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Some Remarks on the Theory of Stochastic Integration *

J. A. Yan

The main purpose of this article is to propose a reasonable definition for the stochastic integration (S.I.) of progressive processes w.r.t. semimartingales. This S.I. generalizes that of predictable processes w.r.t. semimartingales as well as the stochastic Stieltjes integration. This S.I. is proposed in §1. We give also in §1 an exponential formula for semimartingales using this S.I.. The rest of this paper consists of several remarks on the theory of stochastic integration which are mostly of pedagogical interest. In §2 we propose a new construction of the S.I. of predictable processes w.r.t. local martingales. A simple proof of the integration by parts formula is given in §3. Finally, we propose in §4 a short proof of Meyer's theorem on compensated stochastic integrals of local martingales.

§1. S.I. of Progressive Processes w.r.t. Seminartingales

We work on a filtered probability sapce $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ which verifies the usual conditions. We denote by $\mathcal{L}$ the set of all local martingales and $\mathcal{V}$ the set of all adapted processes of finite variation. Let $M \in \mathcal{L}$ and K be a predictable process such that $\sqrt{K^2 \cdot [M,M]}$ is locally integrable. There exists a unique local martingale, denoted by $K_{\dot{m}} M$, such that one has $[K_{\dot{m}} M, N] = K.[M,N]$ for each local martingale N. We call $K_{\dot{m}} M$ the stochastic integral of K w.r.t. M. We denote by $L_m(M)$ the set of all M-integrable predictiable processes. Let $A \in \mathcal{V}$ and H be a progressive process such that for almost all $\omega \in \Omega$ $H.(\omega)$ is Stieltjes integrable w.r.t. $A.(\omega)$ on $[o,t]$, $t \in \mathbb{R}_+$. H is said to be stochastic Stieltjes integrable w.r.t. A and we denote by $H_{\dot{s}} A$ this integral. Then $H_{\dot{s}} A \in \mathcal{V}$. We denote by $I_s(A)$ the set of those progressive processes which are stochastic Stieltjes integrable w.r.t. A.

Lex X be a semimartingale. A predictable process K is said to be X-integrable if there exists a so-called K-decomposition $X = M + A$ with $M \in \mathcal{L}$ and $A \in \mathcal{V}$ such that $K \in L_m(M) \cap I_s(A)$. In this case, we put $K.X = K_{\dot{m}} M + K_{\dot{s}} A$ and call $K.X$ the stochastic integral of K w.r.t. X. $K.X$ doesn't depend on the utilized K-decomposition. We denote by $L(X)$ the

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set of all X-integrable predictable processes. Let $M \in \mathcal{L}$. In general, we have $L_m(M) \subsetneq L(M)$. But for a continuous local martingale M, we have $L_m(M) = L(M)$.

It is very natural to raise the following question: How to define a stochastic integration of progressive processes w.r.t. semimartingales in such a way that it generalizes that of predictable processes w.r.t. semimartingales as well as the stochastic Stieltjes integration. We shall solve this problem in this section.

1.1. The Case of Local Martingales

First of all, we consider the case of local martingales. Recall that for any optional process H there exists always a predictable process K such that $[H \neq K]$ is a thin set.

The following definition is a slight generalization of the one given by Yor [7].

Definition 1.1. *Let $M \in \mathcal{L}$ and H be a progressive process. We denote by ${}^{\circ}H$ the optional projection of H. If there exists a predictable process K such that*

(i) $[{}^{\circ}H \neq K]$ is a thin set;

(ii) $K \in L_m(M)$;

(iii) $\sum_{s \leq t} |H_s - K_s||\Delta M_s| < \infty$, a.s., $\forall t \in \mathbb{R}_+$,

then H is said to be M-integrable and the stochastic integral $H_{\dot{m}}M$ is defined by the following formula:

$$H_{\dot{m}}M = K_{\dot{m}}M + \sum_{s \leq \cdot}(H_s - K_s)\Delta M_s. \tag{1.1}$$

It is easy to prove that $H_{\dot{m}}M$ doesn't depend on the utilized predictable process K verifying conditions (i)-(iii). We denote by $I_m(M)$ the set of all M-integrable progressive processes.

Remark 1. We have $H \in I_m(M) \Longleftrightarrow {}^{\circ}H \in I_m(M)$ and $H_{\dot{m}}M = {}^{\circ}H_{\dot{m}}M$, because $({}^{\circ}H - H)\Delta M = 0$.

Remark 2. Let $M, N \in \mathcal{L}$. In general, we don't have the inclusion $I_m(M) \cap I_m(N) \subset I_m(M + N)$.

Remark 3. Let M be a quai-continuous local martingale. If $H \in I_m(M)$, then any predictable process verifying condition (i) satisfies automatically conditions (ii) and (iii). In consequence, if M and N are two quasi-continuous local martingales, then we have $L_m(M) \cap L_m(N) \subset L_m(M + N)$. This remark is important for our definition of S.I. of progressive processes w.r.t. semimartingales (see Definition 1.4 below).

The following two lemmas are essential for our main results of this section.

Lemma 1.1. *Let $A \in \mathcal{V}$ and $H \in I_s(A)$. Then we have ${}^{\circ}H \in I_s(A)$ and ${}^{\circ}H_{\dot{s}}A = H_{\dot{s}}A$.*

Proof. Since $H_{\dot{s}}A \in \mathcal{V}$, according to a result from the general theory of stochastic processes,

${}^{\circ}H_{\dot{s}}A$ exists and we have ${}^{\circ}H_{\dot{s}}A = (H_{\dot{s}}A)^{\circ} = H_{\dot{s}}A$, where B° stands for the optional dual projection of B.

Remark. Let $A \in \mathcal{V}$ and H be a progressive process. It is possible that ${}^{\circ}H \in I_s(A)$ but $H \notin I_s(A)$.

Lemma 1.2. *Let $M \in \mathcal{L} \cap \mathcal{V}$ and $H \in I_m(M) \cap I_s(M)$. Then we have $H_{\dot{m}}M = H_{\dot{s}}M$.*

Proof. Let K be a predictable process such that conditions (i)–(iii) in Definition 1.1 are satisfied.

Set $A = \sum_{s \leq \cdot} \Delta M_s$. Since $M \in \mathcal{L} \cap \mathcal{V}$, we have $M = A - A^p$ and A^p is continuous, where A^p is the predictable dual projection of A. Thus, we have $\Delta M = \Delta A$ and $H - K \in I_s(A)$ in view of condition (iii). By Lemma 1.1, we have ${}^{\circ}H - K \in I_s(A)$ and

$$(H - K)_{\dot{s}}A = ({}^{\circ}H - K)_{\dot{s}}A = ({}^{\circ}H - K)_{\dot{s}}M.$$

Again by Lemma 1.1, we have ${}^{\circ}H \in I_s(M)$ so that $K \in I_s(M)$. Consequently, according to a property of S.I. of predictable processes w.r.t. local martingales we have $K_{\dot{m}}M = K_{\dot{s}}M$. Finally, we obtain that

$$\begin{aligned} H_{\dot{s}}M &= H_{\dot{s}}A - H_{\dot{s}}A^p = H_{\dot{s}}A - {}^{\circ}H_{\dot{s}}A^p = H_{\dot{s}}A - K_{\dot{s}}A^p \\ &= K_{\dot{s}}(A - A^p) + \sum_{s \leq \cdot}(H_s - K_s)\Delta A_s \\ &= K_{\dot{s}}M + \sum_{s \leq \cdot}(H_s - K_s)\Delta M_s \\ &= K_{\dot{m}}M + \sum_{s \leq \cdot}(H_s - K_s)\Delta M_s = H_{\dot{m}}M. \end{aligned}$$

1.2 The Case of Semimartingales

Lemma 1.2 suggests us to give the following definition.

Definition 1.2. *Let X be a semimartingale. A progressive process H is said to be X-integrable in the restricted sense, if there exists a so-called H-decomposition $X = M + A$ with $M \in \mathcal{L}$ and $A \in \mathcal{V}$ such that $H \in I_m(M) \cap I_s(A)$. In this case, we put*

$$H_{\dot{r}}X = H_{\dot{m}}M + H_{\dot{s}}A \tag{1.2}$$

and call $H_{\dot{r}}X$ the stochastic integral of H w.r.t. X in the restricted sense. By Lemma 1.2, $H_{\dot{r}}X$ doesn't depend on the utilized H-decomposition of X.

We denote by $I_r(X)$ the set of those progressive processes which are X-integrable in the restricted sense. It is easy to see that we have $L(X) \subset I_r(X)$ and $I_s(A) \subset I_r(A)$ for $A \in \mathcal{V}$ and these stochastic integrations coincide. Let X and Y be semimartingales. In general, we don't have the inclusion $I_r(X) \cap I_r(Y) \subset I_r(X+Y)$. Therefore, the above definition of S.I. isn't quite reasonable.

Befor going to give a reasonable definition of S.I. of progressive process w.r.t. semimartingales, we introduce some notations.

We denote by $\mathcal{L}^{da}$ (resp. $\mathcal{V}^{da}$) the collection of those purely discontinuous local martingales (resp. processes of finite variation) which have no jump at totally inaccessible times. We denote by $\mathcal{L}^q$ (resp. $\mathcal{V}^q$) the collection of all quasi-continuous elements of $\mathcal{L}$ (resp. $\mathcal{V}$). We put

$$\mathcal{S}^{da} = \mathcal{L}^{da} + \mathcal{V}^{da}, \mathcal{S}^q = \mathcal{L}^q + \mathcal{V}^q.$$

Then we have $\mathcal{S} = \mathcal{S}^{da} \oplus \mathcal{S}^q$ (direct sum), wher $\mathcal{S}$ is the set of all semimartingales (see [6]). Let $X \in J$. We denote by $X = X^{da} + X^q$ the decomposition of X following $\mathcal{S}^{da} \oplus \mathcal{S}^q$.

Let $X \in \mathcal{S}$. We have $L(X) = L(X^{da}) \cap L(X^q)$ and $H.X = H.X^{da} + H.X^q$. This observation suggests us to define a S.I. of progressive processes w.r.t. semimartingales along this way.

The following lemma characterizes the elements of $L(X)$ for $X \in \mathcal{J}^{da}$.

Lemma 1.3. *Let $X \in \mathcal{S}^{da}$ and H be a predictable process. Then $H \in L(X)$ if and only if there exists a (unique) $Y \in \mathcal{S}^{da}$ such that $\Delta Y = H\Delta X$. If it is the case, one has $H.X = Y$.*

Proof. We only need to prove the sufficency of the condition. Assume that there exists a $Y \in \mathcal{S}^{da}$ such that $\Delta Y = H\Delta X$. Put

$$C_t = \sum_{s \leq t} \Delta X_s I_{\{|\Delta X_s| > 1 \text{ or } |H\Delta X_s| > 1\}}.$$

Then $X - C$ and $Y - H_s C$ are special semimartingales. Let $X - C = M + A$ and $Y - H_s C = N + B$ be their canonical decompositions. Then we have $H(\Delta M + \Delta A) = \Delta N + \Delta B$. By taking predictable projections we get that $H\Delta A = \Delta B$. Thus we have $B = H_s A$ and $H\Delta M = \Delta N$. The latter equality implies that $N = H_m M$. Thus we conclude that $H \in L(X)$ and $Y = H.X$.

The following lemma characterizes the jump of an element of $\mathcal{J}^{da}$.

Lemma 1.4. *Let Z be an accessible process such that $Z_0 = 0$ and $\{Z \neq 0\}$ is a thin set. Then there exists an $X \in \mathcal{S}^{da}$ such that $\Delta X = Z$ if and only if $\sum_{s \leq \cdot} Z_s^2 \in \mathcal{V}$ and $\sum_{s \leq \cdot} |{}^p(ZI_{[|Z| \leq 1]})_s| \in \mathcal{V}$, where ${}^pH$ stands for the predictable projection of H.*

Proof. Assume that Z satisfies the conditions mentioned above. Put

$$A_t = \sum_{s \leq t} {}^p(ZI_{[|Z| \leq 1]})_s.$$

Then $A \in \mathcal{V}$ and A is a predictable process. Put

$$J = ZI_{[|Z| \leq 1]}, \quad K = J - {}^pJ.$$

Then ${}^pK = 0$, and we have

$$\sum_{s \leq t} K_s^2 \leq 2 \sum_{s \leq t} [J_s^2 + ({}^pJ_s)^2] < \infty, \quad a.s.$$

Since the increasing process $\sqrt{\sum_{s\leq \cdot} K_s^2}$ is obviously locally integrable, there exists a unique $M \in \mathcal{L}^{da}$ such that $\Delta M = K$ by a theorem of Chou and Lépingle (see [2]). Put

$$B = \sum_{s\leq \cdot} Z_s I_{[|Z_s|>1]}, \quad X = M + A + B.$$

Then $X \in \mathcal{S}^{da}$ and $\Delta X = Z$. The sufficiency of the conditions is proved. We leave the proof of the necessity part to the reader.

Lemma 1.3 suggests us to give the following definition.

Definition 1.3. *Let $X \in \mathcal{S}^{da}$. A progressive process H is said to be X-integrable if there exists a $Y \in \mathcal{S}^{da}$ such that $\Delta Y = H\Delta X$. In this case we put $H.X = Y$ and call $H.X$ the stochastic integral of H w.r.t. X. We denote by $I(X)$ the set of all X-integrable progressive processes.*

Remark 1. *Let $X \in \mathcal{S}^{da}$. Then $H \in I(X) \Longleftrightarrow {}^{\circ}H \in I(X)$ and we have $H.X = {}^{\circ}H.X$, because $H\Delta X = {}^{\circ}H\Delta X$.*

Remark 2. *Let $X \in \mathcal{S}^{da}$ and H be progressiv eprocess. If there exists a predictable process K such that $K \in L(X)$ and $\sum_{s\leq \cdot} |H_s - K_s||\Delta X_s| \in \mathcal{V}$, then $H \in I(X)$ and we have*

$$H.X = K.X + \sum_{s\leq \cdot}(H_s - K_s)\Delta X_s. \tag{1.3}$$

Now we arrive at a reasonable definition of S.I. of progressive processes w.r.t. semimartingales.

Definition 1.4. *Let $X \in \mathcal{S}^q$. A progressive process H is said to be X-integrable if there exists a so-called H-decomposition $X = M + A$ with $M \in \mathcal{L}^q$ and $A \in \mathcal{V}^q$ such that $H \in I_m(M)$ and $H \in I_s(A)$. In this case, we put*

$$H.X = H_{\dot{m}} M + H_{\dot{s}} A, \tag{1.4}$$

and call $H.X$ the stochastic integral of H w.r.t X. Let $X \in J$. A progressive process H is said to be X-integrable if H is separately X^{da}-and X^q-integrable. In this case, we put $H.X = H.X^{da} + H.X^q$, and call $H.X$ the stochastic integral of H w.r.t. X. We denote by $I(X)$ the set of all X-integrable progressive processes.

Remark 1. Let X and Y be semimartingales. If $H \in I(X) \cap I(Y)$ then $H \in I(X+Y)$ and we have $H.(X+Y) = H.X + H.Y$. Moreover, we have $L(X) \subset I(X)$ and $I_s(A) \subset I(A)$ for $A \in \mathcal{V}$. Therefore Definition 1.4 is more reasonable than Definition 1.2.

Remark 2. Since $I_m(M) = I_m(M^{da}) \cap I_m(M^q)$ for $M \in \mathcal{L}$, we have $I_r(X) \subset I(X)$ and two integrations coincide. Therefor Definition 1.4 is more general than Definition 1.2.

Remark 3. If $H \in I(X)$ then ${}^{\circ}H \in I(X)$ and $H.X = {}^{\circ}H.X$.

The following theorem gives us a usefull creterion for optional integrands.

Theorem 1.1. *Let $X \in \mathcal{S}$ and H be an optional process. If there exists a predictable process K such that*

(i) $[K \neq H]$ is a thin set; (ii) $K \in L(X)$; (iii) $\sum_{s\leq \cdot} |H_s - K_s||\Delta X_s| \in \mathcal{V}$,

then $H \in I(X)$ and we have

$$H.X = K.X + \sum_{s\leq \cdot}(H_s - K_s)\Delta X_s. \tag{1.5}$$

Proof. In view of Remark 2 following Definition 1.1, we may assume $X \in \mathcal{S}^q$. Put

$$A_t = \sum_{s\leq t} \Delta X_s I_{[|\Delta X_s||>1 \text{ or } |K_s \Delta X_s|>1]}.$$

Then $X - A$ and $K.(X - A)$ are special semimartingales. Let $X - A = N + B$ be the canonical decomposition of $X - A$. According to a lemma of Jeulin (see [5]), we have

$$K.(X - A) = K_{\dot{m}}N + K_{\dot{s}}B.$$

Since $X - A$ is quasi-continuous, B is continuous. Therefore, we have $\Delta X = \Delta N + \Delta A$, $\Delta N \Delta A = 0$, and

$$\sum_{s\leq \cdot} |H_s - K_s||\Delta N_s| + \sum_{s\leq \cdot} |H_s - K_s||\Delta A_s| = \sum_{s\leq \cdot} |H_s - K_s||\Delta X_s| \in \mathcal{V}.$$

Consequently, $H_{\dot{m}}N$ exists and we have $K_{\dot{s}}B = H_{\dot{s}}B$. Thus, $H_{\dot{s}}(A + B)$ exists and we get

$$\begin{aligned} &K.X + \sum_{s\leq \cdot}(H_s - K_s)\Delta X_s \\ =&K_{\dot{m}}N + K_{\dot{s}}B + K_{\dot{s}}A + \sum_{s\leq \cdot}(H_s - K_s)(\Delta N_s + \Delta A_s) \\ =&H_{\dot{m}}N + H_{\dot{s}}(A + B). \end{aligned}$$

This means $H \in I(X)$ and we have (1.5).

Remark. In [7], the stochastic integral of H w.r.t. X was defined by (1.5). Theorem 1.1 shows that the present definition of S.I. is more general than that given in [7].

Corollary. *Let $X, Y \in \mathcal{S}$. Then $Y \in I(X)$ and we have*

$$Y.X = Y_-.X + \sum_{s\leq \cdot} \Delta Y_s \Delta X_s. \tag{1.6}$$

Proof. By Theorem 1.1, we have $Y \in I(X^q)$ and

$$Y.X^q = Y_-.X^q + \sum_{s\leq \cdot} \Delta Y_s \Delta X^q_s.$$

Put

$$Z = Y_-.X^{da} + \sum_{s\leq \cdot} \Delta Y_s \Delta X^{da}_s.$$

Then $Z \in \mathcal{S}^{da}$ and $\Delta Z = Y\Delta X^{da}$. Thus $Y \in I(X^{da})$ and $Y.X^{da} = Z$. Consequently, (1.6) holds.

Let $M \in \mathcal{L}^q$. In general, we have $I_m(M) \subsetneq I(M)$. The following theorem shows that we have $I_m(M) = I(M)$ if M is a continuous local martingal. For further result see Theorem 1.3.(6).

Theorem 1.2. *Let M be a continuous local martingale. We have $I_m(M) = I(M)$ and $H.M = H_{\dot m}M = K_{\dot m}M$, where K is any predictable process such that $[{}^\circ H \neq K]$ is a thin set.*

Proof. Assume that $H \in I(M)$. Let $M = N + A$ be a H-decomposition of M. Then $N \in \mathcal{L}^q$ and $A \in \mathcal{L}^q \cap \mathcal{V}$. Put $B_t = \sum_{s\leq t} \Delta A_s$. Then $A = B - B^p$ and B^p is continuous. Let K be a predictable process such that $[{}^\circ H \neq K]$ is a thin set. We have

$$\begin{aligned} H.M =& H_{\dot m}N + H_{\dot s}A = K_{\dot m}N + \sum_{s\leq \cdot}(H_s - K_s)\Delta N_s + H_{\dot s}B - H_{\dot s}B^p \\ =& K_{\dot m}N - (H-K)_{\dot s}B + H_{\dot s}B - K_{\dot s}B^p \\ =& K_{\dot m}N + K_{\dot s}(B - B^p) = K.M. \end{aligned}$$

Since M is a continuous local martingale and K is a predictable process, we have $K.M = K_{\dot m}M$ by Jeulin's lemma. Consequently, we have $H \in I_m(M)$ and $H_{\dot m}M = K_{\dot m}M = H.M$.

Remark. Let M be a continuous local martingale. Then $I_m(M)$ consists of those progressive processes H such that $({}^\circ H)^2 \in I_s([M,M])$. If $H \in I_m(M)$, then $H_{\dot m}M$ is the unique continuous local martingale such that $[H_{\dot m}M, N] = {}^\circ H_{\dot s}[M,N]$ for each continuous local martingale.

We end this sub-section with the following theorem. We leave its proof to the reader.

Theorem 1.3. *Let $X \in \mathcal{S}$ and $H \in I(X)$.*

(1) *We have $\Delta(H.X) = H\Delta X$ and $(H.X)^C = H.X^C$, where X^C stands for the continous local martingale part of X.*

(2) *For any stopping time T we have*

$$(H.X)^T = H.X^T = (HI_{[0,T]}).X$$

$$(H.X)^{T-} = H.X^{T-} = (HI_{[0,T[}).X$$

(3) $[H.X, Y] = H.[X,Y], \forall Y \in \mathcal{S}$

(4) *Let K be a locally bounded predictable process, then we have*

$$K.(H.X) = H.(K.X) = (KH).X.$$

(5) *Let H' be a locally bounded progressive process such that $H' \in I(X)$. Then $H' + H \in I(X)$ and $(H' + H).X = H'.X + H.X$.*

(6) *If X is a continuous semimartingale, then for any predictable process K such that $[{}^\circ H \neq K]$ is a thin set we have $K \in L(X)$ and $K.X = H.X$.*

1.3 An Exponential Formula for Semimartingales

Let X be a semimartingale with $X_0 = 0$. Assume that $[\Delta X = 1]$ is an evanescent set. We consider the following stochastic equation:

$$Y_t = 1 + \int_0^t Y_s dX_s, \tag{1.7}$$

or equivalently (by (1.6))

$$Y_t = 1 + \int_0^t Y_{s-} dX_s + \sum_{s \le t} \Delta Y_s \Delta X_s. \tag{1.8}$$

If Y is a solution of (1.8), we have

$$\Delta Y_t = Y_{t-} \Delta X_t + \Delta Y_t \Delta X_t$$

so that

$$\Delta Y_t \Delta X_t = \frac{Y_{t-} \Delta X_c^2}{1 - \Delta X_t}. \tag{1.9}$$

Put

$$A_t = \sum_{s \le t} \frac{\Delta X_s^2}{1 - \Delta X_s}. \tag{1.10}$$

It is easy to prove that $A \in \mathcal{V}$ and we have

$$\sum_{0 \le s \le t} \Delta Y_s \Delta X_s = \int_0^t Y_{s-} dA_s.$$

Consequently, (Y_t) satisfies the following Doléans-Dade equation

$$Y_t = 1 + \int_0^t Y_{s-} d(X_s + A_s). \tag{1.11}$$

Conversely, if (Y_t) satisfies (1.11), then we have

$$\Delta Y = Y_-(\Delta X + \Delta A) = \frac{Y_- \Delta X}{1 - \Delta X},$$

from which it follows

$$Y_- \Delta A = \Delta Y - Y_- \Delta X = \Delta Y \Delta X.$$

Thus (Y_t) satisfies (1.8).

Therefore we have proved the following

Theorem 1.4. Let X be a semimartingale with $X_0 = 0$. Assume that $[\Delta X = 1]$ is an evanescent set. Then the stochastic equation (1.7) has a unique solution denoted by $e(X)$, which is given by the following formula:

$$e(X)_t = \mathcal{E}(X + A)_t = \exp\{X_t - \frac{1}{2} < X^c, X^c >_t\} \prod_{s \le t} \frac{e^{-\Delta X_s}}{1 - \Delta X_s}, \tag{1.12}$$

where A is defined by (1.10), the product $Z_t = \prod_{s\leq t} \frac{e^{-\Delta X_s}}{1-\Delta X_s}$ is absolutly convergent and (Z_t) is a process of finite variation.

Corollary. (1) Let X and Y be semimrtingales with $X_0 = Y_0 = 0$. Assume that $[\Delta X = 1]$ and $[\Delta Y = 1]$ are evanescent sets. Then we have

$$e(X)e(Y) = e(X + Y - [X,Y]) \tag{1.13}$$

(2) Let X be as above. Then we have

$$e(X)\mathcal{E}(-X + < X^c, X^c >) = 1 \tag{1.14}$$

Remark. Let X be a semimartingale with $X_0 = 0$. If we consider the following stochastic equation

$$Y_t = 1 + \int_0^e Y_s - dX_s + \sum_{s\leq t} \Delta Y_s \Delta X_s I_{[\Delta X_s \neq 1]}, \tag{1.15}$$

then from the above argument we see that (1.15) has a unique solution which is given by the follwoing formula:

$$Y_t = \exp\{X_t - \frac{1}{2} < X^c, X^c >_t\} \prod_{s\leq t} \frac{e^{-\Delta X_s}}{1-\Delta X_s} I_{[\Delta X_s \neq 1]}. \tag{1.16}$$

§2. A Simple Construction of the S.I. w.r.t. Local Martingales

In this section we shall show how to reduce the S.I. of predictable process w.r.t. local martingales to that w.r.t. locally square-integrable martingales.

Let M be a locally square-integrable martingale and H be a predictable process such that the increasing process $H^2.[M,M]$ is locally integrable. Then it is well known that there exists a unique locally square-integrable martingale, denoted by $H.M$ and called the stochastic integral of H w.r.t. M, such that for each local matringale N we have $[H.M, N] = H.[M,N]$. Moreover, one has $\Delta(H.M) = H\Delta M$. The extension of this S.I. to local martingale case has been achieved by Meyer and Doléans-Dade. We propose here a very simple appoach to this extension.

Theorem 2.1. *Let M be a local martingale and H be a predicatable process such that the increasing process $\sqrt{H^2.[M,M]}$ is locally integrable. Then there exists a unique local martingale, denoted by $H.M$ and called the stochastic integral of H w.r.t.M, such that one has $[H.M, N] = H.[M,N]$ for each local martingale N. Moreover, one has $\Delta(H.M) = H\Delta M$.*

Proof. Set

$$A_t = \sum_{s<t} \Delta M_s I_{[|\Delta M_s|>1 \text{ or } |H_s \Delta M_s|>1]}.$$

Since M is a local martingale and $\sqrt{H^2.[M,M]}$ is locally integrable, it is easy to see that A and $H.A$ are of locally integrable variation. Thus, $H.A^p$ exists and we have $(H.A)^p = H.A^p$, where A^p is the predictable dual projection of A. Put

$$V = A - A^p, \ U = M - V$$

We have

$$|\Delta U| \leq |\Delta M - \Delta A| + |\Delta A^p| \leq 1 + |^p(\Delta A)| = 1 + |^p(\Delta A - \Delta M)| \leq 2$$
$$|H\Delta A^p| = |\Delta(H.A)^p| = |^p(H\Delta A)| = |^p(H\Delta A - H\Delta M| \leq 1$$
$$|H\Delta U| \leq |H(\Delta M - \Delta A)| + |H\Delta A^p| \leq 2$$

Therefore, U is a locally square-inegrable martingale, and$H^2.[U,U]$ is locally integrable because $H^2.[U,U] \leq 2H^2.([M,M]+[V,V]) \in \mathcal{V}$. We put

$$H.M = H.U + H.A - H.A^p$$

Then $H.M$ is a local martingale which meets the requirement. The uniqueness is trival.

§3. A Simple Proof of the Integration by Parts Formula

Let X be a semimartingale. It was first discoved by Dellacherie and Meyer in their book [1] that the Ito formula could be deduced easily from the following so-called integration by parts formula: $X^2 = 2X_-.X + [X.X]$. This is a great simplification to the theory of the stochastic integration. However, the proof of the integration by parts formula given in [1] seems to be a little complicated. Now we propose a simple one.

The following lemma is well known and can be easily proved (see Jacod and Shiryaev [2]).

Lemma 3.1. *Let M be a local martingale and A be a predicatable process of finite variation. Then $[A,M]$ and $MA - M_-.A$ are local martingales.*

Now using this lemma we can prove the integration by parts formula.

Theorem 3.1. *Let X be a semimartingale. We have*

$$X^2 = 2X_-.X + [X,X].$$

Proof. Instead of considering X^{T_n-}, where $T_n = \inf\{t : |X_t| > n\}$, we may assume that X is bounded, so that X is a special semimartingale. Let $X = M+A$ be its canonical decomposition. We have, using the fact that $A^2 = 2A_-.A + [A,A]$,

$$\begin{aligned} &X^2 - 2X_-.X - [X,X] \\ &\quad = M^2 + 2MA + A^2 - 2M_-.M - 2A_-.M - 2M_-.A - 2A_-.A \\ &\quad - [M,M] - 2[M,A] - [A,A] \\ &\quad = (M^2 - [M,M]) - 2M_-.M + 2(MA - M_-.A - A_-.M + [M,A]) \end{aligned}$$

By Lemma 3.1, $X^2 - 2X_-.X - [X,X] = B$ is a local martingale. Moreover, B is continuous. On the other hand, just as proved in [1] by the dominated convergence theorem for the S.I., $X^2 - 2X_-.X$ is an increasing process. Therefore, B is of finite variation, so that $B = 0$. The theorem is proved.

§4. A Remark on the Compensated S.I. w.r.t Local Martingales

The so-called compensted S.I. of optional processes w.r.t. local martingales was introduced by Meyer [4]. The main result of this S.I. is the following theorem:

Theorem 4.1. Let M be a local martingale and H be an optional process such that $\sqrt{H^2.[M,M]}$ is locally integrable. Then there exists a unique local martingale, denoted by $H\dot{c}M$, such that for each bounded martingale $N, [H\dot{c}M,N] - H.[M,N]$ is a local martingale. Moreover, one has $\Delta(H\dot{c}M) = H\Delta M -^p (H\Delta M)$, where $^p(H\Delta M)$ is the predictable projection of $H\Delta M$.

Let M^c (resp. M^d) be the continuous (resp. purely discontinuous) local martingale part of M. If H is an optional process such that $\sqrt{H^2.[M,M]}$ is locally integrable, then $H\dot{c}M^c$ and $H\dot{c}M^d$ exist and one has $H\dot{c}M = H\dot{c}M^c + H\dot{c}M^d$. Moreover, for any predictable process K such that $[H \neq K]$ is a thin set one has $H\dot{c}M^c = K.M^c$. Therefore, the compensated S.I. can be reduced to that w.r.t. purely discontinuous local martingales. In the latter case, just as remarked by Jacod [3], one can use a theorems of Chou and Lépingle on the characterization of the jump of a local martingale to give the following general definition of the compensated S.I.

Definition 4.1. Let M be a local martingale and H be an optional process. H is said to be M-integrable in the sense of compensated S.I. (we write $H \in L_c(M)$) if (i) $H^2. < M^c, M^c >$ is an increasing process, and (ii) $^p(H\Delta M)$ exists and $\sqrt{\sum_{s\leq .} Z_s^2}$ is locally integrable, where $Z = H\Delta M -^p (H\Delta M)$. If $H \in L_c(M)$, we put

$$H\dot{c}M = K.M^c + L$$

where K is any predictable process such that $[H \neq K]$ is a thin set and L is the unique purely discontinuous local martingale such that $\Delta L = Z$.

Now we give a simple proof of Theorem 4.1 by using the theorem of Chou-Lépingle.

Proof of Theorem 4.1. We may assume that M is a purely discontinuous local martingale. Assume that $\sqrt{H^2.[M,M]}$ is locally integrable. Set $W = H\Delta M I_{[|H\Delta M|>1]}$, $U = H\Delta M I_{[|H\Delta M|\leq 1]}$, and $A = \sum_{s\leq .} W_s$. Then A is a process of locally integrable variation. We have $^p(W) = {}^p(\Delta A) = \Delta A^p$. Since $H\Delta M = W + U$, $^p(H\Delta M)$ exists.

Set $B = \sum_{s\leq .} U_s^2$. Then B is locally integrable, and we have $\Delta(B^p) = {}^p(U^2)$, so that

$\sum_{s\leq \cdot}({}^pU_s)^2 \leq \sum_{s\leq \cdot} {}^p(U^2)_s \leq \sum_{s\leq \cdot} \Delta B^p_s \leq B^p$. Put $Z = H\Delta M -^p (H\Delta M)$. We obtain that

$$\sum_{s\leq \cdot} Z_s^2 \leq 2(H^2.[M,M] + \sum_{s\leq \cdot} |{}^p(H\Delta M)_s|^2)$$
$$\leq 2[H^2.[M,M] + \sum_{s\leq \cdot}({}^pW_s)^2 + \sum_{s\leq \cdot}({}^pU_s)^2]$$

Therefore, $\sqrt{\sum_{s\leq \cdot} Z_s^2}$ is locally integrable. Let L be the unique purely discontinuous local martingale such that $\Delta L = Z$. Then for any bounded martingale N, the following process V is obviously of locally integrable variation:

$$V = [L, N] - H.[M, N]$$

and we have $\Delta V = -^p(H\Delta M)\Delta N$. Therefore, we obtain that $\Delta(V^p) = {}^p(\Delta V) = 0$. That means V^p is continuous. However, $V = \sum_{s\leq \cdot} \Delta V_s \in \mathcal{V}^{da}$, so we must have $V^p = 0$. Thus, V is a local martingale. Theorem 4.1 is proved.

The following theorem shows that the sufficient condition in Theorem 4.1 is almost necessary.

Theorem 4.2. Let $M \in \mathcal{L}$ and H be an optional process such that $H^2 \in I_s([M,M])$. Then the compensated stochastic integral $H_{\dot{c}}M$ exists if and only if $\sqrt{H^2.[M,M]}$ is locally integrable.

Proof. We only need to prove the necessity of the condition. Assume that $H_{\dot{c}}M$ exists. Let $Z = H\Delta M - {}^p(H\Delta M)$. Then we have

$$A := \sqrt{\sum_{s\leq \cdot}[{}^p(H\Delta M)_s]^2} \leq \sqrt{2}(\sqrt{\sum_{s\leq \cdot} Z_s^2} + \sqrt{H^2.[M,M]}) \in U.$$

Since A is predictable, A is locally integrable. Thus, $\sqrt{H^2.[M,M]}$ is locally integrable, because we have

$$\sqrt{\sum_{s\leq \cdot} H_s^2 \Delta M_s^2} \leq \sqrt{2}(A + \sqrt{\sum_{s\leq \cdot} Z_s^2}).$$

Corollary. Let $M \in \mathcal{L}$ and H be an optional process such that $H \in I_m(M)$. Then $H_{\dot{c}}M$ exists if and only if $H_{\dot{m}}M$ is a special semimartingale. If it is the case and $H.M = N + A$ be its canonical decomposition, then $H_{\dot{c}}M = N$.

Proof. Assume $H_{\dot{m}}M$ is a special semimartingale. Let $H_{\dot{m}}M = N + A$ be the canonical decomposition of $H_{\dot{m}}M$. Then $H\Delta M = \Delta N + \Delta A$ and ${}^p(H\Delta M) = \Delta N$. Thus $\Delta N = H\Delta M - {}^p(H\Delta M)$. On the other hand, we have $N^c = (H_{\dot{m}}M)^c = H_{\dot{m}}M^c = H_{\dot{c}}M^c$. Thus $H_{\dot{c}}M$ exists and $H_{\dot{c}}M = N$. Now assume that $H_{\dot{c}}M$ exists. Let K be a predictable process verifying conditions (i)–(iii) in Definition 1.1. Then from (1.1) it is easy to see that $H^2 \in I_s([M,M])$. Since $H_{\dot{c}}M$ exists, by Theorem 4.2 $\sqrt{H^2.[M,M]}$ is locally integrable. But we have $[H_{\dot{m}}M, H_{\dot{m}}M] = H^2.[M,M]$ by (1.1), thus $H_{\dot{m}}M$ is a special semimartingale.

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