, is shown to be measure-preserving. The ergodicity of this transformation is then established by showing that it is isomorphic to the one-sided, Bernoulli shift-transformation associated with a sequence of independent random variables, each uniformly distributed on the unit interval.

From the absolute value of a martingale, X, there is a unique increasing process that can be subtracted so as to obtain a martingale, Y. Paul Levy (1939) and (1948 , p. 194) discovered that if X is Brownian motion, B, then Y, too, is a Brownian motion. Equivalently, Levy found that the transformation that maps B to Y, is measure-preserving. Marc Yor asked whether it is ergodic. We have not seen how to resolve this question. Possibly as a step towards its resolution, and possibly of interest in its own right, we modify the natural discrete analogue of Levy's transformation and show that, thus modified, it is a measure-preserving transformation isomorphic to the STANDARD transformation, that is, the one-sided, Bernoulli shift-transformation associated with a sequence -- indexed by the nonnegative integers -- of independent random variables, each uniformly distributed on the unit interval. Since, as is well-known, the standard transformation is ergodic, so is the modified-discrete-Levy transformation.

In this note, w is a variable that designates an infinite sequence of integers w0, w1, ...such that w0 = 0 and such that each increment, w(n+1) - wn is +1 or -1.[Here, and elsewhere, when no confusion is possible, parentheses are omitted, so wn is a typographical simplification of w(n)]. The set, W, of all such w is endowed with the usual product sigma-field as well as with that probability measure under which w becomes a symmetric random walk, or, as is easily verified to be equivalent, under which w is martingale-distributed.

The map that transforms w into the martingale obtained by subtracting from w's absolute value an increasing process, Z = Zw, is the DISCRETE LEVY TRANSFORMATION, and is designated by L.

Plainly, L is not measure-preserving. For Lw may pause, that is, may have the same value at two successive moments of time, for instance, the value 0 at times 0 and 1, while w never pauses. However, once the pauses in Lw are excised, as will be clarified below, one obtains a modified sequence, MLw, which, with ease, will be seen to be a symmetric random walk or, equivalently, ML will be seen to be measure-preserving. Record, for later use, that h = Lw is a martingale with +1, 0, and -1 as possible increments.

The Modification. There is some increase in clarity if the modification, M, is defined for a sequence of arbitrary objects, h0, h1, ... It is suggestive to call the integers, 0,1,..., when occurring as arguments of h, moments of time. Mh is simply the sequence obtained when all the pauses, and only the pauses, of h are eliminated. The next two paragraphs make this precise, and introduce some useful terminology and notation.

If hi = h(i-1), then h completes a PAUSE at (time) i. In the contrary case, hi is different from h(i-1), and h completes a CHANGE at i. If h completes infinitely many changes, h is CHANGEABLE. For changeable h, and for each positive integer, j, let lj be the moment at which h completes its jth change, and set l0 equal to 0. Of course, l is strictly increasing, with l0 = 0. So, if for some j, lj = j, then, for all i < j, li = i.

For changeable h, define Mh to be hl, that is, Mh is the composition of h with l. More elaborately, the value of Mh at j is the value of h at the moment that h completes it's jth change, that moment being lj. As is easily verified, the range of M is the set of sequences that complete a change at every positive i, hereafter called the NIMBLE sequences. Summarizing, h is in the domain of M if it is changeable, and is in the range of M if it is nimble. Indeed, Mh is that nimble subsequence of h that is not a proper subsequence of any nimble subsequence of h. Plainly, therefore, M is idempotent, that is, the square of M is M.

If g is a sequence different from h, there is obviously a smallest integer, t = t(g,h), at which g and h disagree, herein called the moment at which (g,h) forks or SEPARATES. Plainly, g or h completes a change at t(g,h). It can happen that not both g and h complete changes at t. If both do complete changes at t, then the pair (g,h) is HARMONIOUS. Let j(g,h) be the number of changes that g completes up to, and including, time t(g,h). Plainly, t(g,h) = t(h,g) if, and only if, the pair (g,h) is harmonious. Suppose that both g and h are in the domain of M. Then, as is easily verified, if (g,h) is harmonious, then (Mg,Mh) also is, and this latter pair separates at j(g,h). So, unless j = lj, that is, unless g completes a change at each positive i < j, (Mg,Mh) separates strictly before (g,h) does. For later reference, this is recorded as:

Lemma 1. If (g,h) is a harmonious pair of distinct, changeable sequences, and if g pauses at any time prior to separation, then (Mg,Mh) is a pair of sequences that separate strictly earlier than (g,h) does.

Henceforth, only h's that are stochastic, almost all of whose realizations are changeable, are of interest. Plainly, those properties of h are possessed by Mh, too. However, though h be Markovian, Mh need not be. Nor does M quite preserve the set of martingales; however, it almost does. For instance, if h is a martingale with uniformly bounded increments, then so is Mh, as is easily seen with, or without, reference to the optional sampling theorem of Doob (1953, p.302). This case is adequate for the purposes of this note, since only +1, 0, and -1 appear as increments henceforth.

Of course, zero is never an increment of Mh. Consequently, if h is an integer-valued martingale whose only possible increments are +1, 0, -1, then Mh is a martingale whose only possible increments are +1 and -1 and is, therefore, a symmetric random walk on the integers. Since Lw is such an h, conclude that Mh, that is, MLw, is a symmetric random walk on the integers or, equivalently, that ML is measure preserving. The MODIFIED DISCRETE LEVY TRANSFORMATION is ML which henceforth is designated by T.

The program remaining is to exhibit a random variable, s, with these three properties: (1) s has infinite entropy, indeed, the distribution of s is nonatomic; (2) s is stochastically independent of T; (3) the T orbit of s separates points. For then, letting I be the mapping, obviously Borel, that associates to w the sequence sw, sTw, ...,(3) states that I is injective, that is, one-to-one. In view of a theorem of M. Souslin (Kuratowski, 1966, p. 487), the range of I, as well as the forward image under I, of every Borel set, is Borel. Since, letting S be the one-sided shift, SI = IT, it is then easily seen that I is an isomorphism of T with S.

Henceforth, s is the sequence of signs of the successive excursions of the original random walk, w. More fully, the set of times that the walk is at the origin constitutes an increasing sequence of times commencing with time 0; this sequence, translated by one unit to the right, is a strictly increasing sequence of times commencing with time 1, the times at which the walk EMERGES from zero; and s is simply the sequence of values of the walk at these emergent times. So, sj is plus or minus one according as the jth excursion of the walk is positive or negative. Moreover, as is easily verified, the random variable, s, has as its distribution that of a sequence of plus, minus ones generated by a fair coin. So, the entropy of s, being the infinite sum of terms each of which is log2, certainly is infinite. Since T is a function of something that is independent of s, namely, of the absolute value of w, T itself is independent of s. What remains to be demonstrated is formulated as the next lemma.

Lemma 2. The orbit of s under T separates points.

Proof: Let v and w be distinct paths of W. There is then a least positive integer t, t = t(v,w) at which v and w are unequal. The program is to prove the lemma by induction on t. If t = 1, s itself, indeed the first coordinate of s, having different values for v and w, separates v from w. Suppose, for all m < t, and all paths v^ and w^ of W for which t(v^,w^) = m, that the T-orbit of s separates v^ from w^; now assume that v,w are such that t(v,w) = t. That is, v agrees with w at all times less than t, but at time t, v and w have unequal values.

Case 1: v(t-1) -- and hence w(t-1) -- is 0. So, at time t, both v and w experience an excursion, and the ordinality of this excursion is the same for v as it is for w. So, the sign of this excursion, being different for v and w, separates v from w. Consequently, for Case 1, s itself separates v from w.

As a preliminary to Case 2, recall that Z denotes the increasing process associated with the absolute value of the random walk, and record:

Fact 1. Zv agrees with Zw up to, and including, time t.

That Fact 1 holds is evident since the increasing process associated with any discrete-time, submartingale (increasing semimartingale) up to, and including any time t, depends only on the value of the submartingale up to t-1, and the absolute value of the symmetric random walk is such a submartingale.

Case 2. v(t-1) --and hence w(t-1) -- is not 0. For this case, t is at least 2, and the absolute value of v agrees with the absolute value of w at all times prior to time t, but not at time t.

For Case 2, make three observations. First, in view of Fact 1, t is minimal with the property that, at t, Lv is unequal to Lw. Second, at t, both Lv and Lw are different from their common value at t-1. Third and last, prior to time t, Lv has had the same value at at least two successive moments of time, namely the value 0 at times 0 and 1; equivalently, Lv pauses prior to its separation from Lw. Therefore, for g = Lv, and h = Lw, the hypotheses of Lemma 1 hold. Consequently, Mg and Mh, that is Tv and Tw, differ at some time m, strictly less than t. Therefore, by the inductive hypothesis, the T-orbit of s separates Tv from Tw. It is now straightforward to verify that the T-orbit of s separates v from w. QED

The proof that T is isomorphic to S is now complete. If s is interpreted as the binary expansion of a uniformly distributed random number, S is seen to be the standard one-sided Bernoulli shift transformation. It is known, and easily verified, that S is isomorphic to the one-sided Bernoulli shift transformation corresponding to any measure isomorphic to the uniform distribution on the Borel subsets of the unit interval, and, in particular, to any nonatomic probability measure on the unit interval. But S is isomorphic to no other Bernoulli shift transformation. Designate the STANDARD two-sided Bernoulli shift transformation for which each coordinate is uniformly distributed by S^. Plainly, S is a homomorph of S^, that is there is a measure-preserving map, P, such that SP = PS^.

Consequently, T is a homomorph of S^, or of any invertible transformation, T^, isomorphic to S^. Actually, S is a homomorph of any two-sided Bernoulli shift trasnsformation of infinite entropy. This is due to Sinai (1962) who discovered that every Bernoulli shift transformation is a homomorph of any other that does not have smaller entropy. That S is such a homomorph can be seen also to follow from a theorem of Ornstein (1974, p.53) who has shown that, in contrast to the one-sided case, every two two-sided Bernoulli shift transformations of infinite entropy are isomorphic or, what is equivalent, that each is isomorphic to S^. Using either route, one sees that the modified discrete Levy transformation, T, is a homomorph of any two-sided Bernoulli shift transformation of infinite entropy.

Can the argument above be modified so as to apply to the Levy transformation, that is, to the case of Brownian motion? That is, can an analogue of s be so defined that the three properties established for s hold also for the analogue? As a preliminary, it is necessary to provide a Borel naming of the set of excursions of a typical Brownian path, w. This can be done in a variety of ways. For one such naming, notice that if the set of excursions of w is provided with the ordering e < f if the excursion e is completed before the excursion f starts, then, for a typical w, this ordering is, or rather is isomorphic to, that of the set of rationals. Of course, there are then many isomorphisms of the set of excursions of w onto the set of rationals. And, as is not difficult to verify, an isomorphism, say E, can be defined so as to be Borel measurable in w. For instance, E(w,1) could be that excursion that covers the time moment, 1; E(w,1/2) could be that excursion that covers the time moment that is midway between 0 and the moment that the excursion E(w,1) begins; and, in a Borel manner, E(w,r) can be defined in an order-preserving manner for all rationals, r, as is easily verified. Alternatively, and in a variety of ways, the excursions can be given a Borel sequential ordering. For whatever Borel ordering, sequential or rational, let s* designate the signs of the excursions. That every such s* satisfies the first two of the three properties satisfied by s is easily seen, and with no change in argument. What remains to be seen is whether all, or at least one, such s* essentially satisfies the third condition, that is, whether, on a Borel set of Brownian probability one, s* is injective. This we have not seen how to settle.

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Department of Mathematics
University of California ,Berkeley
Berkeley, California 94720
415-6427019 or 415-6422781

School of Mathematics
Tel Aviv University
Tel Aviv 69978, Israel
meir@math.tau.ac.il